Mediterr. J. Math. (2018) 15:221
https://doi.org/10.1007/s00009-018-1269-7
1660-5446/18/060001-12
published online November 17, 2018

Mediterranean Journal of Mathematics

A Variational Approach for the Mixed Problem in the Elastostatics of Bodies with Dipolar Structure

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Abstract. In this study, we address the mixed initial boundary value problem in the elastostatics of dipolar bodies. Using the equilibrium equations, we build the operator of dipolar elasticity and prove that this operator is positively defined even in the general case of an elastic inhomogeneous and anisotropic dipolar solid. This helps us to prove the existence of a generalized solution for first boundary value problem and also the uniqueness of the solution. Moreover, relying on this property of the operator of dipolar elasticity to be positively defined, we can apply the known variational method proposed by Mikhlin.

Mathematics Subject Classification. Primary 70G75; Secondary 53A45, 58E30, 74A20.

Keywords. Dipolar bodies, first boundary value problem, operator of dipolar elasticity, generalized solution, variational method.

1. Introduction

The modern theories that take into account the microstructure of the medium have as main aim the elimination of the discrepancies that occur between experiments and the consecrated elasticity theory. The old theory of elasticity is not enough for the best characterization of deformations, when it is necessary to consider the microstructure of the solids. In this regard we can remember some materials with great applicability, such as human bones, graphite, polymers and, in general, some granular bodies with large molecules, and so on. First studies dedicated to these modern theories were published by Eringen, of which there are remarkable papers [1] and [2]. Starting from Eringen's results, a large number of authors have developed a considerable number of studies in order to continue the approaches of different types of micro-structures, of which we recall [3–7]. For instance, based on the results of Eringen, Ciarletta studied in [7] the bending of a plate made of an elastic material, neglecting

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the thermal effect but considering its microstructure. In addition, Iesan and Pompei in paper [3] proposed a solution of Boussinesq–Somigliana–Galerkin type for the boundary value problem in this context. Other issues regarding these materials have been addressed in [8] and [9]. The studies [10] and [11] offer some considerations on waves propagation in the context of the bodies with microstructure. Among the theories dedicated to the microstructure, a special place is occupied by the dipolar structure. To highlight the importance of the dipolar structure, it is enough to see the list of researchers who have dedicated their studies to this modern theory. The importance of the dipolar structure of materials was highlighted by many renowned researchers. At the top of this list we will find the results published in Mindlin [12], Green and Rivlin [13], and so on. We must emphasize that these authors approached in their papers the multipolar structures of the bodies, from which the dipolar structure is obtained as a particular case. Another name to be remembered is that of M. E. Gurtin who elaborated a few works dedicated to the multipolar structures. A conclusive example is offered by the work [14] in which Gurtin and Fried give new formulations for the energy balance, for the force balance and for entropy imbalance to characterize the interface between the bodies and their environment. It is known that in the theory that considers dipolar structure, there are twelve degrees of freedom, for each point of the medium, namely: three translations and nine micro-deformations; see Marin [5]. In addition, each particle of the body is bound to deform uniformly and homogeneously.

To model an elastic body having a dipolar structure, we formulate an initial boundary value problem which becomes more credible after a result is obtained on the uniqueness of the solution for the respective problem.

Lately, many studies have been devoted to both the uniqueness of the solution and other related problems, of which we mention a few: Brun [15], Knops and Payne [16], Levine [17], Wilkes [18], Rionero and Chirita [19], and so on. In a part of previous papers on uniqueness of solutions in elasticity or thermoelasticity, the results are based on the assumptions that the thermoelastic coefficients or elasticity tensors are considered to be positive definite. See for instance the paper [18]. In other studies the same issues of uniqueness is approached using different conservation laws of energy. We wish to mention that the uniqueness result obtained by Green and Laws in the paper [20] was only possible by adding some positive definiteness hypotheses to the restrictions already imposed by thermodynamics. A uniqueness result obtained using medium restrictions is due to Brun [15]. In this paper, Brun employs an identity of Lagrange type together with an ordinary law on energy conservation to deduce the uniqueness of solution, but neglecting the thermal effect.

In the present paper, we consider the first boundary value problem in the context of dipolar structure and address the problem of the existence of a generalized solution for this problem. In addition, the uniqueness of the weak solution is investigated. In addition, we extend variational method proposed by Mikhlin in [21] to cover the new context of elastic dipolar solids. We have also taken into account the results of existence for the classical solutions derived in [22].

By taking into account the dipolar structure, we can obtain a good description for a considerable number of applications in solid mechanics.

2. Notations, Basic Equations and Conditions

Assume that our dipolar elastic body occupies the region B of three-dimensional Euclidean space R^3. The closure of the domain B is denoted by $\bar{B}$ and the boundary of B will be denoted by ∂B, which is a closed, bounded and regular surface. The deformation of the body is referred to a system of Cartesian axes, so that the spatial variables of points from B are denoted by x_j. We also use the time variable t, $t \in [0, \infty)$. When possible, it is omitted to specify the variables on which a function depends. Throughout this paper, we adopt the Cartesian tensor notation. If there is no likelihood of confusion, the spatial variables and the time variable of functions will be omitted. The known convention on summation over repeated subscripts is implied. A superposed dot is used for partial differentiation with respect to time, while a subscript preceded by a comma is used for partial differentiation with respect to the corresponding spatial variable.

In what follows, the theory of elastostatic bodies with a dipolar structure is considered. Our considerations start from the work [13], while the notations and the terminology are inspired by the papers [5] and [6].

To insert the components of strain tensors, ε_{ij}, γ_{ij} and χ_{ijk}, the kinematic equations are used, namely

$$\begin{aligned} \varepsilon_{ij} &= \frac{1}{2}\left(u_{j,i} + u_{i,j}\right), \\ \gamma_{ij} &= u_{j,i} - \varphi_{ij}, \\ \chi_{ijk} &= \varphi_{ij,k}. \end{aligned} \tag{1}$$

The equilibrium equations in our context are

$$\left(\tau_{ij} + \sigma_{ij}\right)_{,j} + F_i = 0, \tag{2}$$

$$\mu_{ijk,i} + \sigma_{jk} + G_{jk} = 0, \tag{3}$$

for any $(x, t) \in B \times [0, \infty)$.

With the help of the constitutive equations, we introduce the stress tensors, namely τ_{ij}, σ_{ij} and μ_{ijk}

$$\begin{aligned} \tau_{ij} &= C_{ijmn}\varepsilon_{mn} + G_{ijmn}\gamma_{mn} + F_{mnrij}\chi_{mnr}, \\ \sigma_{ij} &= G_{ijmn}\varepsilon_{mn} + B_{ijmn}\gamma_{mn} + D_{ijmnr}\chi_{mnr}, \\ \mu_{ijk} &= F_{ijkmn}\varepsilon_{mn} + D_{mnijk}\gamma_{mn} + A_{ijkmnr}\chi_{mnr}, \end{aligned} \tag{4}$$

satisfied for all $(t, x) \in [0, \infty) \times B$. In above equations we also used the following notations

F_i—the body force per unit mass,

G_{jk}—the body couple per unit mass,

C_{ijmn}, G_{ijmn}, B_{ijmn}, F_{ijkmn}, D_{ijkmn}, A_{ijkmnr} - the characteristic constants of material and these elastic coefficients satisfy following symmetry relations

$$\begin{aligned} &C_{mnij} = C_{ijmn} = C_{ijnm},\ B_{ijmn} = B_{mnij},\\ &G_{ijmn} = G_{ijnm},\ F_{ijkmn} = F_{ijknm},\ A_{ijkmnr} = A_{mnrijk}. \end{aligned} \tag{5}$$

Along with the basic equations (1)–(4) we prescribe the initial conditions in the form

$$\begin{aligned} &u_i(x_s,0) = a_i(x_s),\ \dot{u}_i(x_s,0) = b_i(x_s),\\ &\varphi_{jk}(x_s,0) = c_{jk}(x_s),\ \dot{\varphi}_{jk}(x_s,0) = d_{jk}(x_s), \end{aligned} \tag{6}$$

considered for any $(x_k) \in \bar{B}$.

In addition, we will divide the surface ∂B into subsets ∂B_1 and its complement ∂B_1^c, respectively ∂B_2 and its complement ∂B_2^c and suppose that

$$\begin{aligned} &\partial B_1 \cup \partial B_1^c = \partial B_2 \cup \partial B_2^c = \partial B,\\ &\partial B_1 \cap \partial B_1^c = \partial B_2 \cap \partial B_2^c = \emptyset. \end{aligned}$$

The above mentioned mixed problem for elastic dipolar bodies will be complete after we add the boundary conditions in the following form

$$\begin{aligned} &u_i = \tilde{u}_i \text{ on } \partial B_1 \times [0,\infty),\ t_i = \tilde{t}_i \text{ on } \partial B_1^c \times [0,\infty),\\ &\varphi_{jk} = \tilde{\varphi}_{jk} \text{ on } \partial B_2 \times [0,\infty),\ m_{jk} = \tilde{m}_{jk} \text{ on } \partial B_2^c \times [0,\infty), \end{aligned} \tag{7}$$

where t_i and m_{jk} are the components of surface traction and the components of surface couple, that are, respectively, are computed by using the relations

$$t_i = (\tau_{ij} + \sigma_{ij})\, n_j, m_{jk} = \mu_{ijk} n_i.$$

As usual, we denote by n_i the components of the outward normal to the surface ∂B.

The functions a_i, b_i, c_{jk} and d_{jk} from (6) are known in all domain B. At the same time, the functions $\tilde{u}_i$, $\tilde{t}_i$, $\tilde{\varphi}_{jk}$, and $\tilde{m}_{jk}$ from (7) are prescribed and regular functions where they are defined.

Introducing (1) and (4) into Eqs. (2) and (3), we are led to the equations

$$\begin{aligned} &[(C_{ijmn} + G_{ijmn})\, u_{n,m} + (G_{mnij} + B_{ijmn})\,(u_{n,m} - \varphi_{mn})\\ &\quad + (F_{mnrij} + D_{ijmnr})\,\varphi_{nr,m}]_{,j} + F_i = 0,\\ &[F_{ijkmn} u_{n,m} + D_{mnijk}\,(u_{n,m} - \varphi_{mn}) + A_{ijkmnr}\varphi_{nr,m}]_{,i}\\ &\quad + G_{jkmn} u_{m,n} + B_{jkmn}\,(u_{n,m} - \varphi_{mn}) + D_{jkmnr}\varphi_{nr,m} + G_{jk} = 0. \end{aligned} \tag{8}$$

An ordered array (u_i, φ_{jk}) is called a solution for our mixed problem, for any (x,t) in the cylinder $\Omega_0 = B \times [0,\infty)$ if it satisfies the system of differential equations (8) for all $(x,t) \in \Omega_0$, the conditions on the border (7) and the above initial conditions (6).

The regularity hypotheses we need to get our results are the following: To obtain our results, we shall use the following regularity assumptions

(i) the components of the body force F_i and the components of the body couple G_{jk} are continuous functions for any $(x,t) \in \Omega_0 = B \times [0,\infty)$;

(ii) the coefficients from the constitutive equations are continuous functions on $\bar{B}$;
(iii) the given initial functions $a_i,\ b_i,\ c_{jk}$ and d_{jk} are continuous on $\bar{B}$;
(iv) the given boundary functions $\tilde{u}_i$ and $\tilde{\varphi}_{jk}$ are continuous in their respective domains of definition;
(v) the given traction functions $\tilde{t}_i$ and $\tilde{m}_{jk}$ are continuous in time and piecewise regular in their respective domains of definition.

3. Main Results

With a suggestion given by (8), we define the operators T_i by

$$\begin{aligned} T_i u = & -\frac{\partial}{\partial x_j}\left[\left(C_{ijmn}+2G_{ijmn}+B_{ijmn}\right)\frac{\partial u_n}{\partial x_m}\right. \\ & \left. +\left(F_{mnrij}+D_{ijmnr}\right)\frac{\partial \varphi_{nr}}{\partial x_m}-\left(G_{mnij}+B_{ijmn}\right)\varphi_{mn}\right], \\ T_{j+k+3}u = & -\frac{\partial}{\partial x_i}\left[\left(F_{ijkmn}u_{n,m}+D_{mnijk}\right)\frac{\partial u_n}{\partial x_m}+A_{ijkmnr}\frac{\partial \varphi_{nr}}{\partial x_m}\right. \\ & \left. -D_{mnijk}\varphi_{mn}\right]+\left(G_{jkmn}u_{m,n}+B_{jkmn}\right)\frac{\partial u_n}{\partial x_m} \\ & +D_{jkmnr}\frac{\partial \varphi_{nr}}{\partial x_m}-B_{jkmn}\varphi_{mn}, \end{aligned} \tag{9}$$

where we used the notation

$$u=\left(u_i,\varphi_{jk}\right)=\left(u_1,u_2,u_3,\varphi_{11},\varphi_{12},...,\varphi_{33}\right). \tag{10}$$

Taking into account (9) and (10), we can write the system of equations (8) in the form

$$\begin{aligned} & T_i u = F_i,\ i=1,2,3 \\ & T_{j+k+3}u = G_{jk},\ j,k=1,2,3 \end{aligned} \tag{11}$$

With the help of notations

$$\begin{aligned} f &= \left(F_1,F_2,F_3,G_{11},G_{12},...,G_{33}\right), \\ Tu &= \left(T_1u,T_2u,T_3u,T_4u,,...,T_{12}u\right) \end{aligned} \tag{12}$$

the system of equations (11) may be written in a more concise form

$$Tu=f. \tag{13}$$

In the following, we will call T as the operator of the dipolar elasticity.

Now, we will prove an auxiliary result which will be useful in the following. Let us consider two different systems of elastic loadings which act on our body, $\left(F_i^{(\alpha)},\ G_{jk}^{(\alpha)}\right)$ and the two corresponding states $\left(u_i^{(\alpha)},\ \varphi_{jk}^{(\alpha)},\ \varepsilon_{ij}^{(\alpha)},\ \gamma_{ij}^{(\alpha)},\ \chi_{ijk}^{(\alpha)},\ \tau_{ij}^{(\alpha)},\ \sigma_{ij}^{(\alpha)},\ \mu_{ijk}^{(\alpha)}\right)$.

Let us introduce the notation

$$2E_{\alpha,\beta}=\tau_{ij}^{(\alpha)}\varepsilon_{ij}^{(\beta)}+\sigma_{ij}^{(\alpha)}\gamma_{ij}^{(\beta)}+\mu_{ijk}^{(\alpha)}\chi_{ijk}^{(\beta)},\ \alpha,\ \beta=1,2 \tag{14}$$

Proposition 1. *The following two identities hold true*

$$E_{1,2} = E_{2,1} \tag{15}$$

$$2\int_B E_{\alpha,\beta} dV = \int_B \left(F_i^{(\alpha)} u_i^{(\beta)} + G_{jk}^{(\alpha)} \varphi_{jk}^{(\beta)}\right) \mathrm{d}V + \int_{\partial B} \left(t_i^{(\alpha)} u_i^{(\beta)} + m_{jk}^{(\alpha)} \varphi_{jk}^{(\beta)}\right) \mathrm{d}S. \tag{16}$$

Proof. Taking into account (14), the constitutive equations (4) and the symmetry relations (5), we can write

$$\begin{aligned} 2E_{\alpha,\beta} &= C_{ijmn}\varepsilon_{mn}^{(\alpha)}\varepsilon_{ij}^{(\beta)} + G_{ijmn}\left(\varepsilon_{ij}^{(\alpha)}\gamma_{mn}^{(\beta)} + \varepsilon_{ij}^{(\beta)}\gamma_{mn}^{(\alpha)}\right) \\ &\quad + B_{ijmn}\gamma_{ij}^{(\alpha)}\gamma_{mn}^{(\beta)} + F_{mnrij}\left(\varepsilon_{ij}^{(\alpha)}\chi_{mnr}^{(\beta)} + \varepsilon_{ij}^{(\beta)}\chi_{mnr}^{(\alpha)}\right) \\ &\quad + D_{ijmnr}\left(\gamma_{ij}^{(\alpha)}\chi_{mnr}^{(\beta)} + \gamma_{ij}^{(\beta)}\chi_{mnr}^{(\alpha)}\right) + A_{ijkmnr}\chi_{mnr}^{(\alpha)}\chi_{ijk}^{(\beta)}. \end{aligned} \tag{17}$$

If we integrate in (17) over B and use the divergence theorem, we are led to (16). The equality (15) is immediately obtained starting from (14) and taking into account the symmetry relations (5). □

Let us consider two displacements and two tractions

$$\begin{aligned} u &= \left(u_i^{(1)}, \varphi_{jk}^{(1)}\right),\ v = \left(u_i^{(2)}, \varphi_{jk}^{(2)}\right), \\ t(u) &= \left(t_i^{(1)}, m_{jk}^{(1)}\right),\ t(v) = \left(t_i^{(2)}, m_{jk}^{(2)}\right). \end{aligned} \tag{18}$$

In (16) we use the notation $E_{1,2} = E(u,v)$ and will give two properties of the operator of the dipolar elasticity.

Proposition 2. *The operator of the dipolar elasticity satisfies the relations*

$$\int_B (u\,Tv - v\,Tu)\mathrm{d}V = \int_{\partial B} [v\,t(u) - u\,t(v)]\,\mathrm{d}S, \tag{19}$$

$$\int_B u\,Tu\,\mathrm{d}V = 2\int_B E(u)\,\mathrm{d}V - \int_{\partial B} u\,t(u)\,\mathrm{d}S. \tag{20}$$

Proof. From (16) for $\alpha = 1$ and $\beta = 2$, with the help of (18) we deduce

$$2\int_B E(u,v)\,\mathrm{d}V = \int_B v\,Tu\,\mathrm{d}V + \int_{\partial B} v\,t(u)\,\mathrm{d}S \tag{21}$$

Taking into account the symmetries (5), from (17) we obtain the following symmetry relation

$$\begin{aligned} 2E(u,v) &= C_{ijmn}\varepsilon_{mn}(u)\varepsilon_{ij}(v) + G_{ijmn}\left[\varepsilon_{ij}(u)\gamma_{mn}(v) + \varepsilon_{ij}(v)\gamma_{mn}(u)\right] \\ &\quad + B_{ijmn}\gamma_{ij}(u)\gamma_{mn}(v) + F_{mnrij}\left[\varepsilon_{ij}(u)\chi_{mnr}(v) + \varepsilon_{ij}(v)\chi_{mnr}(u)\right] \\ &\quad + D_{ijmnr}\left[\gamma_{ij}(u)\chi_{mnr}(v) + \gamma_{ij}(v)\chi_{mnr}(u)\right] \\ &\quad + A_{ijkmnr}\chi_{mnr}(u)\chi_{ijk}(v) = 2E(v,u), \end{aligned} \tag{22}$$

so that from (21) we can write

$$2\int_B E(v,u)\,\mathrm{d}V = \int_B u\,Tv\,\mathrm{d}V + \int_{\partial B} u\,t(v)\,\mathrm{d}S,$$

and this equality with (21) led to (19).

It is easy to obtain (20) from (21) by using the notation $E(u) = E(u, u)$. □

To achieve our proposed goals, we will have to assume the internal energy density $E(u)$ to be a positive definite quadratic form. This is a natural restriction very often used in continuum mechanics because it is in line with real experiments. In a large number of studies dedicated to the classical theory of elasticity, this hypothesis was intensively used (see, for instance, [23]).

As such, there is a constant $c_1 > 0$ so that

$$\begin{aligned} 2E(u) &= C_{ijmn}\varepsilon_{mn}(u)\varepsilon_{ij}(u) + 2G_{ijmn}\varepsilon_{ij}(u)\gamma_{mn}(u) + B_{ijmn}\gamma_{ij}(u)\gamma_{mn}(u) \\ &\quad + 2F_{mnrij}\varepsilon_{ij}(u)\chi_{mnr}(u) + 2D_{ijmnr}\gamma_{ij}(u)\chi_{mnr}(u) \\ &\quad + A_{ijkmnr}\chi_{mnr}(u)\chi_{ijk}(u) \\ &\geq c_1\left(\sum_{i,j=1}^{3}\left(\varepsilon_{ij}^2(u) + \gamma_{ij}^2(u)\right) + \sum_{i,j,k=1}^{3}\chi_{ijk}^2(u)\right) \end{aligned} \tag{23}$$

To the system of field equations (20) we will add the boundary condition $u = u_0 \;\; on \;\; \partial B$, where $u = (u_i, \varphi_{jk})$ and the value u_0 is prescribed. So we have the following boundary value problem

$$\begin{aligned} &Tu = f \;\; in \;\; B, \\ &u = u_0 \;\; on \;\; \partial B. \end{aligned} \tag{24}$$

It is natural to address the issue of the uniqueness of the solution for the problem (24).

Theorem 1. *The solution of the boundary value problem* (24) *is determined up to a rigid displacement and a rigid dipolar displacement.*

Proof. The difference u^* of two solutions of problem (24), that is,

$$u^* = \left(u_i^{(1)} - u_i^{(2)}), \varphi_{jk}^{(1)} - \varphi_{jk}^{(2)}\right),$$

satisfies the homogeneous equation $Au^* = 0 \;\; in \;\; B$ and the homogeneous boundary condition $u^* = 0 \; on \; \partial B$ so that the equality (20), written for u^*, becomes

$$\int_B E(u^*)\mathrm{d}V = 0,$$

from where, taking into account that the quadratic form $E(u^*)$ is positive definite, we deduce that

$$\varepsilon_{ij}(u^*) = 0, \;\; \gamma_{ij}(u^*) = 0, \;\; \chi_{ijk}(u^*) = 0. \tag{25}$$

According to geometric equations (1), from (25) we are led to the conclusion

$$u^*_{i,j} + u^*_{j,i} = 0, \;\; u^*_{i,j} - \varphi^*_{ij} = 0, \;\; \varphi^*_{ijk,i} = 0, \tag{26}$$

which ends the proof of the theorem. □

In the following, we wish to approach the problem of existence of a solution of the boundary value problem (24). Using an idea given by Mikhlin in [1] we will attach to operator T in $(24)_1$ the functional $\mathcal{F}$ called *functional of energy*, defined by

$$\mathcal{F}(v) = (Tv, v) - 2(v, f). \tag{27}$$

Let us denote by H the domain of the functional $\mathcal{F}$. If we denote by D_T the domain of the operator T and consider the metric generated by the scalar product (Tu, u), then H is a Hilbert space, namely the closure of the domain D_T in relation to this metric.

A known result affirms that if the the operator T is positive definite, then the boundary value problem (24) admits a weak (generalized) solution and this is the point of minimum of the energy functional (27).

We recall that an operator which is symmetric, that is, $(Tu, v) = (u, Tv), \quad \forall u \in D_T$ and positive, that is $(Tu, u) \geq 0, \quad \forall u \in D_T$ is called a positive definite operator if it satisfies the inequality

$$(Tu, u) \geq c_2^2 \|u\|^2, \ c_2 = \text{constant}, \tag{28}$$

for any $u \in D_T$. The main result of our study is the following existence result.

Theorem 2. *The boundary value problem* (24) *has at least one generalized solution.*

Proof. We have to prove that the operator T of the dipolar elasticity is positive definite. More specifically, we will prove that T is symmetric, then T is a positive operator and finally, it is positive definite. In this situation, the scalar product that generates the Hilbert space H is

$$\begin{aligned}
&(u, v) = \int_B uv \, \mathrm{d}V = \int_B \left[\sum_{i=1}^{3} u_i v_i + \sum_{j=1}^{3} \left(\sum_{k=1}^{3} \varphi_{jk} \psi_{jk} \right) \right] \mathrm{d}V, \\
&u = (u_i, \varphi_{jk}) = \left(u_1, u_2, u_3, \varphi_{11}, \varphi_{12}, ..., \varphi_{33} \right), \\
&v = (v_i, \psi_{jk}) = \left(v_1, v_2, v_3, \psi_{11}, \psi_{12}, ..., \psi_{33} \right)
\end{aligned} \tag{29}$$

We will use the notation $u(x) = (u_i(x), \varphi_{jk}(x))$. Let C_0^2 be the usual space of these vector functions, with the following properties:

- they are twice continuous differentiable in the domain $\bar{B}$;
- they vanish on the surface ∂B, the boundary of B.

We will prove that the operator T is positive definite on C_0^2.

First, clearly from (19) we deduce that T is symmetric on the set C_0^2. In addition, from (20) and (23) we can see that T is a positive operator on the set C_0^2. We still have to prove the positive defining property of this operator of elasticity for dipolar structure. Considering that $u \in B_0$, the relation (20) reduces to

$$(Tu, u) = \int_B uTu \, \mathrm{d}V = 2 \int_B E(u) \, \mathrm{d}V,$$

and this, together with (23) provides

$$(Tu,u) \geq c_1 \int_B \left(\sum_{i,j=1}^{3} \left(\varepsilon_{ij}^2(u) + \gamma_{ij}^2(u)\right) + \sum_{i,j,k=1}^{3} \chi_{ijk}^2(u) \right) \mathrm{d}V. \tag{30}$$

Now, we recall the inequality due to Friedrichs

$$\int_B \sum_{i=1}^{3} u_i^2 \mathrm{d}V \leq c_2 \left[\int_B \sum_{i,j=1}^{3} (u_{i,j})^2 \mathrm{d}V + \int_{\partial B} u^2 \mathrm{d}A \right],\ c_2 = \text{constant},\ c_2 > 0. \tag{31}$$

In our situation, because we are on the set C_0^2, (31) reduces to

$$\int_B \sum_{i=1}^{3} u_i^2 \mathrm{d}V \leq c_2 \int_B \sum_{i,j=1}^{3} (u_{i,j})^2 \mathrm{d}V. \tag{32}$$

We also recall the first Korn's inequality

$$\int_B \sum_{i,j=1}^{3} (u_{i,j})^2 \mathrm{d}V \leq c_3 \int_B \sum_{i,j=1}^{3} \varepsilon_{ij}^2 \mathrm{d}V,\ c_3 = constant,\ c_3 > 0. \tag{33}$$

Clearly, from (31) and (32), taking into account the geometric equations (31), we deduce

$$\int_B \sum_{i,j=1}^{3} \left(\varepsilon_{ij}^2 + \gamma_{ij}^2\right) \mathrm{d}V \geq c_4 \int_B \sum_{i=1}^{3} u_i^2 \mathrm{d}V,\ c_4 = constant,\ c_4 > 0. \tag{34}$$

In a similar way, we use the inequality of Friedrichs and the first Korn's inequality to obtain

$$\int_B \sum_{j,k=1}^{3} \varphi_{jk}^2(u) \mathrm{d}V \leq c_5 \int_B \sum_{j,k,i=1}^{3} \varphi_{jk,i}^2(u) \mathrm{d}V,\ c_5 = constant,\ c_5 > 0. \tag{35}$$

If we take into account (34) and (35), from (30) we deduce

$$(Tu,u) \geq c_6 \int_B \sum_{i=1}^{3} u_i^2 \mathrm{d}V + \int_B \sum_{j,k=1}^{3} \varphi_{jk}^2(u) \mathrm{d}V,\ c_6 = constant,\ c_6 > 0.$$

Clearly, this proves that T is an operator positive definite on the space C_0^2.

Further on, we can use a procedure almost identical to that of Mikhlin [1] to find a generalized solution to the boundary value problem (24). Here, we find the idea of introducing a scalar product $\langle u, v \rangle$ to the set C_0^2, defined by

$$\begin{aligned} \langle u, v \rangle &= (Tu, v) \\ &= \int_B \{ C_{ijmn} \varepsilon_{mn}(u) \varepsilon_{ij}(v) + G_{ijmn} \left[\varepsilon_{ij}(u)\gamma_{mn}(v) + \varepsilon_{ij}(v)\gamma_{mn}(u)\right] \\ &\qquad + B_{ijmn}\gamma_{ij}(u)\gamma_{mn}(v) + F_{mnrij}\left[\varepsilon_{ij}(u)\chi_{mnr}(v) + \varepsilon_{ij}(v)\chi_{mnr}(u)\right] \\ &\qquad + D_{ijmnr}\left[\gamma_{ij}(u)\chi_{mnr}(v) + \gamma_{ij}(v)\chi_{mnr}(u)\right] \\ &\qquad + A_{ijkmnr}\chi_{mnr}(u)\chi_{ijk}(v) \} \mathrm{d}V. \end{aligned} \tag{36}$$

Let us denote by H_0 the completion of C_0^2 relative to the scalar product (36). So our problem can be reformulated in the form: find a vector functions in H_0 which satisfies the boundary value problem (24). Next, we apply the procedure of Mikhlin, which means to find a vector function in H_0 which is the minimum of the functional

$$\begin{aligned}\mathcal{F}(u) = \langle u, v\rangle - 2(u, f) = \int_B \{&C_{ijmn}\varepsilon_{mn}(u)\varepsilon_{ij}(v)\\ &+2G_{ijmn}\varepsilon_{ij}(u)\gamma_{mn}(u) + B_{ijmn}\gamma_{ij}(u)\gamma_{mn}(u)\\ &+F_{mnrij}\varepsilon_{ij}(u)\chi_{mnr}(u) + D_{ijmnr}\gamma_{ij}(u)\chi_{mnr}(u)\\ &+A_{ijkmnr}\chi_{mnr}(u)\chi_{ijk}(u) - 2F_i u_i - 2G_{jk}\varphi_{jk}\}\,\mathrm{d}V.\end{aligned}$$

According to [1], the problem of finding the minimum of this function is based exclusively on the positive definiteness of the operator T. This ends the proof of the theorem. □

4. Conclusions

Clearly, in the context of the elastostatic of bodies with dipolar structure, the basic equations and the general boundary data are much more and more complex than in the elasticity of classical medium. However, the main features of the solutions for the mixed problem, formulated in this context, does not change. So the uniqueness result is based only on hypothesis that the internal energy density is a quadratic form which is positive definite. In addition, as in the classical case, the procedure used by Mikhlin is useful in the case of elasticity of dipolar body too. Using Mikhlin's algorithm, we prove that according to this, the boundary value problem admits a weak solution, namely, the minimum of the energy functional.

Acknowledgements

The authors are grateful to the anonymous reviewer for the careful reading of the initial version of this paper and for numerous suggestions that improved the present work. V.D. Rădulescu acknowledges the support through the Project MTM2017-85449-P of the DGISPI (Spain).

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Received: March 6, 2018.
Revised: August 8, 2018.
Accepted: November 8, 2018.