Mediterr. J. Math. 13 (2016), 4783–4804
DOI 10.1007/s00009-016-0775-8
1660-5446/16/064783-22
published online July 26, 2016

Mediterranean Journal of Mathematics

On k-Jet Ampleness of Line Bundles on Hyperelliptic Surfaces

Łucja Farnik

Abstract. We study k-jet ampleness of line bundles on hyperelliptic surfaces using vanishing theorems. Our main result states that on a hyperelliptic surface of an arbitrary type, a line bundle of type (m, m) with $m \geq k + 2$ is k-jet ample.

Mathematics Subject Classification. 14C20, 14F17, 14E25.

Keywords. k-jet ampleness, generation of k-jets, hyperelliptic surfaces, vanishing theorems.

1. Introduction

The concepts of higher order embeddings: k-spandness, k-very ampleness, and k-jet ampleness were introduced and studied in a series of papers by Beltrametti, Francia, and Sommese, see [6–8]. The last notion is of our main interest in the present work.

The problem of k-jet ampleness has been studied on certain types of algebraic surfaces. In [4], Bauer and Szemberg characterise k-jet ample line bundles on abelian surfaces with Picard number 1. For an ample line bundle L on a $K3$ surface, Bauer et al. in [2] and Rams and Szemberg in [21] explore for which n, the line bundle nL is k-jet ample.

There are also several papers, concerning k-jet ampleness in higher dimensions, e.g., [3] studies k-jet ampleness on abelian varieties, [12] on toric varieties, [9] on Calabi–Yau threefolds, and [11] on hyperelliptic varieties.

In this paper, we prove that on a hyperelliptic surface of an arbitrary type, a line bundle of type (m, m) with $m \geq k + 2$ is k-jet ample. Note that a line bundle of type $(k + 2, k + 2)$ is numerically equivalent to $(k + 2)L_1$, where $L_1 = (1, 1)$. By theory of hyperellipic surfaces, we know that L_1 is ample, so our result is consistent with results obtained on other algebraic surfaces with Kodaira dimension 0. Our approach uses vanishing theorems of the higher order cohomology groups—Kawamata–Viehweg Theorem and Norimatsu Lemma.

Proof of the fact that a line bundle of type $(k+2, k+2)$ is k-jet ample on any hyperelliptic surface S which can also be found in [11]. The authors use the fact that S is covered by an abelian surface divided by the group action, and the results of [20]. We provide a self-contained and more elementary proof of this fact. Application of Norimatsu Lemma turns out to be a powerful tool.

2. Preliminaries

Let us set up the notation and basic definitions. We work over the field of complex numbers $\mathbb{C}$. We consider only smooth reduced and irreducible projective varieties. By $D_1 \equiv D_2$, we denote the numerical equivalence of divisors D_1 and D_2. By a curve, we understand an irreducible subvariety of dimension 1. In the notation, we follow [18].

Let X be a smooth projective variety of dimension n. Let L be a line bundle on X, and let $x \in X$.

Definition 2.1. 1. We say that L generates k-jets at x, if the restriction map

$$H^0(X, L) \longrightarrow H^0(X, L \otimes \mathcal{O}_X/m_x^{k+1})$$

is surjective.

2. We say that L is k-jet ample, if for every points $x_1, \ldots, x_r$ the restriction map

$$H^0(X, L) \longrightarrow H^0\left(X, L \otimes \mathcal{O}_X/(m_{x_1}^{k_1} \otimes \cdots \otimes m_{x_r}^{k_r})\right)$$

is surjective, where $\sum_{i=1}^{r} k_i = k + 1$.

Note that 0-jet ampleness is equivalent to being spanned by the global sections, and 1-jet ampleness is equivalent to very ampleness.

The notion of k-jet ampleness generalises the notion of very ampleness and k-very ampleness (see [8, Proposition 2.2]). We recall the definition of k-very ampleness, as we mention this notion it the proof of the main theorem:

Definition 2.2. We say that a line bundle L is k-very ample if for every 0-dimensional subscheme $Z \subset X$ of length $k+1$ the restriction map

$$H^0(X, L) \longrightarrow H^0(X, L \otimes \mathcal{O}_Z)$$

is surjective.

In the other words, k-very ampleness means that the subschemes of length at most $k+1$ impose independent conditions on global sections of L.

We also recall the definition of the Seshadri constant:

Definition 2.3. The Seshadri constant of L at a given point $x \in X$ is the real number

$$\varepsilon(L, x) = \inf \left\{ \frac{LC}{\operatorname{mult}_x C} : \ C \ni x \right\},$$

where the infimum is taken over all irreducible curves $C \subset X$ passing through x.

If $\pi\colon \widetilde{X} \longrightarrow X$ is the blow-up of X at x, and E is an exceptional divisor of the blow-up, then equivalently the Seshadri constant may be defined as (see [18, vol. I, Proposition 5.1.5]):

$$\varepsilon(L,x) = \sup\{\varepsilon :\ \pi^*L - \varepsilon E \text{ is nef}\}.$$

We will use two vanishing theorems for the higher order cohomology groups—Kawamata–Viehweg Vanishing Theorem and Norimatsu Lemma.

Theorem 2.4 (Kawamata–Viehweg Vanishing Theorem; [17], Vanishing Theorem 5.2). *Let D be an nef and big divisor on X. Then*

$$H^i(X, K_X + D) = 0 \quad \text{for } i > 0.$$

Definition 2.5 ([18], vol. II, Definition 9.1.7). We say that $D = \sum D_i$ is a simple normal crossing divisor (or an SNC divisor for short) if D_i is smooth for each i, and D is defined in a neighbourhood of any point in local coordinates $(z_1,\ldots,z_n)$ as $z_1\cdot\ldots\cdot z_k = 0$ for some $k \le n$.

Theorem 2.6 (Norimatsu Lemma; [18], vol. I, Vanishing Theorem 4.3.5). *Let D be an ample divisor on X and let F be an SNC divisor on X. Then*

$$H^i(X, K_X + D + F) = 0 \quad \text{for } i > 0.$$

3. Hyperelliptic Surfaces

First, let us recall the definition of a hyperelliptic surface.

Definition 3.1. A hyperelliptic surface S (sometimes called bielliptic) is a surface with Kodaira dimension equal to 0 and irregularity $q(S) = 1$.

Alternatively (see [5, Definition VI.19]), a surface S is hyperelliptic if $S \cong (A\times B)/G$, where A and B are elliptic curves, and G is an abelian group acting on A by translation and acting on B, such that A/G is an elliptic curve and $B/G \cong \mathbb{P}^1$. G acts on $A\times B$ coordinatewise. Hence, we have the following situation:

$$\begin{array}{ccc} S \cong (A\times B)/G & \xrightarrow{\ \Phi\ } & A/G \\ \Psi\big\downarrow & & \\ B/G \cong \mathbb{P}^1 & & \end{array}$$

where Φ and Ψ are the natural projections.

Hyperelliptic surfaces were classified at the beginning of twentieth century by Bagnera and de Franchis in [10], and, independently, by Enriques and Severi in [13,14]. They showed that there are seven non-isomorphic types of hyperelliptic surfaces, characterised by the action of G on $B \cong \mathbb{C}/(\mathbb{Z}\omega \oplus \mathbb{Z})$ (for details, see [5, VI.20]). For each hyperelliptic surface, we have that the canonical divisor K_S is numerically trivial.

In 1990, Serrano in [22, Theorem 1.4] characterised the group of classes of numerically equivalent divisors $\mathrm{Num}(S)$ for each of the surface's type:

Theorem 3.2 (Serrano). *A basis of the group* $\mathrm{Num}(S)$ *for each of the hyperelliptic surface's type and the multiplicities of the singular fibres in each case are the following*:

Type of a hyperelliptic surface	G	$m_1, \ldots, m_s$	*Basis of* $\mathrm{Num}(S)$
1	$\mathbb{Z}_2$	$2,2,2,2$	$A/2, B$
2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$2,2,2,2$	$A/2, B/2$
3	$\mathbb{Z}_4$	$2,4,4$	$A/4, B$
4	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$2,4,4$	$A/4, B/2$
5	$\mathbb{Z}_3$	$3,3,3$	$A/3, B$
6	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$3,3,3$	$A/3, B/3$
7	$\mathbb{Z}_6$	$2,3,6$	$A/6, B$

Let $\mu = \mathrm{lcm}\{m_1, \ldots, m_s\}$ and let $\gamma = |G|$. Given a hyperelliptic surface, its basis of $\mathrm{Num}(S)$ consists of divisors A/μ and $(\mu/\gamma)\, B$.

We say that L is a line bundle of type (a, b) on a hyperelliptic surface if $L \equiv a \cdot A/\mu + b \cdot (\mu/\gamma)B$.

In $\mathrm{Num}(S)$, we have that $A^2 = 0$, $B^2 = 0$, and $AB = \gamma$. Note that a divisor $b \cdot (\mu/\gamma)\, B \equiv (0, b)$, $b \in \mathbb{Z}$, is effective if and only if $b \cdot (\mu/\gamma) \in \mathbb{N}$ (see [1, Proposition 5.2]).

The following proposition holds:

Proposition 3.3 (see [22], Lemma 1.3). *Let D be a divisor of type (a, b) on a hyperelliptic surface S. Then*

1. $\chi(D) = ab$*;*
2. *D is ample if and only if $a > 0$ and $b > 0$; and*
3. *If D is ample then $h^0(D) = \chi(D) = ab$.*

Now, we recall a bound for the self-intersection of a curve. The adjunction formula, applied to the normalisation of a curve C, implies the following formula:

Proposition 3.4 (Genus formula, [16], Lemma, p. 505). *Let C be a curve on a surface S, passing through $x_1, \ldots, x_r$ with multiplicities respectively $m_1, \ldots, m_r$. Let $g(C)$ denote the genus of the normalisation of C. Then*

$$g(C) \leq \frac{C^2 + C.K_S}{2} + 1 - \sum_{i=1}^{r} \frac{m_i(m_i - 1)}{2}.$$

Note that:

Observation 3.5. *A curve C on a hyperelliptic surface has genus at least 1. Indeed, otherwise, the normalisation of C, of genus zero, would be a covering (via Φ) of an elliptic curve A/G. This contradicts the Riemann–Hurwitz formula.*

4. Main Result

Our main result is the following.

Theorem 4.1. *Let S be a hyperelliptic surface. Let L be a line bundle of type (m,m) with $m \geq k+2$ on S. Then, L is k-jet ample.*

By the results of Mella and Palleschi, see [19, Theorems 3.2–3.4], we know that $L \equiv (a,b)$ with at least one of the coefficients strictly smaller than $k+2$ is not k-very ample on an arbitrary hyperelliptic surface, in particular, it is not k-very ample on a hyperelliptic surface of type 1. A line bundle which is not k-very ample is not k-jet ample. Therefore, the line bundle $L \equiv (k+2, k+2)$ is the first natural object of study.

Proof. We will prove that $L \equiv (k+2, k+2)$ is k-jet ample, and as a consequence, we will get that a line bundle of type (m,m) with $m \geq k+2$ is k-jet ample.

Let $r \geq 1$. We have to check that for each choice of distinct points $x_1, \ldots, x_r \in S$, the map

$$H^0(X, L) \longrightarrow H^0\left(X, L \otimes \mathcal{O}_X/(m_{x_1}^{k_1} \otimes \cdots \otimes m_{x_r}^{k_r})\right)$$

is surjective, where $\sum_{i=1}^{r} k_i = k+1$. □

We consider the standard exact sequence:

$$0 \longrightarrow (K_S + L) \otimes m_{x_1}^{k_1} \otimes \cdots \otimes m_{x_r}^{k_r} \longrightarrow K_S + L$$
$$\longrightarrow (K_S + L) \otimes \mathcal{O}_X/(m_{x_1}^{k_1} \otimes \cdots \otimes m_{x_r}^{k_r}) \longrightarrow 0.$$

By the long sequence of cohomology, surjectivity of the map

$$H^0(K_S + L) \longrightarrow H^0\left((K_S + L) \otimes \mathcal{O}_X/(m_{x_1}^{k_1} \otimes \cdots \otimes m_{x_r}^{k_r})\right)$$

is implied by vanishing of $H^1\left((K_S + L) \otimes m_{x_1}^{k_1} \otimes \cdots \otimes m_{x_r}^{k_r}\right)$.

By the projection formula, we have that

$$H^1\left((K_S + L) \otimes m_{x_1}^{k_1} \otimes \cdots \otimes m_{x_r}^{k_r}\right) \cong H^1\left(\pi^*(K_S + L) - \sum_{i=1}^{r} k_i E_i\right)$$
$$\cong H^1\left(K_{\widetilde{S}} - \sum_{i=1}^{r} E_i + \pi^* L - \sum_{i=1}^{r} k_i E_i\right) \cong H^1\left(K_{\widetilde{S}} + \pi^* L - \sum_{i=1}^{r}(k_i + 1)E_i\right).$$

We will show that $H^1\left(K_{\widetilde{S}} + \pi^* L - \sum_{i=1}^{r}(k_i + 1)E_i\right) = 0$, using vanishing theorems.

We consider, separately, case $r = 1$, and, separately, case $r \geq 2$.

First, let $r = 1$. We show that $\pi^* L - (k+2)E$ is nef and big, and hence, by the Kawamata–Viehweg vanishing theorem, we get that $H^1(K_{\widetilde{S}} + \pi^* L - (k+2)E) = 0$.

We have that

$$\pi^* L - (k+2)E = \pi^*((k+2, k+2)) - (k+2)E = (k+2)\left(\pi^*(1,1) - E\right).$$

By [15, Theorem 3.1], we know that on a hyperelliptic surface, the Seshadri constant of a line bundle of type $(1,1)$ at an arbitrary point x is at least 1. Therefore

$$\sup\{\varepsilon : \pi^*L - \varepsilon E \text{ is nef}\} = \varepsilon(L,x) = (k+2)\cdot\varepsilon\left((1,1),x\right) \geq k+2,$$

hence, the line bundle $\pi^*L-(k+2)E$ is nef. Thus to prove that $\pi^*L-(k+2)E$ is also big, it is enough to show that $\left(\pi^*L-(k+2)E\right)^2 > 0$, which is equivalent to prove that $L^2 > (k+2)^2$. The last inequality holds, as

$$L^2 = (k+2,k+2)^2 = 2(k+2)\cdot(k+2) = 2(k+2)^2.$$

The case $r=1$ is proved.

Now, let $r \geq 2$. We will prove that $H^1\left(K_{\widetilde{S}} + \pi^*L - \sum_{i=1}^{r}(k_i+1)E_i\right) = 0$. The proof will be divided in several cases, depending on the position of points $x_1, \ldots, x_r$.

Let $k=1$. Generation of 1-jets is by definition equivalent to 1-very ampleness. The line bundle $L \equiv (3,3)$ is 1-very ample on any hyperelliptic surface by [19, Theorems 3.2–3.4].

Let $k \geq 2$.

If $k_i = 0$ for some i, then we can consider points $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r$. In this case, without lose of generality, we may take a smaller r. From now on, we assume that $k_i \geq 1$ for every i. Obviously, $r \in [2, k+1]$ as $\sum_{i=1}^{r} k_i = k+1$.

For simplicity, we present a proof for hyperelliptic surfaces of type 1. For surfaces of other types, the proof is analogous. The small differences are listed in Remark 4.15.

The proof consists of a few steps which we describe briefly before we turn to the details.

First, in Case I, we consider a situation where on each singular fibre $A/2$, on each fibre A, and on each fibre B, there are points x_i with the sum of multiplicities k_i equal to at most $\frac{k+1}{2}$.

Then, in Cases II and III, we consider a situation where there exists a fibre $A/2$, respectively A, on which there are some points x_i with the sum of multiplicities k_i greater than $\frac{k+1}{2}$.

In both cases, we have two possibilities: (a) the sum of multiplicities of points lying on any fibre B is smaller than $\frac{k+1}{2}$; and (b) there exists a fibre B for which the sum of multiplicities of points on this fibre is at least $\frac{k+1}{2}$. Therefore, we divide Cases II and III into two subcases: IIa, IIb and IIIa, IIIb, respectively.

Finally, in Case IV, we consider the situation where some points x_i lie on a fixed fibre B and their sum of multiplicities does not exceed $\frac{k+1}{2}$; moreover, for each fibre $A/2$ and for each fibre A, the sum of multiplicities of points lying on this fibre is at most $\frac{k+1}{2}$. This covers all possibilities.

In all the cases described above, we prove that $H^1\left(K_{\widetilde{S}} + \pi^*L - \sum_{i=1}^{r}(k_i+1)E_i\right) = 0$, using Kawamata–Viehweg vanishing theorem in Cases I and IIIa, and Norimatsu lemma in Cases IIa, IIb, IIIb, and IV. In Cases I and IIIa, we show that a divisor $M = \pi^*L - \sum_{i=1}^{r}(k_i+1)E_i$

is big and nef; while in Cases IIa, IIb, IIIb, and IV, we define an appropriate SNC divisor F and prove that a divisor $N = M - F$ is ample, using Nakai–Moishezon criterion.

In all the cases by $C \equiv (\alpha, \beta)$, we denote a reduced irreducible curve on S passing through $x_1, \ldots, x_r$ with multiplicities respectively $m_1, \ldots, m_r$, where $m_i \geq 0$ for all i, and there exists j with $m_j > 0$. We have $\widetilde{C} = \pi^*C - \sum_{i=1}^{r} m_i E_i$.

Define $k_i^W = \begin{cases} k_i \text{ if } x_i \in W \text{ for a reduced fibre } W \text{ of } \Phi \text{ or } \Psi, \\ 0 \text{ otherwise.} \end{cases}$

Let r_W be the number of points from the set $\{x_1, \ldots, x_r\}$ which are contained in W.

Let us now move on to considering the cases in more detail.

Case I. For an arbitrary fibre W (where $W = A/2$, or $W = A$, or $W = B$), the sum of multiplicities of the points x_i lying on this fibre is at most $\frac{k+1}{2}$, that is

$$\sum_{i=1}^{r_W} k_i^W \leq \frac{k+1}{2}.$$

Since $\sum_{i=1}^{r_W} k_i^W \leq \frac{k+1}{2}$, in particular, we have that $r_W \leq \frac{k+1}{2}$.

We will show that the line bundle $M = \pi^*L - \sum_{i=1}^{r}(k_i + 1)E_i$ is big and nef.

Lemma 4.2. *M is a nef line bundle.*

Proof. We ask whether $M\widetilde{C} \geq 0$. We have to check that

$$(\star) = M\widetilde{C} = \left(\pi^*L - \sum_{i=1}^{r}(k_i + 1)E_i\right) \cdot \left(\pi^*C - \sum_{i=1}^{r} m_i E_i\right) \geq 0.$$

Let us consider the following cases:

(1) $C = A/2$, or $C = B$, or $C = A$. Then, for $i = 1, \ldots, r$, we have $m_i = 1$, hence

$$(\star) \geq LC - \sum_{i=1}^{r_W}(k_i^W + 1) \geq (k+2) - \sum_{i=1}^{r_W}(k_i^W + 1)$$
$$\geq (k+2) - \left(\left(\sum_{i=1}^{r_W} k_i^W\right) + r_W\right) \geq (k+2) - \left(\frac{k+1}{2} + \frac{k+1}{2}\right) > 0.$$

(2) C is not a fibre. Hence, $\alpha > 0$ and $\beta > 0$. We prove the inequality $M\widetilde{C} \geq 0$ in Proposition 4.9 at the end of the proof of the main theorem.

$\square$

Lemma 4.3. *M is big.*

Proof. Since M is nef, it is enough to prove that $M^2 > 0$. As $M^2 = 2(k+2)^2 - \sum_{i=1}^{r}(k_i + 1)^2$, we ask whether $2(k+2)^2 > \sum_{i=1}^{r}(k_i + 1)^2$. It suffices to show that

$$2(k+2)^2 > \left(\sum_{i=1}^{r} k_i^2\right) + 2(k+1) + r,$$

where r at the end of the formula is of the greatest possible value $k+1$. Since $\sum_{i=1}^{r} k_i^2 \leq \left(\sum_{i=1}^{r} k_i\right)^2 = (k+1)^2$, our goal is to prove the inequality

$$2(k+2)^2 > (k+1)^2 + 3k + 3,$$

which is elementary. □

Case IIa. There exists a fibre $A/2$ on which there are points $x_1, \ldots, x_s$ with the sum of multiplicities $\sum_{i=1}^{s} k_i > \frac{k+1}{2}$, and on each fibre B, we have $\sum_{i=1}^{r_W} k_i^W < \frac{k+1}{2}$.

Obviously, for any other fibre $A/2$ and for any fibre A, we have $\sum_{i=1}^{r_W} k_i^W < \frac{k+1}{2}$.

We write $M = \pi^* L - \sum_{i=1}^{r}(k_i+1) = \pi^*(A/2) - \sum_{i=1}^{s} E_i + \pi^*((k+1, k+2)) - \sum_{i=1}^{s} k_i E_i - \sum_{i=s+1}^{r}(k_i+1)E_i$, so $M = N + F$, where $N = \pi^*((k+1, k+2)) - \sum_{i=1}^{s} k_i E_i - \sum_{i=s+1}^{r}(k_i+1)E_i$ and $F = \pi^*(A/2) - \sum_{i=1}^{s} E_i = \widetilde{A/2}$. Clearly, F is a smooth and reduced divisor, and hence, F is an SNC divisor. It remains to show that N is ample.

Lemma 4.4. *N is ample.*

Proof. We check that $N^2 > 0$, and $N\widetilde{C} > 0$.

Let us estimate N^2

$$N^2 = 2(k+1)(k+2) - \sum_{i=1}^{r} k_i^2 - 2\sum_{i=s+1}^{r} k_i - (r-s) \geq k^2 + 2k + 1 + s > 0,$$

as $\sum_{i=s+1}^{r} k_i < \frac{k+1}{2}$, $\sum_{i=1}^{r} k_i^2 \leq \left(\sum_{i=1}^{r} k_i\right)^2 = (k+1)^2$ and $r \leq k+1$.

Now, we check whether

$$\begin{aligned}(\star) &= N\widetilde{C}\\ &= \left(\pi^*((k+1, k+2)) - \sum_{i=1}^{s} k_i E_i - \sum_{i=s+1}^{r}(k_i+1)E_i\right) \cdot \left(\pi^* C - \sum_{i=1}^{r} m_i E_i\right) > 0.\end{aligned}$$

We consider the following cases:

(1) C is the fibre $A/2$ for which $\sum_{i=1}^{s} k_i > \frac{k+1}{2}$. Then, $m_i = 1$ for all i, and

$$(\star) = k + 2 - \sum_{i=1}^{s} k_i \geq k + 2 - (k+1) = 1 > 0,$$

as $\sum_{i=1}^{s} k_i \leq \sum_{i=1}^{r} k_i = k+1$.

(2) C is a different fibre $A/2$, or $C = A$, or $C = B$. All $m_i = 1$, hence

$$(\star) \geq k + 1 - \sum_{i=1}^{r_W}(k_i^W + 1) > k + 1 - \left(\frac{k+1}{2} + \frac{k+1}{2}\right) = 0,$$

as $\sum_{i=1}^{r_W} k_i^W < \frac{k+1}{2}$ and $r_W < \frac{k+1}{2}$.

(3) C is not a fibre—see Proposition 4.9.

$\square$

Case IIb. Points $x_1, \ldots, x_s$ lie on a fixed fibre $A/2$ with $\sum_{i=1}^{s} k_i > \frac{k+1}{2}$; moreover, there exists a fibre B, such that $x_s, \ldots, x_t$ lie on B with $\sum_{i=s}^{t} k_i \geq \frac{k+1}{2}$.

We define $F := \widetilde{A/2} + \widetilde{B} = \widetilde{A/2 + B}$. Clearly, F is an SNC divisor. We define $N := M - F = \pi^*((k+1, k+1)) + E_s - \sum_{i=1}^{t} k_i E_i - \sum_{i=t+1}^{r} (k_i+1) E_i$. It remains to check that N is ample.

Lemma 4.5. *N is ample.*

Proof. Analogously to Case IIa, we show that $N^2 > 0$, and that $N\widetilde{C} > 0$.

We estimate N^2 from below, using inequalities $\sum_{i=t+1}^{r} k_i \leq \frac{k+1}{2}$, $\sum_{i=1}^{r} k_i^2 \leq (k+1)^2$, and $r \leq k+1$.

$$\begin{aligned} N^2 &= 2k^2 + 4k + 1 - \sum_{i=1}^{r} k_i^2 - 2 \sum_{i=t+1}^{r} k_i - (r - t) \\ &\geq 2k^2 + 4k + 1 - (k+1)^2 - 2 \cdot \frac{k+1}{2} - (k + 1 - t) = k^2 + t - 2. \end{aligned}$$

Hence, $N^2 > 0$, because $k \geq 2$.

Now, we check that

$$(\star) = N\widetilde{C} > 0.$$

We consider the following cases:

(1) C is the fibre $A/2$ for which $\sum_{i=1}^{s} k_i > \frac{k+1}{2}$. Then

$$(\star) = k + 1 + 1 - \sum_{i=1}^{s} k_i \geq k + 2 - (k+1) = 1 > 0,$$

as $\sum_{i=1}^{s} k_i \leq k + 1$.

(2) C is a different fibre $A/2$, or $C = A$. Then

$$(\star) \geq k + 1 - \sum_{i=1}^{r_W} \left(k_i^W + 1\right) > k + 1 - \left(\frac{k+1}{2} + \frac{k+1}{2}\right) = 0,$$

as $\sum_{i=1}^{r_W} k_i^W < \frac{k+1}{2}$ and $r_W < \frac{k+1}{2}$.

(3) C is the fibre B for which $\sum_{i=s}^{t} k_i \geq \frac{k+1}{2}$. Then

$$(\star) = (k+1) + 1 - \sum_{i=s}^{t} k_i \geq k + 2 - (k+1) = 1 > 0.$$

(4) C is a different fibre B. Then

$$(\star) \geq (k+1) - \sum_{i=1}^{r_W} (k_i^W + 1) > (k+1) - \left(\frac{k+1}{2} + \frac{k+1}{2}\right) = 0,$$

as $\sum_{i=1}^{r_W} k_i^W < \frac{k+1}{2}$ and $r_W < \frac{k+1}{2}$.

(5) C is not a fibre—see Proposition 4.9.

$\square$

Case IIIa. Points $x_1, \ldots, x_s$ lie on a fixed fibre A with $\sum_{i=1}^{s} k_i > \frac{k+1}{2}$, and for each fibre B, we have $\sum_{i=1}^{r_W} k_i^W < \frac{k+1}{2}$.

Obviously, in this case, for any other fibre A and for any fibre $A/2$, we have $\sum_{i=1}^{r_W} k_i^W < \frac{k+1}{2}$.

Let $M = \pi^* L - \sum_{i=1}^{r} (k_i + 1) E_i$. We have already showed in Case I that $M^2 > 0$. It remains to prove that M jest nef.

Lemma 4.6. *M is nef.*

Proof. We ask whether

$$(\star) = M\widetilde{C} \geq 0.$$

We consider the following cases:

(1) C is the fibre A for which $\sum_{i=1}^{s} k_i > \frac{k+1}{2}$. Then

$$\begin{aligned}(\star) &= 2(k+2) - \sum_{i=1}^{s}(k_i+1) \geq 2(k+2) - \sum_{i=1}^{r} k_i - s \\ &\geq 2(k+2) - (k+1) - (k+1) = 2 > 0.\end{aligned}$$

(2) C is a different fibre A, or $C = A/2$, or $C = B$. Then

$$(\star) \geq k+2 - \sum_{i=1}^{r_W}(k_i^W + 1) \geq k+2 - \left(\frac{k+1}{2} + \frac{k+1}{2}\right) = 1 > 0.$$

(3) C is not a fibre—see Proposition 4.9. $\square$

Case IIIb. Points $x_1, \ldots, x_s$ lie on a fixed fibre A with $\sum_{i=1}^{s} k_i > \frac{k+1}{2}$, and there exists a fibre B, such that $x_s, \ldots, x_t$ lie on B and $\sum_{i=s}^{t} k_i \geq \frac{k+1}{2}$.

We define $F := \widetilde{B} = \pi^* B - \sum_{i=s}^{t} E_i$. Clearly, F is an SNC divisor. Let $N := M - F = \pi^*((k+2, k+1)) - \sum_{i=1}^{s-1}(k_i+1)E_i - \sum_{i=s}^{t} k_i E_i - \sum_{i=t+1}^{r}(k_i + 1)E_i$. It remains to check that N is ample.

Lemma 4.7. *N is ample.*

Proof. Let us estimate N^2:

$$\begin{aligned}N^2 &= 2(k+2)(k+1) - \sum_{i=1}^{s-1}(k_i+1)^2 - \sum_{i=s}^{t} k_i^2 - \sum_{i=t+1}^{r}(k_i+1)^2 \\ &= 2(k+2)(k+1) - \sum_{i=1}^{r} k_i^2 - 2\left(\sum_{i=1}^{s-1} k_i - \sum_{i=t+1}^{r} k_i\right) - (s-1) - (r-t) \\ &\geq 2k^2 + 6k + 4 - (k+1)^2 - 2(k+1) - s + 1 - r + t \\ &\geq k^2 + 2k + 2 - (k+1) - (k+1) + t = k^2 + t > 0.\end{aligned}$$

Now, we check that

$$(\star) = N\widetilde{C} > 0.$$

Let us consider the following cases:

(1) C is the fibre A for which $\sum_{i=1}^{s} k_i > \frac{k+1}{2}$. Then

$$(\star) = 2(k+1) - \sum_{i=1}^{s-1}(k_i+1) - k_s \geq 2(k+1) - \sum_{i=1}^{s} k_i - (s-1)$$
$$\geq 2(k+1) - (k+1) - (k+1) + 1 = 1 > 0.$$

(2) C is the fibre B for which $\sum_{i=s}^{t} k_i \geq \frac{k+1}{2}$. Then

$$(\star) = k+2 - \sum_{i=s}^{t} k_i \geq k+2-(k+1) = 1 > 0.$$

(3) C is a different fibre A or a different fibre B, or $C = A/2$. Then

$$(\star) \geq k+1 - \sum_{i=1}^{r_W}(k_i^W+1) > k+1 - \left(\frac{k+1}{2} + \frac{k+1}{2}\right) = 0.$$

(4) C is not a fibre—see Proposition 4.9. □

Case IV. Points $x_1, \ldots, x_s$ lie on a fixed fibre B with $\sum_{i=1}^{s} k_i > \frac{k+1}{2}$, and for each fibre $A/2$ and for each fibre A, the sum of multiplicities of the points lying on this fibre does not exceed $\frac{k+1}{2}$.

We define $F := \widetilde{B} = \pi^*B - \sum_{i=1}^{s} E_i$. Of course, F is an SNC divisor. We define $N := M - F = \pi^*((k+2, k+1)) - \sum_{i=1}^{s} k_i E_i - \sum_{i=s+1}^{r}(k_i+1)E_i$. It remains to prove that N is ample.

Lemma 4.8. *N is ample.*

Proof. Analogously to Case IIa,

$$N^2 = k^2 + 2k + 1 + s > 0.$$

We have to check that

$$(\star) = N\widetilde{C} > 0.$$

Let us consider the following cases:

(1) C is the fibre B for which $\sum_{i=1}^{s} k_i > \frac{k+1}{2}$. Then

$$(\star) = k+2 - \sum_{i=1}^{s} k_i \geq k+2-(k+1) = 1 > 0.$$

(2) C is a different fibre B or $C = A$. Then

$$(\star) \geq k+2 - \sum_{i=1}^{r_W}(k_i^W+1) \geq k+2 - \left(\frac{k+1}{2} + \frac{k+1}{2}\right) = 1 > 0.$$

(3) $C = A/2$.
If $r_W < \frac{k+1}{2}$, then

$$(\star) = k+1 - \sum_{i=1}^{r_W}(k_i^W+1) > k+1 - \left(\frac{k+1}{2} + \frac{k+1}{2}\right) = 0.$$

Otherwise, $r_W = \frac{k+1}{2}$. Hence, the situation is as follows: s points whose sum of multiplicities is greater than $\frac{k+1}{2}$ lie on the fixed fibre B, and there exists a fibre $A/2$ with $\frac{k+1}{2}$ points whose sum of multiplicities does not exceed $\frac{k+1}{2}$, which implies that all the multiplicities of points lying on $A/2$ equal 1. Hence, $x_1, \ldots, x_s$ lie on B and $x_s, \ldots, x_r$ lie on $A/2$. Therefore

$$\begin{aligned}(\star) &= k+1-k_s-\sum_{i=s+1}^{r}(k_i+1) = k+1-1-2(r-s)\\ &= k+2-2\cdot\frac{k+1}{2} = 1>0.\end{aligned}$$

(4) C is not a fibre—see Proposition 4.9. □

We have considered all the possible positions of $x_1, \ldots, x_r$. Hence, the proof will be completed if we show that for a curve C not being a fibre the inequality, respectively, $M\widetilde{C} \geq 0$ or $N\widetilde{C} > 0$ holds. We prove this fact in the proposition below.

Proposition 4.9. *We have the following inequalities for a curve C which is not a fibre*:

- $M\widetilde{C} \geq 0$ *in Cases I and IIIa, and*
- $N\widetilde{C} > 0$ *in Cases IIa, IIb, IIIb, and IV.*

Proof. By assumption $C \equiv (\alpha, \beta)$ with $\alpha > 0$ and $\beta > 0$. We have to prove that

- $\left(\left(\sum_{i=1}^{r} k_i\right)+1\right)(\alpha+\beta)-\sum_{i=1}^{r}(k_i+1)m_i \geq 0$ in Cases I and IIIa;
- $\left(\sum_{i=1}^{r} k_i\right)(\alpha+\beta)+\alpha-\sum_{i=1}^{s}k_im_i-\sum_{i=s+1}^{r}(k_i+1)m_i > 0$ in Case IIa;
- $\left(\sum_{i=1}^{r} k_i\right)(\alpha+\beta)+m_s-\sum_{i=1}^{t}k_im_i-\sum_{i=t+1}^{r}(k_i+1)m_i > 0$ in Case IIb;
- $\left(\sum_{i=1}^{r} k_i\right)(\alpha+\beta)+\beta-\sum_{i=1}^{s-1}(k_i+1)m_i-\sum_{i=s}^{t}k_im_i-\sum_{i=t+1}^{r}(k_i+1)m_i > 0$ in Case IIIb;
- $\left(\sum_{i=1}^{r} k_i\right)(\alpha+\beta)+\beta-\sum_{i=1}^{s}k_im_i-\sum_{i=s+1}^{r}(k_i+1)m_i > 0$ in Case IV.

Observe that in all the situations above, Proposition 4.9 will be proved if we show that

$$(\star) \qquad \left(\sum_{i=1}^{r} k_i\right)(\alpha+\beta) \geq \sum_{i=1}^{r}(k_i+1)m_i.$$

Let $D \equiv (4,4)$. Since $h^0(D) = 4\cdot 4 = 16$, for an arbitrary point x, there exists a divisor $D_x \in |D|$, such that $\operatorname{mult}_x D_x = 5$ (vanishing up to order 5 imposes 15 conditions). Hence, there are two possibilities: either $\alpha \leq 4$ and

$\beta \leq 4$, and then, C and D_x may have a common component C (the curve C is irreducible); or $\alpha > 4$ or $\beta > 4$, and then, by Bézout's Theorem

$$4(\alpha+\beta) = (\alpha, \beta).(4,4) = CD = CD_x \geq \operatorname{mult}_x C \cdot \operatorname{mult}_x D_x \geq 5m_i.$$

Proof of the proposition in each case will be completed in Lemmas 4.10 and 4.12. □

Lemma 4.10. *Proposition 4.9 holds if $\alpha > 4$ or $\beta > 4$.*

Proof. We begin with a useful observation: □

Observation 4.11. *Let S be a hyperelliptic surface of any type, let $C \equiv (\alpha, \beta)$, where $\alpha > 4$ or $\beta > 4$. To prove $(\star)$, it suffices to prove the inequality*

$$(\star\star) \qquad r(\alpha+\beta) \geq 2\sum_{i=1}^{r} m_i.$$

Proof. We have already observed that by Bézout's Theorem, for every $i \in \{1, \dots, r\}$, we have that $\alpha + \beta \geq \frac{5}{4} m_i \geq m_i$, and hence

$$(k_i - 1)(\alpha+\beta) \geq (k_i - 1)m_i.$$

Summing up these r inequalities with inequality $(\star\star)$, we obtain the inequality $(\star)$. □

Now, we will show that if $\alpha > 4$ or $\beta > 4$, then $(\star\star)$ is satisfied for $r \geq 3$. Let us denote i-th inequality in the inequality $(\star\star)$ by $(\star\star_i)$, that is

$$(\star\star_i) \qquad (\alpha+\beta) \geq 2m_i.$$

The inequality $(\star\star_i)$ is satisfied for $m_i < 4$. Indeed, C is not a fibre, and by assumption, we have $\alpha \geq 5$ or $\beta \geq 5$, and hence, $\alpha + \beta \geq 5 + 1 = 6 \geq 2m_i$ for $m_i \leq 3$. Therefore, we may assume that $m_i \geq 4$.

We delete from $(\star\star)$ all the inequalities $(\star\star_i)$ with $m_i < 4$ and consider a modified inequality $(\star\star)$, possibly with a smaller number of points r. It may even happen that $r < 2$ in the modified $(\star\star)$.

Now, we will prove $(\star\star)$ assuming that $m_i \geq 4$ for all i, and $r \geq 3$. Equivalently, we want to show that

$$r^2(\alpha+\beta)^2 \geq 4\left(\sum_{i=1}^{r} m_i\right)^2.$$

By inequality between means, it is enough to check that

$$r^2(\alpha+\beta)^2 \geq 4r\sum_{i=1}^{r} m_i^2.$$

It suffices to check that

$$r\,(2\alpha\beta) \geq 2\sum_{i=1}^{r} m_i^2.$$

By the genus formula, $2\alpha\beta \geq \sum_{i=1}^{r} m_i^2 - \sum_{i=1}^{r} m_i$ (see Proposition 3.4 and Observation 3.5), and hence, it is enough to prove that

$$r\left(\sum_{i=1}^{r} m_i^2 - \sum_{i=1}^{r} m_i\right) \geq 2\sum_{i=1}^{r} m_i^2,$$

$$(r-2)\left(\sum_{i=1}^{r} m_i^2\right) - r\sum_{i=1}^{r} m_i \geq 0.$$

We assume that $m_i \geq 4$, so $\sum_{i=1}^{r} m_i^2 \geq 4\sum_{i=1}^{r} m_i$. Hence, it is enough to show that

$$4(r-2)\left(\sum_{i=1}^{r} m_i\right) - r\sum_{i=1}^{r} m_i \geq 0,$$

$$(3r-8)\cdot(4r) \geq 0,$$

which is obviously true for $r \geq 3$.

To finish the proof of Lemma, it remains to check that the assertion holds for $r < 3$. If $r = 0$, then every inequality $(\star\star_i)$ is satisfied, which together with Observation 4.11 completes the proof of the lemma. Hence, we have to consider cases $r = 1$ (when in the inequality $(\star\star)$, all but one inequalities $(\star\star_i)$ hold), and $r = 2$.

Let $r = 2$. We prove Lemma in Cases I and IIIa, and separately, in all the remaining cases.

Let us consider Cases I and IIIa. We have to prove the inequality

$$(k_1 + k_2 + 1)(\alpha + \beta) - (k_1 + 1)m_1 - (k_2 + 1)m_2 \geq 0.$$

Analogously to Observation 4.11, it suffices to prove the inequality $(1+1)(\alpha+\beta) + (\alpha+\beta) \geq 2m_1 + 2m_2$. Indeed, adding two fulfilled inequalities of the form $(k_i - 1)(\alpha+\beta) \geq (k_i - 1)m_i$, we obtain the assertion.

Therefore, we have to check that

$$3(\alpha+\beta) \geq 2m_1 + 2m_2.$$

In the linear system of divisor $D \equiv (4, 4)$, for each points x_1, x_2, there exists such a divisor $D_{x_1,x_2} \equiv D$ that its multiplicity at x_1 equals 4, and multiplicity at x_2 equals 2.

Since $\alpha > 4$ or $\beta > 4$ and the curve C is irreducible, C is not a component of D. Therefore, by Bézout's Theorem, we have

$$4(\alpha+\beta) = CD = CD_{x_1,x_2} \geq 4m_1 + 2m_2. \tag{4.1}$$

Analogously

$$4(\alpha+\beta) \geq 2m_1 + 4m_2.$$

Summing up two inequalities above, we obtain

$$8(\alpha+\beta) \geq 6m_1 + 6m_2.$$

Hence

$$3(\alpha+\beta) \geq \frac{9}{4}m_1 + \frac{9}{4}m_2 \geq 2m_1 + 2m_2,$$

and the assertion is proved.

Now let us consider Cases IIa, IIb, IIIb, and IV. There are two possibilities:

(a) One of the points x_1, x_2 (without loss of generality x_2) lies respectively: on the fixed fibre $A/2$ in Case IIa, on the intersection of the fixed fibres $A/2$ and B in Case IIb, on the intersection of the fixed fibres A and B in Case IIIb, and on the fixed fibre B in Case IV. Then, the desired inequalities $N\widetilde{C} > 0$ are implied by the inequality

$$(k_1 + k_2)(\alpha + \beta) \geq (k_1 + 1)m_1 + k_2 m_2.$$

By observation analogous to Observation 4.11 it suffices to show that

$$2(\alpha+\beta) \geq 2m_1 + m_2.$$

The assertion holds by inequality (4.1).

(b) None of the points x_1, x_2 lies on the fixed fibre, respectively, $A/2$, $A/2$ intersected with B, A intersected with B, and B. By assumption of Case, respectively, IIa, IIb, IIIb, and IV, at the beginning, there was a point x_3 on those fixed fibres or on the intersection of the fixed fibres, and the inequality $(\star\star_3)$ is satisfied. We restore this inequality, and we want to prove that

$$(k_1 + k_2 + k_3)(\alpha + \beta) \geq (k_1 + 1)m_1 + (k_2 + 1)m_2 + k_3 m_3.$$

By observation analogous to Observation 4.11, it is enough to prove that

$$3(\alpha+\beta) \geq 2m_1 + 2m_2 + m_3.$$

In the linear system of divisor $D \equiv (4, 4)$, for each points x_1, x_2, and x_3, there exists such a divisor $D_{x_1,x_2,x_3} \equiv D$ that its multiplicity at x_1 equals 3, its multiplicity at x_2 equals 3, and its multiplicity at x_3 equals 2.

By Bézout's Theorem, we obtain

$$\begin{aligned} 4(\alpha+\beta) = CD = CD_{x_1,x_2,x_3} &\geq \sum_{i=1}^{3} \operatorname{mult}_{x_i} C \cdot \operatorname{mult}_{x_i} D_{x_1,x_2,x_3} \\ &\geq 3m_1 + 3m_2 + 2m_3. \end{aligned} \tag{4.2}$$

Hence

$$3(\alpha+\beta) \geq \frac{9}{4}m_1 + \frac{9}{4}m_2 + \frac{6}{4}m_3 \geq 2m_1 + 2m_2 + m_3,$$

and we are done.

Now let $r = 1$. At the beginning, the number of points r was at least 2, hence while deleting from the inequality $(\star\star)$ the inequalities $(\star\star_i)$ with $m_i \leq 3$, we deleted all by one inequalities.

We restore one of the deleted inequalities $(\star\star_i)$: an arbitrary one in Cases I and IIIa; in Cases IIa, IIb, IIIb, and IV—the inequality corresponding to x_i lying, respectively, on the fixed fibre $A/2$, on the intersection of the fixed fibres $A/2$ and B, on the intersection of the fixed fibres A and B, and on the fixed fibre B (if x_1 is not in such a position; an arbitrary inequality $(\star\star_i)$ otherwise). We obtained the inequality with $r = 2$ which was already proved in subcase (a). □

Lemma 4.12. *Proposition 4.9 holds if $\alpha \leq 4$ and $\beta \leq 4$.*

Proof. We denote

$$(\star_i) \qquad (\alpha + \beta)k_i \geq (k_i + 1)m_i.$$

Observe the following property: □

Remark 4.13. For an arbitrary m_i the inequality $(\star_i)$ is satisfied if $m_i \leq \frac{1}{2}(\alpha + \beta)$. Indeed, $2m_i \geq \left(1 + \frac{1}{k_i}\right) m_i$ for all k_i.

In particular, if $m_i = 0$ or $m_i = 1$, then the inequality $(\star_i)$ is satisfied, since $\alpha \geq 1$ and $\beta \geq 1$.

The multiplicities of C at $x_1, \ldots, x_r$ satisfy genus formula, i.e., $2\alpha\beta \geq \sum_{i=1}^{r} m_i^2 - \sum_{i=1}^{r} m_i$ (see Proposition 3.4 and Observation 3.5). In particular, for any x_i, we have an upper bound $2\alpha\beta \geq m_i^2 - m_i$.

If the inequality $(\star_i)$ holds for some m_i, then it holds also for all multiplicities $n_i < m_i$. After renumbering the points, we assume that the multiplicities are decreasing: $m_1 \geq m_2 \geq \cdots \geq m_r$.

In the table below, for every curve $C \equiv (\alpha, \beta)$ with $0 < \alpha \leq 4$ and $0 < \beta \leq 4$, we present the following quantities:

– an upper bound for the maximal possible multiplicity m_i at any point x_i, obtained from genus formula,

– an upper bound for the multiplicity m_i, for which by Remark 4.13, the inequality $(\star_i)$ holds,

– all possible values of m_1 greater than the number from the previous column, and for each such m_1, the greatest possible value of m_2 and m_i for $i > 2$ (or an upper bound in the latter case), all obtained from genus formula.

Case no.	$C \equiv (\alpha, \beta)$	$\max m_i$	$\max m_i$ such that $(\star_i)$ holds	m_1	m_2	m_i for $i > 2$
i	$(4, 4)$	6	4	6 5	2 4	1
ii	$(4, 3)$ or $(3, 4)$	5	3	5 4	2 3	≤ 2
iii	$(4, 2)$ or $(2, 4)$	4	3	4	2	≤ 2
iv	$(4, 1)$ or $(1, 4)$	3	2	3	2	1
v	$(3, 3)$	4	3	4	3	1
vi	$(3, 2)$ or $(2, 3)$	4	2	4 3	1 3	1
vii	$(3, 1)$ or $(1, 3)$	3	2	3	1	1
viii	$(2, 2)$	3	2	3	2	1
ix	$(2, 1)$ or $(1, 2)$	2	1	2	2	1
x	$(1, 1)$	2	1	2	1	1

Note that in each case (i)–(x), the inequality $(\star_i)$ holds for each multiplicity m_i where $i > 2$. Therefore, we would be done if we could prove $(\star)$ for $r = 2$ and points x_1, x_2 with multiplicities m_1, m_2 listed in the table.

Simple computations give us the following.

Remark 4.14. 1. For $r = 2$, if $m_1 + m_2 \leq \alpha + \beta$, then $(\star)$ holds.

2. Moreover for $r = 2$, if $m_1 + m_2 \leq \alpha + \beta + 1$, then the inequality from Cases I and IIIa holds, i.e., $(\alpha + \beta)\left(\sum_{i=1}^{r} k_i + 1\right) \geq \sum_{i=1}^{r}(k_i + 1)m_i$.

Let us consider the cases listed in the table. In cases (ii), (iii), (iv), and (vii), the inequality $(\star)$ holds by Remark 4.14(2). In cases (i), (v), (vi), (viii), and (ix), we repeat the reasoning with restoring a certain inequality $(\star_3)$, if it is needed. Case (x) is the most subtle—in some subcases, we have to restore two inequalities $(\star_3)$ and $(\star_4)$.

We explain the cases (i) and (x) in detail.

Case (i): $C \equiv (4, 4)$.

It is enough to prove $(\star)$ in two subcases: $r = 2$, $m_1 = 6$, $m_2 = 2$, and $r = 2$, $m_1 = 5$, $m_2 = 4$. In the first subcase, we get the assertion by Remark 4.14(1). In the second subcase, in some situations, we have to restore the fulfilled inequality $(\star_3)$ to prove $(\star)$. Our reasoning differs with respect to cases of the main theorem:

– In Cases I and IIIa, the inequality $(\star)$ holds by Remark 4.14(2).

– In Cases IIa, IIIb, and IV, there are two possibilities:

- One of the points x_1, x_2 (without loss of generality x_2) lies respectively: on the fixed fibre $A/2$, in Case IIa, on the intersection of the fixed fibres A and B, in Case IIIb, and on the fixed fibre B, in Case IV. Then, it is enough to prove that

$$(\alpha + \beta)(k_1 + k_2) + \min\{\alpha, \beta\} > (k_1 + 1)m_1 + k_2 m_2,$$

$$8(k_1 + k_2) + 4 > 5(k_1 + 1) + 4k_2,$$

$$3k_1 + 4k_2 > 1.$$

The last inequality holds as $k_i \geq 1$ for all i.

- None of the points x_1, x_2 lies on the fixed fibre, respectively, $A/2$, A intersected with B, and B. By assumption of Cases respectively IIa, IIIb, and IV; at the beginning, there was a point x_3 on the fixed fibre or on the intersection of the fixed fibres, and the maximal possible by genus formula multiplicity m_3 is equal to 1. We restore the inequality $(\star_3)$, and we want to prove that

$$(\alpha + \beta)(k_1 + k_2 + k_3) + \min\{\alpha, \beta\} > (k_1 + 1)m_1 + (k_2 + 1)m_2 + k_3 m_3,$$

$$3k_1 + 4k_2 + 7k_3 > 5.$$

The inequality holds.

– In Case IIb, there are also two possibilities:

- One of the points x_1, x_2 is the intersection point of the fixed fibres $A/2$ and B. Then, the inequality

$$(\alpha + \beta)(k_1 + k_2) + m_2 > (k_1 + 1)m_1 + k_2 m_2,$$

$$3k_1 + 4k_2 > 1$$

obviously, holds.

- None of the points x_1, x_2 lies on the intersection of the fixed fibres $A/2$ and B. Then, by assumption of Case IIb, there was a point x_3 with $m_3 = 1$ on this intersection. We restore the inequality $(\star_3)$ and have to check that the following inequality is satisfied

$$(\alpha+\beta)(k_1+k_2+k_3)+m_3 > (k_1+1)m_1+(k_2+1)m_2+k_3m_3,$$

$$3k_1+4k_2+7k_3 > 8,$$

which is true.

Case (x): $C \equiv (1,1)$.

We consider a situation $r = 2$, $m_1 = 2$, and $m_2 = 1$. If needed, we restore one or two inequalities $(\star_i)$ and prove $(\star)$ for $r = 3$, $m_1 = 2$, $m_2 = 1$, $m_3 = 1$ or for $r = 4$, $m_1 = 2$, $m_2 = 1$, $m_3 = 1$, $m_4 = 1$. Here, come the details:

– In Cases I and IIIa, the inequality $(\star)$ holds by Remark 4.14(2).

– In Cases IIa, IIIb, and IV, there are two possibilities:

- One of the points x_1, x_2 (say x_2) lies respectively: on the fixed fibre $A/2$ in Case IIa, on the intersection of the fixed fibres A and B in Case IIIb, and on the fixed fibre B in Case IV. Then, it suffices to prove that

$$(\alpha+\beta)(k_1+k_2)+\min\{\alpha,\beta\} > (k_1+1)m_1+k_2m_2$$

which gives

$$k_2 > 1.$$

If $k_2 = 1$, then respectively: on the fixed fibre $A/2$, on the fixed fibre B, but outside the intersection of the fixed fibres A and B (otherwise $k = 1$, but we have excluded such a situation at the beginning of main theorem's proof), on the fixed fibre B, there was originally at least one more point, as for points on the fibre $\sum k_i^W > \frac{k+1}{2}$. If x_1 is the described point, it suffices to prove that

$$(\alpha+\beta)(k_1+k_2)+\min\{\alpha,\beta\} > k_1m_1+k_2m_2,$$

$$k_2+1 > 0.$$

The inequality holds. If the described point is different from x_1, then it has multiplicity $m_3 = 1$. We have to check that

$$(\alpha+\beta)(k_1+k_2+k_3)+\min\{\alpha,\beta\} > (k_1+1)m_1+k_2m_2+k_3m_3,$$

$$k_2+k_3 > 1.$$

The inequality holds.

- None of the points x_1, x_2 lies on the fixed fibre, respectively, $A/2$, A intersected with B, and B. By assumption of Case, respectively, IIa, IIIb, and IV; at the beginning, there was a point x_3 on the fixed fibre or on the intersection of the fixed fibres, and the maximal possible by genus formula multiplicity m_3 is equal to 1. We restore the inequality $(\star_3)$, and we have to check that

$$(\alpha+\beta)(k_1+k_2+k_3)+\min\{\alpha,\beta\} > (k_1+1)m_1 \\ +(k_2+1)m_2+k_3m_3,$$

$$k_2+k_3>2.$$

If $k_3=1$, then by assumption of Case IIa, IIIb, and IV, respectively, on the fixed fibre $A/2$, on the fixed fibre B, but outside the intersection of A and B, on the fixed fibre B, there is a point x_4. In Case IIIb, x_4 may be equal to one of the points x_1, x_2—we obtain an inequality which was already proved. Otherwise in Case IIIb, and in Cases IIa and IV, x_4 is different from x_1, x_2, x_3, and $m_4=1$. We have to prove that

$$(\alpha+\beta)(k_1+k_2+k_3+k_4)+m_3 > (k_1+1)m_1 \\ +(k_2+1)m_2+k_3m_3+k_4m_4,$$

$$k_2+k_3+k_4>2.$$

The inequality holds.

– In Case IIb, there are two possibilities as well:

- One of the points x_1, x_2 (say x_2) is the intersection point of the fixed fibres $A/2$ and B. Then, we have to prove that

$$(\alpha+\beta)(k_1+k_2)+m_2 > (k_1+1)m_1+k_2m_2,$$

$$k_2>1.$$

If $k_2=1$, then by assumption of Case IIb, there is at least one more point x_3 on the fixed fibre $A/2$. If $x_3=x_1$, then it is enough to check that

$$(\alpha+\beta)(k_1+k_2)+m_2 > k_1m_1+k_2m_2,$$

$$k_2+1>0.$$

The inequality holds. Otherwise, $m_3=1$, and it suffices to show that

$$(\alpha+\beta)(k_1+k_2+k_3)+m_2 > (k_1+1)m_1+k_2m_2+k_3m_3,$$

$$k_2+k_3>1.$$

The inequality holds.
- None of the points x_1, x_2 lies in the intersection point of the fixed fibres $A/2$ and B. Then, on this intersection, there is a point x_3 with $m_3=1$. Hence, it is enough to prove that

$$(\alpha+\beta)(k_1+k_2+k_3)+m_3 > (k_1+1)m_1+(k_2+1)m_2+k_3m_3,$$

$$k_2+k_3>2.$$

If $k_3=1$, then there is one more point x_4 on the fixed fibre $A/2$. If x_4 is equal to x_1 or x_2, then we get the already proved inequality

$$(\alpha+\beta)(k_1+k_2+k_3)+m_3 > (k_1+1)m_1+k_2m_2+k_3m_3.$$

If x_4 is different from x_1, x_2, and x_3, then $m_4 = 1$, and we obtain the inequality

$$(\alpha+\beta)(k_1+k_2+k_3+k_4)+m_3 > (k_1+1)m_1 \\ +(k_2+1)m_2+k_3m_3+k_4m_4,$$

which has been proved as well. □

The proof of the Lemmas 4.10 and 4.12 is completed, and hence, the main theorem has been proved. □

We have presented the detailed proof for hyperelliptic surfaces of type 1. Below, we list the small differences which occur in the proof for hyperelliptic surfaces of types 2–7.

Remark 4.15.

- For a hyperelliptic surface of even type, there is no Case IIb nor IIIb, as the divisor $(\mu/\gamma)B \equiv (0,1)$ is not effective on such a surfaces. Hence, while proving that, respectively, $M\widetilde{C} \geq 0$ and $N\widetilde{C} > 0$, we do not consider the curve $C \equiv (\mu/\gamma)B$.
- For a hyperelliptic surface of type 3, 4, and 7, there are more types of singular fibres mA/μ. Hence, in Cases I, IIa, IIIa, and IV, and for a surfaces of type 3 and 7 also in Cases IIb and IIIb, we have to consider the intersection of C with singular fibres of all admissible types mA/μ, but they are estimated from below by the intersection with a fibre A/μ.
- For a hyperelliptic surface of type 3, 4, and 7, we consider additional cases—points $x_1, \ldots, x_s$ lie on a fixed singular fibre mA/μ, where $1 < m \leq \frac{\mu}{2}$. These cases are analogous to Cases IIIa and IIIb.

Proof of Theorem 4.1 for $r = 1$ gives us the following:

Corollary 4.16. *Let S be a hyperelliptic surface. Let L be a line bundle of type $(k+2, k+2)$ on S. Then, L generates k-jets at any point x.*

Acknowledgements

The author would like to thank Tomasz Szemberg and Halszka Tutaj-Gasińska for many helpful discussions and improving the readability of the paper.

References

[1] Aprodu, M.: An Appel–Humbert theorem for hyperelliptic surfaces. J. Math. Kyoto Univ. **38**(1), 101–121 (1998)

[2] Bauer, Th., Di Rocco, S., Szemberg, T.: Generation of jets on $K3$ surfaces. J. Pure Appl. Algebra **146**(1), 17–27 (2000)

[3] Bauer, Th., Szemberg, T.: Higher order embeddings of abelian varieties. Math. Z. **224**, 449–455 (1997)

[4] Bauer, Th., Szemberg, T.: Primitive higher order embeddings of abelian surfaces. Trans. Am. Math. Soc. **349**, 1675–1683 (1997)

[5] Beauville, A.: Complex Algebraic Surfaces. London Mathematical Society Student Texts, vol. 34, 2nd edn. Cambridge University Press (1996)

[6] Beltrametti, M.C., Francia, P., Sommese, A.J.: On Reider's method and higher order embeddings. Duke Math. J. **58**, 425–439 (1989)

[7] Beltrametti, M.C., Sommese, A.J.: On k-spannedness for projective surfaces. In: Sommese, A. J., Biancofiore, A., Livorni, E. L. (eds.) Algebraic Geometry (L'Aquila 1988). Lecture Notes in Mathematics, vol. 1417, pp. 24–51. Springer, Berlin, Heidelberg (1990)

[8] Beltrametti, M.C., Sommese, A.J.: On k-jet ampleness. In: Ancona, V., Silva, A. (eds.) Complex Analysis and Geometry, pp. 355–376. Plenum Press, New York (1993)

[9] Beltrametti, M.C., Szemberg, T.: On higher order embeddings of Calabi–Yau threefolds. Arch. Math. **74**, 221–225 (2000)

[10] Bagnera, G., de Franchis, M.: Sur les surfaces hyperelliptiques. C. R. Acad. Sci. **145**, 747–749 (1907)

[11] Chintapalli, S., Iyer, J.: Embedding theorems on hyperelliptic varieties. Geom. Dedic. **171**(1), 249–264 (2014)

[12] Di Rocco, S.: Generation of k-jets on toric varieties. Math. Z. **231**, 169–188 (1999)

[13] Enriques, F., Severi, F.: Mémoire sur les surfaces hyperelliptiques. Acta Math. **32**, 283–392 (1909)

[14] Enriques, F., Severi, F.: Mémoire sur les surfaces hyperelliptiques. Acta Math. **33**, 321–403 (1910)

[15] Farnik, Ł.: A note on Seshadri constants of line bundles on hyperelliptic surfaces. Arch. Math. (2016). doi:10.1007/s00013-016-0938-7

[16] Griffiths, P., Harris, J.: Principles of Algebraic Geometry. Wiley, New York (1978)

[17] Lazarsfeld, R.: Lectures on linear series, with the assistance of Guillermo Fernández del Busto. In: Kollár, J. (ed.) Complex Algebraic Geometry (Park City, UT, 1993), pp. 161–219. American Mathematical Society, Providence, RI (1997)

[18] Lazarsfeld, R.: Positivity in Algebraic Geometry I and II. Springer, Berlin, Heidelberg (2004)

[19] Mella, M., Palleschi, M.: The k-very ampleness on an elliptic quasi bundle. Abh. Math. Semin. Univ. Hambg. **63**, 215–226 (1993)

[20] Pareschi, G., Popa, M.: Regularity on abelian varieties II: basic results on linear series and defining equations. J. Algebraic Geom. **13**, 167–193 (2004)

[21] Rams, S., Szemberg, T.: Simultaneous generation of jets on K3 surfaces. Arch. Math. **83**, 353–359 (2004)

[22] Serrano, F.: Divisors of bielliptic surfaces and embeddings in $\mathbb{P}^4$. Math. Z. **203**, 527–533 (1990)

Łucja Farnik
Faculty of Mathematics and Computer Science
Jagiellonian University
Łojasiewicza 6
30-348 Kraków
Poland
e-mail: lucja.farnik@gmail.com

Received: October 30, 2015.
Revised: July 11, 2016.
Accepted: July 18, 2016.