




The Yagita Invariant of Symplectic Groups of Large Rank

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Abstract. Fix a prime p , and let \mathcal{O} denote a subring of \mathbb{C} that is either integrally closed or contains a primitive p th root of 1. We determine the Yagita invariant at the prime p for the symplectic group $\mathrm{Sp}(2n, \mathcal{O})$ for all $n \geq p - 1$.

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1. Introduction

The Yagita invariant $p^\circ(G)$ of a discrete group G is an invariant that generalizes the period of the p -local Tate–Farrell cohomology of G , in the following sense: it is a numerical invariant defined for any G that is equal to the period when the p -local cohomology of G is periodic. Yagita considered finite groups [6], and Thomas extended the definition to groups of finite vcd [5]. In [3] the definition was extended to arbitrary groups and $p^\circ(G)$ was computed for $G = \mathrm{GL}(n, \mathcal{O})$ for \mathcal{O} any integrally closed subring of \mathbb{C} and for sufficiently large n (depending on \mathcal{O}).

In [2], one of us computed the Yagita invariant for $\mathrm{Sp}(2(p+1), \mathbb{Z})$. Computations from [3] were used to provide an upper bound, and computations with finite subgroups and with mapping class groups were used to provide a lower bound [4]. The action of the mapping class group of a surface upon the first homology of the surface gives a natural symplectic representation of the mapping class group of a genus $p+1$ surface inside $\mathrm{Sp}(2(p+1), \mathbb{Z})$. In the current paper, we compute $p^\circ(\mathrm{Sp}(2n, \mathcal{O}))$ for each $n \geq p - 1$ for each \mathcal{O} for which $p^\circ(\mathrm{GL}(n, \mathcal{O}))$ was computed in [3]. By using a greater range of finite subgroups, we avoid having to consider mapping class groups.

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Throughout the paper, we fix a prime p . Before stating our main result, we recall the definitions of the symplectic group $\text{Sp}(2n, R)$ over a ring R , and of the Yagita invariant $p^\circ(G)$, which depends on the prime p as well as on the group G . The group $\text{Sp}(2n, R)$ is the collection of invertible $2n \times 2n$ matrices M over R such that

$$M^T J M = J, \text{ where } J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Here, M^T denotes the transpose of the matrix M , and as usual I_n denotes the $n \times n$ identity matrix. Equivalently, $M \in \text{Sp}(2n, R)$ if M defines an isometry of the antisymmetric bilinear form on R^{2n} defined by $\langle x, y \rangle := x^T J y$. If C is cyclic of order p , then the group cohomology ring $H^*(C; \mathbb{Z})$ has the form

$$H^*(C; \mathbb{Z}) \cong \mathbb{Z}[x]/(px), \quad x \in H^2(C; \mathbb{Z}).$$

If C is a cyclic subgroup of G of order p , define $n(C)$ a positive integer or infinity to be the supremum of the integers n such that the image of $H^*(G; \mathbb{Z}) \rightarrow H^*(C; \mathbb{Z})$ is contained in the subring $\mathbb{Z}[x^n]$. Now, define

$$p^\circ(G) := \text{lcm}\{2n(C) : C \leq G, |C| = p\}.$$

It is easy to see that if $H \leq G$, then $p^\circ(H)$ divides $p^\circ(G)$ [3, Prop. 1].

2. Results

In the following theorem statement and throughout the paper, we let ζ_p be a primitive p th root of 1 in \mathbb{C} and we let \mathcal{O} denote a subring of \mathbb{C} with $F \subseteq \mathbb{C}$ as its field of fractions. We assume that either $\zeta_p \in \mathcal{O}$ or that \mathcal{O} is integrally closed in \mathbb{C} . We define $l := |F[\zeta_p] : F|$, the degree of $F[\zeta_p]$ as an extension of F . For $t \in \mathbb{R}$ with $t \geq 1$, we define $\psi(t)$ to be the largest integer power of p less than or equal to t .

Theorem 1. *With notation as above, for each $n \geq p - 1$, the Yagita invariant $p^\circ(\text{Sp}(2n, \mathcal{O}))$ is equal to $2(p-1)\psi(2n/l)$ for l even and equal to $2(p-1)\psi(n/l)$ for l odd.*

By the main result of [3], the above is equivalent to the statement that $p^\circ(\text{Sp}(2n, \mathcal{O})) = p^\circ(\text{GL}(2n, \mathcal{O}))$ when l is even and $p^\circ(\text{Sp}(2n, \mathcal{O})) = p^\circ(\text{GL}(n, \mathcal{O}))$ when l is odd. By definition $\text{Sp}(2n, \mathcal{O})$ is a subgroup of $\text{GL}(2n, \mathcal{O})$ and there is an inclusion $\text{GL}(n, \mathcal{O}) \rightarrow \text{Sp}(2n, \mathcal{O})$ defined by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix},$$

and so for any n , $p^\circ(\text{GL}(n, \mathcal{O}))$ divides $p^\circ(\text{Sp}(2n, \mathcal{O}))$, which in turn divides $p^\circ(\text{GL}(2n, \mathcal{O}))$.

Before we start, we recall two standard facts concerning symplectic matrices that will be used in the proof of Corollary 3: if M is in the symplectic group, then $\det(M) = 1$ and M is conjugate to the inverse of its transpose $(M^{-1})^T = (M^T)^{-1}$. We shall use the notation \mathbb{F}_p^\times to denote the multiplicative group of units in the field \mathbb{F}_p .

Proposition 2. *Let $f(X)$ be a polynomial over the field \mathbb{F}_p and suppose that 0 is not a root of f , but that f factors as a product of linear polynomials over \mathbb{F}_p . If there is a polynomial g and an integer n so that $f(X) = g(X^n)$, then n has the form $n = mp^q$ for some m dividing $p - 1$ and some integer $q \geq 0$. If p is odd and for each $i \in \mathbb{F}_p^\times$, the multiplicity of i as a root of f is equal to that of $-i$, then m is even.*

Proof. The only part of this that is not contained in [3, Prop. 6] is the final statement. Since $(1 - iX)(1 + iX) = 1 - i^2X^2$ is a polynomial in X^2 , the final statement follows. For the benefit of the reader, we sketch the rest of the proof. If $n = mp^q$ where p does not divide m , then $g(X^n) = g(X^m)^{p^q}$, so we may assume that $q = 0$. If $g(Y) = 0$ has roots y_i , then the roots of $g(X^m) = 0$ are the roots of $y_i - X^m = 0$. Since p does not divide m , these polynomials have no repeated roots; since their roots are assumed to lie in \mathbb{F}_p it is now easy to show that m divides $p - 1$. □

Corollary 3. *With notation as in Theorem 1, let G be a subgroup of $\text{Sp}(2n, F)$. Then the Yagita invariant $p^\circ(G)$ divides the number given for $p^\circ(\text{Sp}(2n, \mathbb{O}))$ in the statement of Theorem 1.*

Proof. As in [3, Cor. 7], for each $C \leq G$ of order p , we use the total Chern class to give an upper bound for the number $n(C)$ occurring in the definition of $p^\circ(G)$. If C is cyclic of order p , then C has p distinct irreducible complex representations, each one dimensional. If we write $H^*(C; \mathbb{Z}) = \mathbb{Z}[x]/(px)$, then the total Chern classes of these representations are $1 + ix$ for each $i \in \mathbb{F}_p$, where $i = 0$ corresponds to the trivial representation. The total Chern class of a direct sum of representations is the product of the total Chern classes, and so when viewed as a polynomial in $\mathbb{F}_p[x] = H^*(C; \mathbb{Z}) \otimes \mathbb{F}_p$, the total Chern class of any faithful representation $\rho : C \rightarrow \text{GL}(2n, \mathbb{C})$ is a non-constant polynomial of degree at most $2n$ all of whose roots lie in \mathbb{F}_p^\times . Now, let F be a subfield of \mathbb{C} with $l = |F[\zeta_p] : F|$ as in the statement. The group C has $(p - 1)/l$ non-trivial irreducible representations over F , each of dimension l , and the total Chern classes of these representations have the form $1 - ix^l$, where i ranges over the $(p - 1)/l$ distinct l th roots of unity in \mathbb{F}_p . In particular, the total Chern class of any representation $\rho : C \rightarrow \text{GL}(2n, F) \leq \text{GL}(2n, \mathbb{C})$ is a polynomial in x^l whose x -degree is at most $2n$. If ρ has image contained in $\text{Sp}(2n, \mathbb{C})$, then it factors as $\rho = \iota \circ \tilde{\rho}$ with $\tilde{\rho} : C \rightarrow \text{Sp}(2n, \mathbb{C})$ and ι is the inclusion of $\text{Sp}(2n, \mathbb{C})$ in $\text{GL}(2n, \mathbb{C})$. In this case, the matrix representing a generator for C is conjugate to the transpose of its own inverse; in particular, it follows that the multiplicities of the irreducible complex representations of C with total Chern classes $1 + ix$ and $1 - ix$ must be equal for each i . Hence in this case, if p is odd, the total Chern class of the representation $\rho = \iota \circ \tilde{\rho}$ is a polynomial in x^2 . If $p = 2$ (which implies that $l = 1$), then the total Chern class of any representation $\rho : C \rightarrow \text{GL}(2n, \mathbb{C})$ has the form $(1 + x)^i$, where i is equal to the number of non-trivial irreducible summands. Since $\text{Sp}(2n, \mathbb{C}) \leq \text{SL}(2n, \mathbb{C})$, it follows that for symplectic representations i must be even, and so for $p = 2$ the total Chern class is a polynomial in x^2 .

In summary, let $\tilde{\rho}$ be a faithful representation of C in $\text{Sp}(2n, F)$. In the case when l is odd, then the total Chern class of $\tilde{\rho}$ is a non-constant

polynomial $f(y) \tilde{=} f(x)$ in $y = x^{2l}$ such that $f(x)$ has degree at most $2n$, $\tilde{f}(y)$ has degree at most n/l , and all roots of f, \tilde{f} lie in \mathbb{F}_p^\times . In the case when l is even, the total Chern class of ρ is a non-constant polynomial $\tilde{f}(y) = f(x)$ in $y = x^l$ such that $f(x)$ has degree at most $2n$, $\tilde{f}(y)$ has degree at most $2n/l$, and all roots of both lie in \mathbb{F}_p^\times . By Proposition 2, it follows that each $n(C)$ is a factor of the number given for $p^\circ(\mathrm{Sp}(2n, \mathcal{O}))$, and hence the claim. \square

Lemma 4. *Let $H \leq G$ with $|G : H| = m$, and let ρ be a symplectic representation of H on $V = \mathcal{O}^{2n}$. The induced representation $\mathrm{Ind}_H^G(\rho)$ is a symplectic representation of G on $W := \mathcal{O}G \otimes_{\mathcal{O}H} V \cong \mathcal{O}^{2mn}$.*

Proof. Let $e_1, \dots, e_n, f_1, \dots, f_n$ be the standard basis for $V = \mathcal{O}^{2n}$, so that the bilinear form $\langle v, w \rangle := v^T J w$ on V is given by

$$\langle e_i, e_j \rangle = 0 = \langle f_i, f_j \rangle, \quad \langle e_i, f_j \rangle = -\langle f_i, e_j \rangle = \delta_{ij}.$$

The representation ρ is symplectic if and only if each $\rho(h)$ preserves this bilinear form.

Let t_1, \dots, t_m be a left transversal to H in G , so that $\mathcal{O}G = \bigoplus_{i=1}^m t_i \mathcal{O}H$ as right $\mathcal{O}H$ -modules. Define a bilinear form $\langle \cdot, \cdot \rangle_W$ on W by

$$\left\langle \sum_{i=1}^m t_i \otimes v^i, \sum_{i=1}^m t_i \otimes w^i \right\rangle_W := \sum_{i=1}^m \langle v^i, w^i \rangle.$$

To see that this bilinear form is preserved by the $\mathcal{O}G$ -action on W , fix $g \in G$ and define a permutation π of $\{1, \dots, m\}$ and elements $h_1, \dots, h_m \in H$ by the equations $gt_i = t_{\pi(i)}h_i$. Now for each i, j with $1 \leq i, j \leq m$,

$$\begin{aligned} \langle \mathrm{Ind}(\rho(g))t_i \otimes v, \mathrm{Ind}(\rho(g))t_j \otimes w \rangle_W &= \langle t_{\pi(i)} \otimes \rho(h_i)v, t_{\pi(j)} \otimes \rho(h_j)w \rangle_W \\ &= \delta_{\pi(i)\pi(j)} \langle \rho(h_i)v, \rho(h_j)w \rangle \\ &= \delta_{ij} \langle \rho(h_i)v, \rho(h_i)w \rangle \\ &= \delta_{ij} \langle v, w \rangle \\ &= \langle t_i \otimes v, t_j \otimes w \rangle_W. \end{aligned}$$

To see that $\langle \cdot, \cdot \rangle_W$ is symplectic, define basis elements $E_1, \dots, E_{mn}, F_1, \dots, F_{mn}$ for W by the equations

$$E_{n(i-1)+j} := t_i \otimes e_j, \quad \text{and} \quad F_{n(i-1)+j} := t_i \otimes f_j, \quad \text{for } 1 \leq i \leq m, \quad 1 \leq j \leq n.$$

It is easily checked that for $1 \leq i, j \leq mn$

$$\langle E_i, E_j \rangle_W = 0 = \langle F_i, F_j \rangle_W, \quad \langle E_i, F_j \rangle_W = -\langle F_i, E_j \rangle_W = \delta_{ij},$$

and so with respect to this basis for W , the bilinear form $\langle \cdot, \cdot \rangle_W$ is the standard symplectic form. \square

Proposition 5. *With notation as in Theorem 1, the Yagita invariant $p^\circ(\mathrm{Sp}(2n, \mathcal{O}))$ is divisible by the number given in the statement of Theorem 1.*

Proof. To give lower bounds for $p^\circ(\mathrm{Sp}(2n, \mathcal{O}))$, we use finite subgroups. Firstly, consider the semidirect product $H = C_p \rtimes C_{p-1}$, where C_{p-1} acts faithfully on C_p ; equivalently, this is the group of affine transformations of the line over \mathbb{F}_p . It is well known that the image of $H^*(G; \mathbb{Z})$ inside $H^*(C_p; \mathbb{Z}) \cong \mathbb{Z}[x]/(px)$

is the subring generated by x^{p-1} . It follows that $2(p-1)$ divides $p^\circ(G)$ for any G containing H as a subgroup. The group H has a faithful permutation action on p points, and hence a faithful representation in $GL(p-1, \mathbb{Z})$, where \mathbb{Z}^{p-1} is identified with the kernel of the H -equivariant map $\mathbb{Z}\{1, \dots, p\} \rightarrow \mathbb{Z}$. Since $GL(p-1, \mathbb{Z})$ embeds in $Sp(2(p-1), \mathbb{Z})$, we deduce that H embeds in $Sp(2n, \mathcal{O})$ for each \mathcal{O} and for each $n \geq p-1$.

To give a lower bound for the p -part of $p^\circ(Sp(2n, \mathcal{O}))$, we use the extraspecial p -groups. For p odd, let $E(p, 1)$ be the non-abelian p -group of order p^3 and exponent p , and let $E(2, 1)$ be the dihedral group of order 8. (Equivalently in each case $E(p, 1)$ is the Sylow p -subgroup of $GL(3, \mathbb{F}_p)$.) For $m \geq 2$, let $E(p, m)$ denote the central product of m copies of $E(p, 1)$, so that $E(p, m)$ is one of the two extraspecial groups of order p^{2m+1} . Yagita showed that $p^\circ(E(p, m)) = 2p^m$ for each m and p [6]. The centre and commutator subgroup of $E(p, m)$ are equal and have order p , and the abelianization of $E(p, m)$ is isomorphic to C_p^{2m} . The irreducible complex representations of $E(p, m)$ are well understood: there are p^{2m} distinct one-dimensional irreducibles, each of which restricts to the centre as the trivial representation, and there are $p-1$ faithful representations of dimension p^m , each of which restricts to the centre as the sum of p^m copies of a single (non-trivial) irreducible representation of C_p . The group $G = E(p, m)$ contains a subgroup H isomorphic to C_p^{m+1} , and each of its faithful p^m -dimensional representations can be obtained by inducing up a one-dimensional representation $H \rightarrow C_p \rightarrow GL(1, \mathbb{C})$.

According to B\"urigisser, C_p embeds in $Sp(2l, \mathcal{O})$ (resp. in $Sp(l, \mathcal{O})$ when l is even) provided that \mathcal{O} is integrally closed in \mathbb{C} [1]. Here as usual, $l := |F[\zeta_p], F|$ and F is the field of fractions of \mathcal{O} . If instead $\zeta_p \in \mathcal{O}$, then $l = 1$ and clearly C_p embeds in $GL(1, \mathcal{O})$ and hence also in $Sp(2, \mathcal{O}) = Sp(2l, \mathcal{O})$. Taking this embedding of C_p and composing it with any homomorphism $H \rightarrow C_p$, we get a symplectic representation ρ of H on \mathcal{O}^{2l} for any l (resp. on \mathcal{O}^l for l even). For a suitable homomorphism, we know that $\text{Ind}_H^G(\rho)$ is a faithful representation of G on \mathcal{O}^{2lp^m} (resp. on \mathcal{O}^{lp^m} for l even) and by Lemma 4 we see that $\text{Ind}_H^G(\rho)$ is symplectic. Hence, we see that $E(m, p)$ embeds as a subgroup of $Sp(2lp^m, \mathcal{O})$ for any l and as a subgroup of $Sp(lp^m, \mathcal{O})$ in the case when l is even. Since $p^\circ(E(m, p)) = 2p^m$, this shows that $2p^m$ divides $p^\circ(Sp(2lp^m, \mathcal{O}))$ always and that $2p^m$ divides $p^\circ(Sp(lp^m, \mathcal{O}))$ in the case when l is even. □

Corollary 3 and Proposition 5 together complete the proof of Theorem 1.

We finish by pointing out that we have not computed $p^\circ(Sp(2n, \mathcal{O}))$ for general \mathcal{O} when $n < p-1$; to do this one would have to know which metacyclic groups $C_p \rtimes C_k$ with k coprime to p admit low-dimensional symplectic representations.

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