On Brannan's Conjecture

Róbert Szász

Abstract. We prove the Brannan conjecture for particular values of the parameter. The basic tool of the study is an integral representation published in a recent work (Barnard et al., in Proc Am Math Soc 143(5):2117–2128, 2015).

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1. Introduction

We consider the following Mac-Laurin development

$$\frac{(1+xz)^\alpha}{(1-z)^\beta} = \sum_{n=0}^{\infty} A_n(\alpha, \beta, x) z^n, \quad (1)$$

where $\alpha > 0$, $\beta > 0$, $x = e^{i\theta}$, $\theta \in [-\pi, \pi]$, and $z \in U$. The radius of convergence of the series (1) is equal to 1. The author conjectured in [5] that if $\alpha > 0$, $\beta > 0$ and $|x| = 1$, then

$$|A_{2n-1}(\alpha, \beta, x)| \leq A_{2n-1}(\alpha, \beta, 1),$$

where n is a natural number. Partial results regarding this question were already proved in [1, 2, 5, 8].

The case $\beta = 1$, $\alpha \in (0, 1)$ is still open. Regarding this case, partial results were obtained in [3, 4, 6, 7]. We will prove some partial results regarding the case $\beta = 1$, and $\alpha \in (0, 1)$. We will use an integral representation proved in [3], and the fact that the conjecture was proved for $n \leq 26$ in [6].

2. Preliminaries

We introduce the notation:

$$A_n(\alpha, 1, x) = A_n(\alpha, x).$$

It is easily seen that

$$\begin{aligned} A_{2n-1}(\alpha, x) &= \sum_{k=0}^{2n-1} \frac{(-\alpha)_k (-x)^k}{k!} = 1 + \frac{\alpha}{1!} x - \frac{\alpha(1-\alpha)}{2!} x^2 + \frac{\alpha(1-\alpha)(2-\alpha)}{3!} x^3 \\ &\quad - \frac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)}{4!} x^4 + \dots + \frac{\alpha(1-\alpha)(2-\alpha) \dots (2n-2-\alpha)}{(2n-1)!} x^{2n-1}. \end{aligned}$$

(2)

We denote

$$B(t, \theta) = \frac{\cos \theta + t + t^{2n-1} \cos 2n\theta + t^{2n} \cos(2n-1)\theta}{1 + t^2 + 2t \cos \theta},$$

$$C(t, \theta) = \frac{\sin \theta + t^{2n-1} \sin 2n\theta + t^{2n} \sin(2n-1)\theta}{1 + t^2 + 2t \cos \theta}.$$

Lemma 1. For $\alpha \in (0, 1)$, the following equality holds:

$$\begin{aligned} -\Gamma(\alpha)\Gamma(-\alpha)A_{2n-1}(\alpha, e^{i\theta}) &= \int_0^1 F(t) \left[\frac{1}{\alpha} + \sum_{k=1}^{2n-1} (-t)^{k-1} e^{ki\theta} \right] dt \\ &= \int_0^1 F(t) \left[\frac{1}{\alpha} + B(t, \theta) + iC(t, \theta) \right] dt, \end{aligned}$$

where $F'(t) = -t^{-1-\alpha}(1-t)^{\alpha-1}$, and $F(1) = 0$.

Proof. We use two known equalities to prove the assertion of the lemma. The first one is the following:

$$A_n(\alpha, x) = 1 + \frac{1}{\Gamma(\alpha)\Gamma(-\alpha)} \int_0^1 \left(\sum_{k=1}^n \frac{(-tx)^k}{k} \right) t^{-\alpha-1} (1-t)^{\alpha-1} dt, \quad (3)$$

which has been deduced in [3], while the second one is the well-known equality $B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$. Replacing $p = \alpha$ and $q = 1 - \alpha$, we get

$$\int_0^1 t^{\alpha-1} (1-t)^{-\alpha} dt = \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)} = -\alpha\Gamma(\alpha)\Gamma(-\alpha) = \int_0^1 t^{-\alpha} (1-t)^{\alpha-1} dt.$$

This equality and (3) imply that

$$\Gamma(\alpha)\Gamma(-\alpha)A_n(\alpha, x) = \frac{1}{\alpha} \int_0^1 tF'(t)dt - \int_0^1 \left(\sum_{k=1}^n \frac{(-tx)^k}{k} \right) F'(t)dt,$$

or equivalently

$$-\Gamma(\alpha)\Gamma(-\alpha)A_n(\alpha, x) = \int_0^1 F'(t) \left(\frac{-t}{\alpha} + \sum_{k=1}^n \frac{(-tx)^k}{k} \right) dt.$$

Now integrating by parts and using that $\lim_{t \rightarrow 0} F(t) \left(\frac{t}{\alpha} + \sum_{k=1}^n \frac{(-tx)^k}{k} \right) = 0$ and $\lim_{t \rightarrow 1} F(t) \left(\frac{t}{\alpha} + \sum_{k=1}^n \frac{(-tx)^k}{k} \right) = 0$, we infer that

$$-\Gamma(\alpha)\Gamma(-\alpha)A_n(\alpha, x) = \int_0^1 F(t) \left(\frac{1}{\alpha} + \sum_{k=1}^n (-t)^{k-1} x^k \right) dt.$$

We get the desired equality replacing n by $2n - 1$ and $x = e^{i\theta}$.

$$\begin{aligned} -\Gamma(\alpha)\Gamma(-\alpha)A_{2n-1}(\alpha, e^{i\theta}) &= \int_0^1 F(t) \left(\frac{1}{\alpha} + \sum_{k=1}^{2n-1} (-t)^{k-1} x^k \right) dt \\ &= \int_0^1 F(t) \left(\frac{1}{\alpha} + \frac{x + x^{2n} t^{2n-1}}{1 + xt} \right) dt \\ &= \int_0^1 F(t) \left(\frac{1}{\alpha} + \frac{e^{i\theta} + t + e^{2ni\theta} t^{2n-1} + e^{(2n-1)i\theta} t^{2n}}{1 + t^2 + 2t \cos \theta} \right) dt. \end{aligned}$$

□

Remark 1. We note that the condition $\alpha \in (0, 1)$ implies the existence of the integrals in the previous lemma and its proof.

We denote

$$\begin{aligned} \Phi(\theta) &:= -\Gamma(\alpha)\Gamma(-\alpha)A_{2n-1}(\alpha, e^{i\theta}) \\ &= \int_0^1 F(t) \left[\frac{1}{\alpha} + \frac{\cos \theta + t + t^{2n-1} \cos 2n\theta + t^{2n} \cos(2n-1)\theta}{1 + t^2 + 2t \cos \theta} \right. \\ &\quad \left. + i \frac{\sin \theta + t^{2n-1} \sin 2n\theta + t^{2n} \sin(2n-1)\theta}{1 + t^2 + 2t \cos \theta} \right] dt. \end{aligned} \quad (4)$$

Since $|\Phi(\theta)|^2 \in \mathbb{R}$, it follows that

$$\begin{aligned} |\Phi(\theta)|^2 &= \Phi(\theta) \overline{\Phi(\theta)} \\ &= \left[\int_0^1 F(t) \left(\frac{1}{\alpha} + \frac{\cos \theta + t + t^{2n-1} \cos 2n\theta + t^{2n} \cos(2n-1)\theta}{1 + t^2 + 2t \cos \theta} \right. \right. \\ &\quad \left. \left. + i \frac{\sin \theta + t^{2n-1} \sin 2n\theta + t^{2n} \sin(2n-1)\theta}{1 + t^2 + 2t \cos \theta} \right) dt \right] \\ &\quad \left[\int_0^1 F(v) \left(\frac{1}{\alpha} + \frac{\cos \theta + v + v^{2n-1} \cos 2n\theta + v^{2n} \cos(2n-1)\theta}{1 + v^2 + 2v \cos \theta} \right. \right. \\ &\quad \left. \left. - i \frac{\sin \theta + v^{2n-1} \sin 2n\theta + v^{2n} \sin(2n-1)\theta}{1 + v^2 + 2v \cos \theta} \right) dv \right] \\ &= \int_0^1 \int_0^1 F(t) F(v) \left[\left(\frac{1}{\alpha} + \frac{\cos \theta + t + t^{2n-1} \cos 2n\theta + t^{2n} \cos(2n-1)\theta}{1 + t^2 + 2t \cos \theta} \right) \right. \\ &\quad \left(\frac{1}{\alpha} + \frac{\cos \theta + v + v^{2n-1} \cos 2n\theta + v^{2n} \cos(2n-1)\theta}{1 + v^2 + 2v \cos \theta} \right) \\ &\quad + \left(\frac{\sin \theta + t^{2n-1} \sin 2n\theta + t^{2n} \sin(2n-1)\theta}{1 + t^2 + 2t \cos \theta} \right) \\ &\quad \left. \left(\frac{\sin \theta + v^{2n-1} \sin 2n\theta + v^{2n} \sin(2n-1)\theta}{1 + v^2 + 2v \cos \theta} \right) \right] dt dv. \end{aligned} \quad (5)$$

Lemma 2. (a) Let $f, g : [0, 1] \rightarrow \mathbb{R}$ be two continuous functions. If f is a decreasing function, and if there is a point $t^* \in (0, 1)$ such that $g(t) \geq 0$, $t \in (0, t^*)$ and $g(t) \leq 0$, $t \in (t^*, 1)$, then

$$\int_0^1 f(t)g(t)dt \geq f(t^*) \int_0^1 g(t)dt.$$

(b) The Chebyshev inequality for integrals. If f and g are monotonic functions with different monotony, then

$$\int_0^1 f(t)g(t)dt \leq \int_0^1 f(t)dt \int_0^1 g(t)dt,$$

and in case of the same monotony we have

$$\int_0^1 f(t)g(t)dt \geq \int_0^1 f(t)dt \int_0^1 g(t)dt.$$

Proof. We have

$$\begin{aligned} \int_0^1 f(t)g(t)dt &= \int_0^{t^*} f(t)g(t)dt + \int_{t^*}^1 f(t)g(t)dt \geq \int_0^{t^*} f(t^*)g(t)dt \\ &\quad + \int_{t^*}^1 f(t^*)g(t)dt = f(t^*) \int_0^1 g(t)dt. \end{aligned}$$

□

Lemma 3. If $\theta \in [0, \frac{\pi}{2}]$, and $n \geq 27$, then

$$\begin{aligned} \text{(a)} \quad & \int_0^1 F(t) \left(B(t, 0) - B(t, \theta) \right) dt \\ & \geq \int_0^1 F(t) \frac{\frac{1}{2}(1 - \cos \theta) + t^{2n-1}(1 - \cos 2n\theta) + t^{2n}(1 - \cos(2n-1)\theta)}{(1+t)(1+t^2+2t \cos \theta)} dt, \end{aligned} \quad (6)$$

$$\begin{aligned} \text{(b)} \quad & \int_0^1 F(t) \left(1 + B(t, \theta) \right) dt \\ & \geq \int_0^1 F(t) \frac{(1+t)(1 + \cos \theta) + t^{2n-1}(1 + \cos 2n\theta) + t^{2n}(1 + \cos(2n-1)\theta)}{1 + t^2 + 2t \cos \theta} dt. \end{aligned} \quad (7)$$

Proof. According to Lemma 2 (b), we have

$$\int_0^1 \frac{t^{2n}}{1+t} dt \leq \int_0^1 t^{2n} dt \int_0^1 \frac{1}{1+t} dt = \frac{\ln 2}{2n+1}. \quad (8)$$

We use assertion (a) of Lemma 2 putting $f(t) = \frac{F(t)}{1+t^2+2t \cos \theta}$ and $g(t) = \frac{\frac{1}{2}-t-2t^{2n}}{1+t}$. If $\theta \in [0, \frac{\pi}{2}]$, then the mapping $f : [0, 1] \rightarrow [0, \infty)$ is strictly decreasing and we get

$$\begin{aligned} \int_0^1 F(t) \frac{\frac{1}{2} - t - 2t^{2n}}{(1+t)(1+t^2+2t\cos\theta)} dt &\geq \frac{F(t_n)}{1+t_n^2+2t_n\cos\theta} \\ \int_0^1 \left(\frac{\frac{1}{2}-t}{1+t} - 2 \frac{t^{2n}}{1+t} \right) dt &\geq \frac{F(t_n)}{1+t_n^2+2t_n\cos\theta} \left(\frac{3}{2} \ln 2 - 1 - \frac{2\ln 2}{2n+1} \right) > 0, \end{aligned} \quad (9)$$

where t_n denotes the unique root of the equation $\frac{1}{2} - t - 2t^{2n} = 0$, in the interval $t \in (0, 1)$. The following equality holds:

$$\begin{aligned} \int_0^1 F(t) \left(B(t, 0) - B(t, \theta) \right) dt &= \int_0^1 F(t) \frac{\frac{1}{2}(1 - \cos\theta)}{(1+t)(1+t^2+2t\cos\theta)} dt \\ &+ \int_0^1 F(t) \frac{(t^{2n} + t^{2n-1})(1 - \cos 2n\theta)}{(1+t)(1+t^2+2t\cos\theta)} dt \\ &+ \int_0^1 F(t) \frac{(t^{2n} + t^{2n+1})(1 - \cos(2n-1)\theta)}{(1+t)(1+t^2+2t\cos\theta)} dt \\ &+ \int_0^1 F(t) \frac{(\frac{1}{2} - t - 2t^{2n})(1 - \cos\theta)}{(1+t)(1+t^2+2t\cos\theta)} dt. \end{aligned} \quad (10)$$

The equality (10) and the inequality (9) imply that

$$\begin{aligned} \int_0^1 F(t) \left(B(t, 0) - B(t, \theta) \right) dt &\geq \int_0^1 F(t) \frac{\frac{1}{2}(1 - \cos\theta)}{(1+t)(1+t^2+2t\cos\theta)} dt \\ &+ \int_0^1 F(t) \frac{t^{2n-1}(1 - \cos 2n\theta)}{(1+t)(1+t^2+2t\cos\theta)} dt + \int_0^1 F(t) \frac{t^{2n}(1 - \cos(2n-1)\theta)}{(1+t)(1+t^2+2t\cos\theta)} dt. \end{aligned} \quad (11)$$

We use again assertion (a) of Lemma 2 putting $f(t) = \frac{F(t)}{1+t^2+2t\cos\theta}$ and $g(t) = t^2 - 2t^{2n} - t^{2n-1}$. If $\theta \in [0, \frac{\pi}{2}]$, then f is strictly decreasing and we get

$$\begin{aligned} \int_0^1 F(t) \frac{t^2 - t^{2n} - t^{2n-1}}{1+t^2+2t\cos\theta} dt &\geq \frac{F(t_n)}{1+t_n^2+2t_n\cos\theta} \int_0^1 (t^2 - t^{2n} - t^{2n-1}) dt \\ &= \frac{F(t_n)}{1+t_n^2+2t_n\cos\theta} \left(\frac{1}{3} - \frac{1}{2n+1} - \frac{1}{2n} \right) > 0. \end{aligned} \quad (12)$$

To finish the proof of the second inequality, we take notice of the fact that in case $\theta \in [0, \frac{\pi}{2}]$ each member of the following sum is positive:

$$\begin{aligned} \int_0^1 F(t) \left(1 + B(t, \theta) \right) dt &= \int_0^1 F(t) \frac{(1+t)(1+\cos\theta)}{1+t^2+2t\cos\theta} dt \\ &+ \int_0^1 F(t) \frac{t^{2n-1}(1+\cos 2n\theta)}{(1+t)(1+t^2+2t\cos\theta)} dt + \int_0^1 F(t) \frac{t^{2n}(1+\cos(2n-1)\theta)}{1+t^2+2t\cos\theta} dt \\ &+ \int_0^1 F(t) \frac{t^2 - t^{2n} - t^{2n-1}}{1+t^2+2t\cos\theta} dt + \int_0^1 F(t) \frac{t\cos\theta}{1+t^2+2t\cos\theta} dt. \end{aligned}$$

Thus, we get

$$\int_0^1 F(t) (1 + B(t, \theta)) dt \geq \int_0^1 F(t) \frac{(1+t)(1+\cos\theta)}{(1+t)(1+t^2+2t\cos\theta)} dt \\ + \int_0^1 F(t) \frac{t^{2n-1}(1+\cos 2n\theta)}{(1+t)(1+t^2+2t\cos\theta)} dt + \int_0^1 F(t) \frac{t^{2n}(1+\cos(2n-1)\theta)}{(1+t)(1+t^2+2t\cos\theta)} dt,$$

and the proof is done. $\square$

Lemma 4. If $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3}]$, then

$$\frac{5}{2} + \frac{t + \cos\theta}{1 + t^2 + 2t\cos\theta} \geq \frac{50}{23} \frac{(1+t)(1+\cos\theta)}{1 + t^2 + 2t\cos\theta}, \quad (\forall) t \in [0, 1]. \quad (13)$$

Proof. The inequality (13) is equivalent to

$$f(t) = \frac{5}{2}t^2 + t \left(\frac{65}{23} \cos\theta - \frac{27}{23} \right) + \frac{15}{46} - \frac{27}{23} \cos\theta \geq 0, \quad (\forall) t \in [0, 1].$$

The function f has a minimum point at $t^* = -\frac{65}{115} \cos\theta + \frac{27}{115} \in (0, 1)$, for every $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3}]$. Thus, we get

$$f(t) \geq f(t^*) = \frac{15}{46} - \frac{27}{23} \cos\theta - \frac{1}{10} \left(\frac{27 - 65 \cos\theta}{23} \right)^2, \quad (\forall) t \in [0, 1] \text{ and } \theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3} \right].$$

Consequently, to prove (13) we have to show that

$$\frac{15}{46} - \frac{27}{23} \cos\theta - \frac{1}{10} \left(\frac{27 - 65 \cos\theta}{23} \right)^2 \geq 0, \quad (\forall) \theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3} \right].$$

Using the notation $x = \frac{27 - 65 \cos\theta}{23}$, we get

$$\frac{15}{46} - \frac{27}{23} \cos\theta - \frac{1}{10} \left(\frac{27 - 65 \cos\theta}{23} \right)^2 = -\frac{483}{2990} + \frac{27}{65}x - \frac{1}{10}x^2. \\ \theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3} \right] \Rightarrow x \in \left[\frac{27}{23}, \frac{119}{46} \right]. \quad (14)$$

The equality 14 implies that

$$\frac{5}{12} - \frac{13}{12} \cos\theta - \frac{5}{2} \left(\frac{13 - 35 \cos\theta}{60} \right)^2 \geq \min_{x \in [\frac{27}{23}, \frac{119}{46}]} \left\{ -\frac{483}{2990} + \frac{27}{65}x - \frac{1}{10}x^2 \right\} \\ \geq \min_{x \in \left[\frac{3}{2}, \frac{13}{5} \right]} \left\{ -\frac{1}{5} + \frac{5}{13}x - \frac{1}{10}x^2 \right\} = \min \left\{ g\left(\frac{3}{2}\right), g\left(\frac{13}{5}\right) \right\} \\ = g\left(\frac{13}{5}\right) = \frac{31}{250}, \quad \theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3} \right],$$

where $g(x) = -\frac{1}{5} + \frac{5}{13}x - \frac{1}{10}x^2$, and consequently (13) holds. $\square$

Lemma 5. *If $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3}]$, and $n \geq 27$, then*

$$\begin{aligned} & \int_0^1 F(t)(B(t, 0) - B(t, \theta))dt \\ & \geq \int_0^1 F(t) \frac{\frac{27}{50}(1 - \cos \theta) + t^{2n-1}(1 - \cos 2n\theta) + t^{2n}(1 - \cos(2n-1)\theta)}{(1+t)(1+t^2+2t \cos \theta)} dt \end{aligned} \quad (15)$$

$$\begin{aligned} & \int_0^1 F(t)(2 + B(t, 0) + B(t, \theta))dt \geq \int_0^1 F(t) \\ & \frac{\frac{50}{27}(1+t)(1 + \cos \theta) + 2t^{2n-1}(1 + \cos 2n\theta) + 2t^{2n}(1 + \cos(2n-1)\theta)}{1 + t^2 + 2t \cos \theta} dt. \end{aligned} \quad (16)$$

Proof. We use assertion (a) of Lemma 2, putting $f(t) = \frac{F(t)}{(1+t)(1+t^2+2t \cos \theta)}$ and $g(t) = \frac{27}{50} - t - 2t^{2n}$. If $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3}]$, the mapping $f : [0, 1] \rightarrow [0, \infty)$ is strictly decreasing and we get

$$\begin{aligned} & \int_0^1 F(t) \frac{\frac{27}{50} - t - 2t^{2n}}{(1+t)(1+t^2+2t \cos \theta)} dt \\ & \geq \frac{F(t_n)}{(1+t_n)(1+t_n^2+2t_n \cos \theta)} \int_0^1 \left(\frac{27}{50} - t - 2t^{2n} \right) dt, \quad (17) \\ & > \frac{F(t_n)}{(1+t_n)(1+t_n^2+2t_n \cos \theta)} \left(\frac{27}{50} - \frac{1}{2} - \frac{2}{2n+1} \right) > 0 \end{aligned}$$

where t_n denotes the unique root of the equation $\frac{27}{50} - t - 2t^{2n} = 0$, in the interval $(0, 1)$.

$$\begin{aligned} \int_0^1 F(t)(B(t, 0) - B(t, \theta))dt &= \int_0^1 F(t) \frac{\frac{23}{50}(1 - \cos \theta)}{(1+t)(1+t^2+2t \cos \theta)} dt \\ &+ \int_0^1 F(t) \frac{(t^{2n} + t^{2n-1})(1 - \cos 2n\theta)}{(1+t)(1+t^2+2t \cos \theta)} dt \\ &+ \int_0^1 F(t) \frac{(t^{2n} + t^{2n+1})(1 - \cos(2n-1)\theta)}{(1+t)(1+t^2+2t \cos \theta)} dt \\ &+ \int_0^1 F(t) \frac{\left(\frac{27}{50} - t - 2t^{2n} \right)(1 - \cos \theta)}{(1+t)(1+t^2+2t \cos \theta)} dt. \end{aligned} \quad (18)$$

The equality (18) and the inequality (17) imply (15).

To prove the second inequality, we remark that

$$\begin{aligned}
 \int_0^1 F(t)(2 + B(t, 0) + B(t, \theta))dt &= \int_0^1 F(t) \left(\frac{5}{2} + \frac{1+t^{2n-1}}{1+t} - \frac{1}{2} + B(t, \theta) \right) dt \\
 &= \int_0^1 F(t) \left(\frac{5}{2} + \frac{1+2t^{2n-1}-t}{2(1+t)} \right. \\
 &\quad \left. + \frac{t + \cos \theta + t^{2n} \cos(2n-1)\theta + t^{2n-1} \cos(2n\theta)}{1+t^2+2t \cos \theta} \right) dt \\
 &= \int_0^1 F(t) \left(\frac{5}{2} + \frac{t + \cos \theta}{1+t^2+2t \cos \theta} \right. \\
 &\quad \left. + \frac{1+2t^{2n-1}-t}{2(1+t)} + \frac{t^{2n} \cos(2n-1)\theta + t^{2n-1} \cos(2n\theta)}{1+t^2+2t \cos \theta} \right) dt \quad (19) \\
 &= \int_0^1 F(t) \left(\frac{5}{2} + \frac{t + \cos \theta}{1+t^2+2t \cos \theta} \right. \\
 &\quad \left. + \frac{2t^{2n}(1 + \cos(2n-1)\theta) + 2t^{2n-1}(1 + \cos(2n\theta))}{1+t^2+2t \cos \theta} \right) dt \\
 &\quad + \int_0^1 F(t) \frac{2t^{2n-1}}{1+t} dt \\
 &\quad + \int_0^1 F(t) \left(\frac{1-t}{2(1+t)} - \frac{t^{2n}(2 + \cos(2n-1)\theta) + t^{2n-1}(2 + \cos(2n\theta))}{1+t^2+2t \cos \theta} \right) dt,
 \end{aligned}$$

and

$$\begin{aligned}
 &\int_0^1 F(t) \left(\frac{1-t}{2(1+t)} - \frac{t^{2n}(2 + \cos(2n-1)\theta) + t^{2n-1}(2 + \cos(2n\theta))}{1+t^2+2t \cos \theta} \right) dt \\
 &\geq \int_0^1 F(t) \left(\frac{1-t}{2(1+t)} - \frac{2(t^{2n-1} + t^{2n}) + t^{2n-1} \left| \cos(2n-1)\theta + t \cos(2n\theta) \right|}{1+t^2-t} \right) dt. \quad (20)
 \end{aligned}$$

If $\cos(2n-1)\theta$ and $\cos 2n\theta$ have different signs, then we have $|\cos(2n-1)\theta + t \cos(2n\theta)| \leq 1$. If $\cos(2n-1)\theta$ and $\cos 2n\theta$ have the same sign, then it follows that $|\cos(2n-1)\theta + t \cos(2n\theta)| \leq |\cos(2n-1)\theta + \cos(2n\theta)| = 2|\cos(2n-1/2)\theta| \cos \theta/2 \leq 2 \cos \theta/2 \leq \sqrt{2}$ for every $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3}]$. Thus, we infer

$$\begin{aligned}
 &\int_0^1 F(t) \left(\frac{1-t}{2(1+t)} - \frac{2(t^{2n-1} + t^{2n}) + t^{2n-1} \left| \cos(2n-1)\theta + t \cos(2n\theta) \right|}{1+t^2-t} \right) dt \\
 &\geq \int_0^1 F(t) \left(\frac{1-t}{2(1+t)} - \frac{(4+\sqrt{2})t^{2n-1}}{1+t^2-t} \right) dt
 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^1 \frac{F(t)}{2(1+t^3)} (1-2t+2t^2-t^3-2(4+\sqrt{2})t^{2n-1}(1+t))dt \\
 &\geq \int_0^1 \frac{F(t)}{2(1+t^3)} (1-2t+2t^2-t^3-4(4+\sqrt{2})t^{2n-1})dt. \quad (21)
 \end{aligned}$$

Let $g(t) = 1-2t+2t^2-t^3-4(4+\sqrt{2})t^{2n-1}$ and $f(t) = \frac{F(t)}{2(1+t^3)}$. The functions f and g are strictly decreasing on $[0, 1]$ and according to the assertion (a) of Lemma 2 there is a point $t^* \in (0, 1)$ (the unique root of the equation $g(t) = 0$) such that the following inequality holds:

$$\begin{aligned}
 &\int_0^1 \frac{F(t)}{2(1+t^3)} (1-2t+2t^2-t^3-4(4+\sqrt{2})t^{2n-1})dt \\
 &\geq \frac{F(t^*)}{2(1+(t^*)^3)} \int_0^1 (1-2t+2t^2-t^3-4(4+\sqrt{2})t^{2n-1})dt \\
 &= \frac{F(t^*)}{2(1+(t^*)^3)} \left[\frac{5}{12} - \frac{2(4+\sqrt{2})}{n} \right] > 0, \quad \theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3} \right], \quad n \geq 27. \quad (22)
 \end{aligned}$$

Now, inequalities (20)–(22) imply

$$\begin{aligned}
 &\int_0^1 F(t) \left(\frac{1-t}{2(1+t)} - \frac{t^{2n}(2+\cos(2n-1)\theta) + t^{2n-1}(2+\cos(2n\theta))}{1+t^2+2t\cos\theta} \right) dt \\
 &\geq \frac{F(t^*)}{2(1+(t^*)^3)} \left[\frac{5}{12} - \frac{2(4+\sqrt{2})}{n} \right] > 0, \quad \theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3} \right], \quad n \geq 27. \quad (23)
 \end{aligned}$$

Finally inequality (23), Lemma 4 and equality (19) imply (16), and the proof is done. $\square$

Lemma 6. *The following inequality holds:*

$$6 \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{3 \cdot 2}\right) \left(1 - \frac{1}{3 \cdot 3}\right) \dots \left(1 - \frac{1}{3 \cdot 53}\right) < \frac{4}{3}.$$

Proof. We introduce the notation

$$\begin{aligned}
 N &= \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{3 \cdot 2}\right) \left(1 - \frac{1}{3 \cdot 3}\right) \dots \left(1 - \frac{1}{3 \cdot 53}\right) \\
 &= \frac{1 \cdot 3 - 1}{1 \cdot 3} \cdot \frac{2 \cdot 3 - 1}{2 \cdot 3} \cdot \frac{3 \cdot 3 - 1}{3 \cdot 3} \cdot \frac{4 \cdot 3 - 1}{4 \cdot 3} \cdot \dots \cdot \frac{53 \cdot 3 - 1}{53 \cdot 3}. \quad (24)
 \end{aligned}$$

We will use the following implication $\alpha > 0, a > b > 1 \Rightarrow \frac{b}{a} < \frac{b+\alpha}{a+\alpha}$. Using this implication, we get

$$N < \frac{1 \cdot 3 - 1}{1 \cdot 3} \cdot \frac{2 \cdot 3}{2 \cdot 3 + 1} \cdot \frac{3 \cdot 3}{3 \cdot 3 + 1} \cdot \frac{4 \cdot 3}{4 \cdot 3 + 1} \cdot \dots \cdot \frac{53 \cdot 3}{53 \cdot 3 + 1} \quad (25)$$

and

$$N < \frac{1 \cdot 3 - 1}{1 \cdot 3} \cdot \frac{2 \cdot 3 + 1}{2 \cdot 3 + 2} \cdot \frac{3 \cdot 3 + 1}{3 \cdot 3 + 2} \cdot \frac{4 \cdot 3 + 1}{4 \cdot 3 + 2} \cdots \frac{53 \cdot 3 + 1}{53 \cdot 3 + 2}. \quad (26)$$

Multiplying (24), (25), and (26), we get

$$N^3 < \left(\frac{2}{3}\right)^3 \frac{5}{53 \cdot 3 + 2}.$$

Thus to finish the proof, we have to check the inequality

$$\left(\frac{2}{3}\right)^3 \frac{5}{53 \cdot 3 + 2} < \left(\frac{2}{9}\right)^3$$

which is immediate. $\square$

3. Main Result

Theorem 1. *If n is a natural number, $n \geq 27$ and $\alpha \in (0, 1)$ then the following inequality holds:*

$$|A_{2n-1}(\alpha, e^{i\theta})| \leq |A_{2n-1}(\alpha, 1)|, \quad \text{for all } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \quad (27)$$

Proof. According to (4) and (5), the inequality (27) is equivalent to

$$|\Phi(0)|^2 \geq |\Phi(\theta)|^2, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Leftrightarrow |\Phi(0)|^2 - |\Phi(\theta)|^2 \geq 0, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \quad (28)$$

We denote

$$B(t, \theta) = \frac{\cos \theta + t + t^{2n-1} \cos 2n\theta + t^{2n} \cos(2n-1)\theta}{1 + t^2 + 2t \cos \theta},$$

$$C(t, \theta) = \frac{\sin \theta + t^{2n-1} \sin 2n\theta + t^{2n} \sin(2n-1)\theta}{1 + t^2 + 2t \cos \theta}.$$

The equality (5) implies that

$$\begin{aligned} \Phi^2(0) - |\Phi(\theta)|^2 &= \int_0^1 \int_0^1 F(t)F(v) \left[\left(\frac{1}{\alpha} + B(t, 0) \right) \left(\frac{1}{\alpha} + B(v, 0) \right) \right. \\ &\quad \left. - \left(\frac{1}{\alpha} + B(t, \theta) \right) \left(\frac{1}{\alpha} + B(v, \theta) \right) - C(t, \theta)C(v, \theta) \right] dt dv \\ &= \int_0^1 \int_0^1 F(t)F(v) \left[\frac{1}{\alpha} (B(t, 0) - B(t, \theta)) + \frac{1}{\alpha} (B(v, 0) - B(v, \theta)) \right. \\ &\quad \left. + B(t, 0)B(v, 0) - B(t, \theta)B(v, \theta) - C(t, \theta)C(v, \theta) \right] dt dv. \end{aligned} \quad (29)$$

Since $\Phi^2(0) - |\Phi(\theta)|^2$ is an even function with respect to θ , we have to prove (28) only for $\theta \in [0, \frac{\pi}{2}]$.

Lemma 3, (a) implies that in case $\theta \in [0, \frac{\pi}{2}]$, the following inequality holds: $\int_0^1 F(t)(B(t, 0) - B(t, \theta))dt \geq 0$. Thus, the condition $\alpha \in (0, 1)$, implies that

$$\begin{aligned} \Phi^2(0) - |\Phi(\theta)|^2 &\geq \Lambda(\theta) = \int_0^1 \int_0^1 F(t)F(v)[(B(t, 0) - B(t, \theta)) \\ &\quad + (B(v, 0) - B(v, \theta)) + B(t, 0)B(v, 0) - B(t, \theta)B(v, \theta) - C(t, \theta)C(v, \theta)]dtdv. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \Lambda(\theta) &= \int_0^1 \int_0^1 F(t)F(v)[(B(t, 0) - B(t, \theta))(1 + B(v, \theta)) \\ &\quad + (B(v, 0) - B(v, \theta))(1 + B(t, \theta))]dtdv \\ &\quad - \int_0^1 \int_0^1 F(t)F(v)C(t, \theta)C(v, \theta)dtdv \\ &\quad + \int_0^1 \int_0^1 F(t)F(v)(B(t, 0) - B(t, \theta))(B(v, 0) - B(v, \theta))dtdv \\ &= \int_0^1 \int_0^1 F(t)F(v)[(B(t, 0) - B(t, \theta))(1 + B(v, \theta)) \\ &\quad + (B(v, 0) - B(v, \theta))(1 + B(t, \theta))]dtdv \\ &\quad - \int_0^1 \int_0^1 F(t)F(v)C(t, \theta)C(v, \theta)dtdv \\ &\quad + \int_0^1 F(t)(B(t, 0) - B(t, \theta))dt \int_0^1 F(v)(B(v, 0) - B(v, \theta))dv \\ &\geq \int_0^1 \int_0^1 F(t)F(v)[(B(t, 0) - B(t, \theta))(1 + B(v, \theta)) \\ &\quad + (B(v, \theta) - B(v, \theta))(1 + B(t, \theta))]dtdv \\ &\quad - \int_0^1 \int_0^1 F(t)F(v)C(t, \theta)C(v, \theta)dtdv. \end{aligned}$$

This inequality implies

$$\begin{aligned} \Phi^2(0) - |\Phi(\theta)|^2 &\geq \int_0^1 F(t)(B(t, 0) - B(t, \theta))dt \int_0^1 F(v)(1 + B(v, \theta))dv \\ &\quad + \int_0^1 F(t)(1 + B(t, \theta))dt \int_0^1 F(v)(B(v, 0) - B(v, \theta))dv \\ &\quad - \int_0^1 \int_0^1 F(t)F(v)C(t, \theta)C(v, \theta)dtdv. \end{aligned} \quad (30)$$

The inequality between the arithmetic and geometric means leads to

$$\begin{aligned}
 & \int_0^1 F(t) \left(B(t, 0) - B(t, \theta) \right) dt \int_0^1 F(v) \left(1 + B(v, \theta) \right) dv \\
 & + \int_0^1 F(t) \left(1 + B(t, \theta) \right) dt \int_0^1 F(v) \left(B(v, 0) - B(v, \theta) \right) dv \\
 & - \int_0^1 \int_0^1 F(t) F(v) C(t, \theta) C(v, \theta) dt dv \\
 & \geq 2 \left[\int_0^1 F(t) (B(t, 0) - B(t, \theta)) dt \int_0^1 F(v) (1 + B(v, \theta)) dv \right. \\
 & \left. \int_0^1 F(t) (1 + B(t, \theta)) dt \int_0^1 F(v) (B(v, 0) - B(v, \theta)) dv \right]^{\frac{1}{2}} \quad (31) \\
 & - \int_0^1 \int_0^1 F(t) F(v) C(t, \theta) C(v, \theta) dt dv \\
 & = 2 \int_0^1 F(t) \left(1 + B(t, \theta) \right) dt \int_0^1 F(t) \left(B(t, 0) - B(t, \theta) \right) dt \\
 & - \left(\int_0^1 F(t) C(t, \theta) dt \right)^2.
 \end{aligned}$$

We apply Lemma 3 and get

$$\begin{aligned}
 & 2 \int_0^1 F(t) \left(1 + B(t, \theta) \right) dt \int_0^1 F(t) \left(B(t, 0) - B(t, \theta) \right) dt - \left(\int_0^1 F(t) C(t, \theta) dt \right)^2 \\
 & \geq 2 \left(\int_0^1 F(t) \frac{\frac{1}{2}(1 - \cos \theta) + t^{2n-1}(1 - \cos 2n\theta) + t^{2n}(1 - \cos(2n-1)\theta)}{(1+t)(1+t^2+2t \cos \theta)} dt \right. \\
 & \left. \int_0^1 F(t) \frac{(1+t)(1 + \cos \theta) + t^{2n-1}(1 + \cos 2n\theta) + t^{2n}(1 + \cos(2n-1)\theta)}{1+t^2+2t \cos \theta} dt \right. \\
 & \left. - \left(\int_0^1 F(t) C(t, \theta) dt \right)^2 \right). \quad (32)
 \end{aligned}$$

The Cauchy–Schwarz inequality for integrals implies

$$\begin{aligned}
 & 2 \int_0^1 F(t) \frac{\frac{1}{2}(1 - \cos \theta) + t^{2n-1}(1 - \cos 2n\theta) + t^{2n}(1 - \cos(2n-1)\theta)}{(1+t)(1+t^2+2t \cos \theta)} dt \\
 & \int_0^1 F(t) \frac{(1+t)(1 + \cos \theta) + t^{2n-1}(1 + \cos 2n\theta) + t^{2n}(1 + \cos(2n-1)\theta)}{1+t^2+2t \cos \theta} dt \\
 & - \left(\int_0^1 F(t) C(t, \theta) dt \right)^2 \\
 & \geq 2 \left(\int_0^1 F(t) \sqrt{\frac{\frac{1}{2}(1 - \cos \theta) + t^{2n-1}(1 - \cos 2n\theta) + t^{2n}(1 - \cos(2n-1)\theta)}{(1+t)(1+t^2+2t \cos \theta)}} dt \right.
 \end{aligned}$$

$$\sqrt{\frac{(1+t)(1+\cos\theta)+t^{2n-1}(1+\cos 2n\theta)+t^{2n}(1+\cos(2n-1)\theta)}{1+t^2+2t\cos\theta}}dt\Big)^2 - \left(\int_0^1 F(t)C(t,\theta)dt\right)^2. \quad (33)$$

Putting $a_1^2 = \frac{\frac{1}{2}(1-\cos\theta)}{(1+t)(1+t^2+2t\cos\theta)}$, $b_1^2 = \frac{(1+t)(1+\cos\theta)}{1+t^2+2t\cos\theta}$ and so on in the inequality

$$\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2} \geq |a_1b_1|+|a_2b_2|+|a_3b_3|,$$

we get

$$\begin{aligned} & 2\left(\int_0^1 F(t)\sqrt{\frac{\frac{1}{2}(1-\cos\theta)+t^{2n-1}(1-\cos 2n\theta)+t^{2n}(1-\cos(2n-1)\theta)}{(1+t)(1+t^2+2t\cos\theta)}}\right. \\ & \quad \left.\sqrt{\frac{(1+t)(1+\cos\theta)+t^{2n-1}(1+\cos 2n\theta)+t^{2n}(1+\cos(2n-1)\theta)}{1+t^2+2t\cos\theta}}dt\right)^2 \\ & \quad - \left(\int_0^1 F(t)C(t,\theta)dt\right)^2 \\ & \geq 2\left(\int_0^1 F(t)\frac{\sqrt{\frac{1}{2}(1+t)|\sin\theta|+t^{2n-1}|\sin 2n\theta|+t^{2n}|\sin(2n-1)\theta|}}{(1+t^2+2t\cos\theta)\sqrt{1+t}}dt\right)^2 \\ & \quad - \left(\int_0^1 F(t)C(t,\theta)dt\right)^2. \end{aligned} \quad (34)$$

On the other hand, we have

$$\begin{aligned} & 2\left(\int_0^1 F(t)\frac{\sqrt{\frac{1}{2}(1+t)|\sin\theta|+t^{2n-1}|\sin 2n\theta|+t^{2n}|\sin(2n-1)\theta|}}{(1+t^2+2t\cos\theta)\sqrt{1+t}}dt\right)^2 \\ & \quad - \left(\int_0^1 F(t)C(t,\theta)dt\right)^2 \\ & \geq \left(\int_0^1 F(t)\frac{\sqrt{(1+t)|\sin\theta|+\sqrt{2}t^{2n-1}|\sin 2n\theta|+\sqrt{2}t^{2n}|\sin(2n-1)\theta|}}{(1+t^2+2t\cos\theta)\sqrt{1+t}}dt\right)^2 \\ & \quad - \left(\int_0^1 F(t)C(t,\theta)dt\right)^2 \\ & \geq \left(\int_0^1 F(t)\frac{|\sin\theta|+t^{2n-1}|\sin 2n\theta|+t^{2n}|\sin(2n-1)\theta|}{1+t^2+2t\cos\theta}dt\right)^2 \\ & \quad - \left(\int_0^1 F(t)\frac{\sin\theta+t^{2n-1}\sin 2n\theta+t^{2n}\sin(2n-1)\theta}{1+t^2+2t\cos\theta}dt\right)^2 \geq 0, \quad \theta \in \left[0, \frac{\pi}{2}\right]. \end{aligned} \quad (35)$$

Finally, the inequalities (30)–(35) imply that

$$\Phi^2(0) - |\Phi(\theta)|^2 \geq 0, \quad \theta \in \left[0, \frac{\pi}{2}\right],$$

and consequently inequalities (28) and (27) hold in case $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. $\square$

Theorem 2. *If n is a natural number, $n \geq 27$ and $\alpha \in (0, 1)$, then the following inequality holds:*

$$|A_{2n-1}(\alpha, e^{i\theta})| \leq |A_{2n-1}(\alpha, 1)|, \quad \text{for all } \theta \in \left[-\frac{2\pi}{3}, -\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]. \quad (36)$$

Proof. Equality (29) can be rewritten as follows:

$$\begin{aligned} & 2\left(\Phi^2(0) - |\Phi(\theta)|^2\right) \\ &= \int_0^1 F(t) \left(B(t, 0) - B(t, \theta)\right) dt \int_0^1 F(v) \left(\frac{2}{\alpha} + B(v, 0) + B(v, \theta)\right) dv \\ & \quad + \int_0^1 F(v) \left(B(v, 0) - B(v, \theta)\right) dv \int_0^1 F(t) \left(\frac{2}{\alpha} + B(t, 0) + B(t, \theta)\right) dt \\ & \quad - 2 \int_0^1 \int_0^1 F(t) F(v) C(t, \theta) C(v, \theta) dt dv. \end{aligned} \quad (37)$$

Since $2\left(\Phi^2(0) - |\Phi(\theta)|^2\right)$ defines an even function to prove (36), it is enough to show that

$$\Phi^2(0) - |\Phi(\theta)|^2 \geq 0, \quad \theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right].$$

The inequality between the arithmetic and geometric means and the condition $\alpha \in (0, 1]$ imply

$$\begin{aligned} & 2\left(\Phi^2(0) - |\Phi(\theta)|^2\right) \geq 2 \left[\int_0^1 F(t) \left(B(t, 0) - B(t, \theta)\right) dt \int_0^1 F(v) \left(2 + B(v, 0) \right. \right. \\ & \quad \left. \left. + B(v, \theta)\right) dv \int_0^1 F(v) \left(B(v, 0) - B(v, \theta)\right) dv \int_0^1 F(t) \left(2 + B(t, 0) + B(t, \theta)\right) dt \right]^{\frac{1}{2}} \\ & \quad - 2 \int_0^1 \int_0^1 F(t) F(v) C(t, \theta) C(v, \theta) dt dv \\ &= 2 \int_0^1 F(t) \left(B(t, 0) - B(t, \theta)\right) dt \int_0^1 F(t) \left(2 + B(t, 0) + B(t, \theta)\right) dt \\ & \quad - 2 \left(\int_0^1 F(t) C(t, \theta) dt \right)^2. \end{aligned} \quad (38)$$

Lemma 5 and inequality (38) lead to

$$\begin{aligned} & \Phi^2(0) - |\Phi(\theta)|^2 \\ & \geq \int_0^1 F(t) \frac{\frac{12}{25}(1 - \cos \theta) + t^{2n-1}(1 - \cos 2n\theta) + t^{2n}(1 - \cos(2n-1)\theta)}{(1+t)(1+t^2+2t \cos \theta)} dt \end{aligned}$$

$$\int_0^1 F(t) \frac{\frac{25}{12}(1+t)(1+\cos\theta) + 2t^{2n-1}(1+\cos 2n\theta) + 2t^{2n}(1+\cos(2n-1)\theta)}{1+t^2+2t\cos\theta} dt - \left(\int_0^1 F(t) C(t, \theta) dt \right)^2. \quad (39)$$

We apply twice the Cauchy–Schwarz inequality just like in the proof of the previous theorem and we get

$$\begin{aligned} & \int_0^1 F(t) \frac{\frac{12}{25}(1-\cos\theta) + t^{2n-1}(1-\cos 2n\theta) + t^{2n}(1-\cos(2n-1)\theta)}{(1+t)(1+t^2+2t\cos\theta)} dt \\ & \int_0^1 F(t) \frac{\frac{25}{12}(1+t)(1+\cos\theta) + 2t^{2n-1}(1+\cos 2n\theta) + 2t^{2n}(1+\cos(2n-1)\theta)}{1+t^2+2t\cos\theta} dt \\ & \geq \left(\int_0^1 F(t) \frac{\sqrt{1+t}|\sin\theta| + \sqrt{2}t^{2n-1}|\sin 2n\theta| + \sqrt{2}t^{2n}|\sin(2n-1)\theta|}{(1+t^2+2t\cos\theta)\sqrt{1+t}} dt \right)^2. \end{aligned} \quad (40)$$

The following inequality obviously holds:

$$\begin{aligned} & \left(\int_0^1 F(t) \frac{\sqrt{1+t}|\sin\theta| + \sqrt{2}t^{2n-1}|\sin 2n\theta| + \sqrt{2}t^{2n}|\sin(2n-1)\theta|}{(1+t^2+2t\cos\theta)\sqrt{1+t}} dt \right)^2 \\ & \geq \left(\int_0^1 F(t) \frac{\sin\theta + t^{2n-1}\sin 2n\theta + t^{2n}\sin(2n-1)\theta}{1+t^2+2t\cos\theta} dt \right)^2, \end{aligned}$$

and is equivalent to

$$\begin{aligned} & \left(\int_0^1 F(t) \frac{\sqrt{1+t}|\sin\theta| + \sqrt{2}t^{2n-1}|\sin 2n\theta| + \sqrt{2}t^{2n}|\sin(2n-1)\theta|}{(1+t^2+2t\cos\theta)\sqrt{1+t}} dt \right)^2 \\ & \geq \left(\int_0^1 F(t) C(t, \theta) dt \right)^2. \end{aligned} \quad (41)$$

Finally, (39)–(41) imply $\Phi^2(0) - |\Phi(\theta)|^2 \geq 0$, $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3}]$, and the proof is done. $\square$

Theorem 3. *If n is a natural number, $n \geq 27$ and $\alpha \in [\frac{1}{3}, 1)$, then the following inequality holds:*

$$A_{2n-1}(\alpha, 1) \geq |A_{2n-1}(\alpha, e^{i\theta})|, \quad \text{for all } \theta \in \left[-\pi, -\frac{2\pi}{3}\right] \cup \left[\frac{2\pi}{3}, \pi\right]. \quad (42)$$

Proof. We will use the Taylor formula with an integral remainder. Let $f : (-1, 1) \rightarrow \mathbb{R}$ be a $2n$ times derivable function. If $x \in (-1, 1)$, then

$$\begin{aligned} f(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(2n-1)}(0)}{(2n-1)!}x^{2n-1} \\ &\quad + \frac{x^{2n}}{(2n-1)!} \int_0^1 (1-t)^{2n-1} f^{(2n)}(xt) dt. \end{aligned}$$

Let f be the function defined by $f : (-1, 1) \rightarrow \mathbb{R}$, $f(x) = (1+x)^\alpha$, $\alpha \in (0, 1)$ and we get

$$(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots + \frac{\alpha(\alpha-1)\dots(\alpha-2n+2)}{(2n-1)!}x^{2n-1} \\ + x^{2n} \frac{\alpha(\alpha-1)\dots(\alpha-2n+1)}{(2n-1)!} \int_0^1 (1-t)^{2n-1}(1+xt)^{\alpha-2n} dt.$$

Since the mapping $g : U \rightarrow \mathbb{C}$, $g(z) = (1+z)^\alpha$ is well defined (we take the principal branch of the multi valued function), it follows that the equality

$$(1+z)^\alpha = 1 + \frac{\alpha}{1!}z + \frac{\alpha(\alpha-1)}{2!}z^2 + \dots + \frac{\alpha(\alpha-1)\dots(\alpha-2n+2)}{(2n-1)!}z^{2n-1} \\ + z^{2n} \frac{\alpha(\alpha-1)\dots(\alpha-2n+1)}{(2n-1)!} \int_0^1 (1-t)^{2n-1}(1+zt)^{\alpha-2n} dt$$

holds for every $z \in U$. The mapping $g : U \rightarrow \mathbb{C}$, $g(z) = (1+z)^\alpha$ is radially continuous and so we infer that

$$(1+e^{i\theta})^\alpha = 1 + \frac{\alpha}{1!}e^{i\theta} + \frac{\alpha(\alpha-1)}{2!}e^{2i\theta} + \dots + \frac{\alpha(\alpha-1)\dots(\alpha-2n+2)}{(2n-1)!}e^{(2n-1)i\theta} \\ + e^{2ni\theta} \frac{\alpha(\alpha-1)\dots(\alpha-2n+1)}{(2n-1)!} \int_0^1 (1-t)^{2n-1}(1+e^{i\theta}t)^{\alpha-2n} dt, \quad \theta \in (-\pi, \pi).$$

Taking the absolute value, this equality implies that

$$|A_{2n-1}(\alpha, e^{i\theta})| \\ = \left| 1 + \frac{\alpha}{1!}e^{i\theta} + \frac{\alpha(\alpha-1)}{2!}e^{2i\theta} + \dots + \frac{\alpha(\alpha-1)\dots(\alpha-2n+2)}{(2n-1)!}e^{(2n-1)i\theta} \right| \\ \leq \frac{\alpha(1-\alpha)(2-\alpha)\dots(2n-1-\alpha)}{(2n-1)!} \int_0^1 (1-t)^{2n-1} \left| 1 + e^{i\theta}t \right|^{\alpha-2n} dt \\ + \left| 1 + e^{i\theta} \right|^\alpha \leq \left| 1 + e^{i\theta} \right|^\alpha + \alpha(1-\alpha) \left(1 - \frac{\alpha}{2} \right) \left(1 - \frac{\alpha}{3} \right) \dots \left(1 - \frac{\alpha}{2n-1} \right) \\ \int_0^1 (1-t)^{2n-1} \left| 1 + e^{i\theta}t \right|^{\alpha-2n} dt, \quad \theta \in (-\pi, \pi).$$

On the other hand, we have

$$A_{2n-1}(\alpha, 1) = 1 + \frac{\alpha}{1!} - \frac{\alpha(1-\alpha)}{2!} + \frac{\alpha(1-\alpha)(2-\alpha)}{3!} \\ - \frac{\alpha(1-\alpha)(2-\alpha)(3-\alpha)}{4!} + \dots + \frac{\alpha(1-\alpha)\dots(2n-2-\alpha)}{(2n-1)!} \geq 1 + \frac{\alpha(1+\alpha)}{2}.$$

Thus to prove (42), we have to show that

$$1 + \frac{\alpha(1+\alpha)}{2} \geq \left| 1 + e^{i\theta} \right|^\alpha + \alpha(1-\alpha) \left(1 - \frac{\alpha}{2}\right) \left(1 - \frac{\alpha}{3}\right) \cdots \left(1 - \frac{\alpha}{2n-1}\right) \int_0^1 (1-t)^{2n-1} \left| 1 + e^{i\theta} t \right|^{\alpha-2n} dt, \quad \theta \in \left[\frac{2\pi}{3}, \pi\right]. \quad (43)$$

We denote $x = -\cos \theta$, and the inequality (43) will be equivalent to

$$1 + \frac{\alpha(1+\alpha)}{2} \geq \left(2 - 2x\right)^{\frac{\alpha}{2}} + \alpha(1-\alpha) \left(1 - \frac{\alpha}{2}\right) \left(1 - \frac{\alpha}{3}\right) \cdots \left(1 - \frac{\alpha}{2n-1}\right) \int_0^1 \left(\frac{1-t}{\sqrt{1+t^2-2tx}}\right)^{2n-1} \left(\sqrt{1+t^2-2tx}\right)^{\alpha-1} dt, \quad x \in \left[\frac{1}{2}, 1\right].$$

This inequality can be rewritten in the following form:

$$1 + \alpha \geq \frac{(2-2x)^{\frac{\alpha}{2}} - 1}{\frac{\alpha}{2}} + 2(1-\alpha) \left(1 - \frac{\alpha}{2}\right) \left(1 - \frac{\alpha}{3}\right) \cdots \left(1 - \frac{\alpha}{2n-1}\right) \int_0^1 \left(\frac{1-t}{\sqrt{1+t^2-2tx}}\right)^{2n-1} (\sqrt{1+t^2-2tx})^{\alpha-1} dt, \quad x \in \left[\frac{1}{2}, 1\right]. \quad (44)$$

It is easily seen that the expression

$$2(1-\alpha) \left(1 - \frac{\alpha}{2}\right) \left(1 - \frac{\alpha}{3}\right) \cdots \left(1 - \frac{\alpha}{2n-1}\right) \int_0^1 \left(\frac{1-t}{\sqrt{1+t^2-2tx}}\right)^{2n-1} \left(\sqrt{1+t^2-2tx}\right)^{\alpha-1} dt$$

is decreasing with respect to n and increasing with respect to x ; moreover, the expression $\frac{(2-2x)^{\frac{\alpha}{2}} - 1}{\frac{\alpha}{2}}$ is negative. Consequently, to prove (44) it is enough to show that the following inequality holds:

$$1 + \alpha \geq 2(1-\alpha) \left(1 - \frac{\alpha}{2}\right) \left(1 - \frac{\alpha}{3}\right) \cdots \left(1 - \frac{\alpha}{53}\right) \int_0^1 \left(1-t\right)^{\alpha-1} dt. \quad (45)$$

A simple calculation shows that (45) is equivalent to

$$1 + \alpha \geq \frac{2}{\alpha} (1-\alpha) \left(1 - \frac{\alpha}{2}\right) \left(1 - \frac{\alpha}{3}\right) \cdots \left(1 - \frac{\alpha}{53}\right). \quad (46)$$

Since the left hand side is increasing and the right hand side is decreasing provided that $\alpha \in (0, 1)$, it follows that to prove inequality (46) for $\alpha \in [\frac{1}{3}, 1]$ it is enough to be checked in case $\alpha = \frac{1}{3}$, that is,

$$\frac{4}{3} \geq 6 \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{3 \cdot 2}\right) \left(1 - \frac{1}{3 \cdot 3}\right) \cdots \left(1 - \frac{1}{3 \cdot 53}\right).$$

According to Lemma 6, this inequality holds and the proof is done. $\square$

4. Concluding Remarks

The following two corollaries are the proof of the Brannan conjecture in two different particular cases.

Theorems 1, 2 and the result of [6] imply the following corollary.

Corollary 1. *If $x \in \mathbb{C}$ with $|\arg x| \leq \frac{2\pi}{3}$, and $|x| = 1$, then the inequality*

$$|A_{2n-1}(\alpha, x)| \leq A_{2n-1}(\alpha, 1)$$

holds for every $\alpha \in (0, 1)$, $n \in \mathbb{N}^$.*

Theorems 1, 2 and 3 and the result of [6] imply the following corollary. This corollary is the solution of the Brannan conjecture in case $\alpha \in [\frac{1}{3}, 1)$.

Corollary 2. *The inequality*

$$|A_{2n-1}(\alpha, x)| \leq A_{2n-1}(\alpha, 1)$$

holds for every $\alpha \in [\frac{1}{3}, 1)$, $n \in \mathbb{N}^$ and $x \in \mathbb{C}$, with $|x| = 1$.*

The Brannan's conjecture holds if $\alpha \in [\frac{1}{3}, 1)$ and remains an open question in case $\alpha \in (0, \frac{1}{3})$.

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Róbert Szász

Department of Mathematics and Informatics
Sapientia Hungarian University of Transylvania
Târgu-Mureş
Romania
e-mail: rszasz@ms.sapientia.ro

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