



Sharp L^p Bound for Holomorphic Functions on the Unit Disc

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Abstract. For any $1 < p < \infty$ and any $X, Y \in \mathbb{R}$ satisfying $|X| \leq Y$, we determine the optimal constant $C_p(X, Y)$ such that the following holds. If F is a holomorphic function on the unit disc satisfying $\operatorname{Re} F(0) = X$ and $\|\operatorname{Re} F\|_{L^p(\mathbb{T})} = Y$, then

$$\|F\|_{L^p(\mathbb{T})} \geq C_p(X, Y).$$

This can be regarded as a reverse version of the classical estimates of Riesz and Essén. The proof rests on the exploitation of certain families of special subharmonic functions on the plane.

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1. Introduction

Let u, v be two harmonic functions on the unit disc \mathbb{D} , satisfying Cauchy–Riemann equations and normalized so that $v(0) = 0$. A classical problem, which interested many mathematicians at the beginning of the previous century, is the following: How is the size of v controlled by the size of u ? Here, the size of a function is measured, for instance, by the L^p norm on the unit circle \mathbb{T} equipped with the normalized Haar measure m . In other words, for which $1 \leq p \leq \infty$ is there a finite constant C_p , depending only on p , such that

$$\|v\|_{L^p(\mathbb{T})} \leq C_p \|u\|_{L^p(\mathbb{T})} \tag{1.1}$$

for all u, v as above? This question was answered by Riesz [17, 18]: the above estimate holds with some $C_p < \infty$ if and only if $1 < p < \infty$. This result is of fundamental importance to harmonic and complex analysis and has been modified and extended in numerous directions (cf. [2, 4–6, 9, 11, 23] to name just a few). Moreover, the methods presented in the works of Riesz have led to the development of many areas of research (e.g., interpolation

theory, functional analysis) and have had a profound influence on the shape of contemporary mathematics.

The question about the best value of C_p has gained some interest in the literature. It was answered partially by Gohberg and Krupnik [8]: by the use of a clever inductive argument, it was shown there that $C_p = \cot(\pi/(2p))$ if $p = 2, 4, 8, \dots$. The identification of C_p in the full range $p \in (1, \infty)$ is due to Pichorides [16] and Cole (unpublished: see Gamelin [7]): we have $C_p = \cot(\pi/(2p^*))$, where $p^* = \max\{p, p/(p-1)\}$. There are several papers which treat related sharp estimates for conjugate harmonic functions on the disc; see e.g., Aarão and O'Neill [1], Davis [5], Janakiraman [10], Nazarov and Treil [14], Osękowski [15], and consult references therein.

One can look at the estimate (1.1) from a slightly different perspective. Obviously, u can be regarded as the real part of the holomorphic function $u+iv$ on the unit disc. Consequently, by the triangle inequality, the inequality (1.1) is equivalent to the following statement: if F is a holomorphic function on \mathbb{D} satisfying the normalization condition $\text{Im } F(0) = 0$, then

$$\|F\|_{L^p(\mathbb{T})} \leq E_p \|\text{Re } F\|_{L^p(\mathbb{T})}, \quad 1 < p < \infty, \quad (1.2)$$

for some finite E_p depending only on p . Actually, as Essén [6] proved, the choice $E_p = \sin^{-1}(\pi/(2p^*))$ is optimal. See also Tomaszewski [21] for the sharp weak-type counterpart of this estimate.

The purpose of this paper is to study a certain reverse version of (1.2). Clearly, if F is a holomorphic function on the unit disc (with no additional assumptions on $\text{Im } F(0)$), then we have

$$\|F\|_{L^p(\mathbb{T})} \geq \|\text{Re } F\|_{L^p(\mathbb{T})}. \quad (1.3)$$

Of course, this bound is sharp: equality holds for constant real functions. However, one can study the following more sophisticated version of this problem: namely, find the sharp analog of (1.3) subject to the restrictions

$$\text{Re } F(0) = X \quad \text{and} \quad \|\text{Re } F\|_{L^p(\mathbb{T})} = Y. \quad (1.4)$$

Clearly, the answer to this question provides us with more detailed information on the behavior of the operator $F \mapsto \text{Re } F$. Such a type of problems appears in many places in the literature, in the study of other classical operators and objects in harmonic analysis. See e.g., Melas [12], Melas et al. [13] and Slavin et al. [19] for related problems concerning the dyadic maximal operators; consult Burkholder [3] for related results for martingale transforms and the Haar system on $[0, 1]$; Vasyunin [22] studied similar questions for A_p -weights on the real line; Slavin and Vasyunin [20] investigated similar problems for BMO functions on \mathbb{R} ; and more.

Coming back to (1.3) and the restriction (1.4), we easily see that if $Y \neq |X|$, then the lower bound for $\|F\|_{L^p(\mathbb{T})}$ can be improved. For instance, suppose that $p = 2$ and put $u = \text{Re } F$ and $v = \text{Im } F$. Since u, v satisfy Cauchy–Riemann equations, we may write

$$\begin{aligned}
 \|F\|_{L^2(\mathbb{T})} &= \left(\|u\|_{L^2(\mathbb{T})}^2 + \|v\|_{L^2(\mathbb{T})}^2 \right)^{1/2} \\
 &= \left(\|u\|_{L^2(\mathbb{T})}^2 + \|v - v(0)\|_{L^2(\mathbb{T})}^2 + |v(0)|^2 \right)^{1/2} \\
 &= \left(\|u\|_{L^2(\mathbb{T})}^2 + \|u - u(0)\|_{L^2(\mathbb{T})}^2 + |v(0)|^2 \right)^{1/2} \\
 &= \left(2\|u\|_{L^2(\mathbb{T})}^2 - |u(0)|^2 + |v(0)|^2 \right)^{1/2} \geq (2Y^2 - X^2)^{1/2}
 \end{aligned}$$

and equality can hold, for instance, if we take $F(z) = z\sqrt{2Y^2 - 2X^2} + X$. What about other values of p ? This question is answered in Theorem 1.1 to formulate which we will need some auxiliary notation. Let X, Y be two real numbers satisfying $|X| \leq Y$. For $1 < p \leq 2$, define

$$C_p(X, Y) = \left[\sin^{-p} \left(\frac{\pi}{2p} \right) (Y^p - |X|^p) + |X|^p \right]^{1/p},$$

while for $2 < p < \infty$, let

$$C_p(X, Y) = \frac{Y}{\cos \phi_p},$$

where ϕ_p is the unique number $\phi \in [0, \pi/(2p))$, satisfying

$$\left(\frac{|X|}{Y} \right)^p = \frac{\cos(p\phi)}{\cos^p \phi}. \tag{1.5}$$

The existence and uniqueness of ϕ_p follow from the fact that the right-hand side of (1.5) is a continuous and strictly decreasing function of ϕ , which takes value 1 at $\phi = 0$ and converges to 0 as $\phi \rightarrow \pi/(2p)$.

We are ready to state our main result.

Theorem 1.1. *Let $1 < p < \infty$. Then for any holomorphic function $F : \mathbb{D} \rightarrow \mathbb{C}$, satisfying $\operatorname{Re} F(0) = X$ and $\| \operatorname{Re} F \|_{L^p(\mathbb{T})} = Y$, we have*

$$\|F\|_{L^p(\mathbb{T})} \geq C_p(X, Y). \tag{1.6}$$

For each p, X and Y , the number $C_p(X, Y)$ cannot be replaced by a smaller number.

The proof of this statement rests on the existence of certain families of subharmonic functions on the plane. In the next section, we study the case $1 < p \leq 2$ of the above theorem, and Sect. 2 of the paper is devoted to the case $2 < p < \infty$. In the final part of the paper, we sketch some ideas which lead to the discovery of special functions used in Sects. 2 and 3.

2. The Case $1 < p \leq 2$

As we have mentioned above, the proof will rest on the existence of certain special subharmonic functions. Introduce $U_p : [0, \infty)^2 \rightarrow \mathbb{R}$ by

$$U_p(x, y) = \begin{cases} R^p - \sin^{-p} \left(\frac{\pi}{2p} \right) |x|^p & \text{if } \theta \leq \frac{\pi}{2} - \frac{\pi}{2p}, \\ \cot \left(\frac{\pi}{2p} \right) R^p \cos \left(p \left(\theta - \frac{\pi}{2} \right) \right) & \text{if } \theta > \frac{\pi}{2} - \frac{\pi}{2p}. \end{cases}$$

Here $x = R \cos \theta$, $y = R \sin \theta$, where $R \geq 0$ and $\theta \in [0, \pi/2]$, stand for the polar coordinates. Let us extend U_p to the whole plane \mathbb{R}^2 by the requirement $U_p(x, y) = U_p(-x, y) = U_p(x, -y)$ for all $x, y \in \mathbb{R}$. One easily checks that the function U_p is continuous; further properties of U_p are gathered in a lemma below.

Lemma 2.1. *The function U_p enjoys the following properties:*

- (i) *We have $U_p(x, y) \geq U_p(x, 0)$ for all $x, y \in \mathbb{R}$.*
- (ii) *For all $x, y \in \mathbb{R}$ we have the majorization*

$$U_p(x, y) \leq R^p - \sin^{-p} \left(\frac{\pi}{2p} \right) |x|^p. \tag{2.1}$$

- (iii) *The function U_p is subharmonic.*

Proof. (i) By the symmetry of U_p , it is enough to prove that $U_{py}(x, y) \geq 0$ for $x, y \geq 0$. This estimate is evident if $\theta \leq \pi/2 - \pi/(2p)$. If $\theta > \pi/2 - \pi/(2p)$, then, using the identities $R_y = y/R$ and $\theta_y = x/R^2$, we compute that

$$U_{py}(x, y) = p \cot \left(\frac{\pi}{2p} \right) R^{p-1} \sin \left(\theta + p \left(\frac{\pi}{2} - \theta \right) \right).$$

This is positive, since $\theta + p(\pi/2 - \theta) \in (0, \pi)$.

- (ii) By the symmetry of U_p , we may restrict ourselves to nonnegative x and y . Furthermore, we may assume that $\theta \in [\pi/2 - \pi/(2p), \pi/2]$, since for the remaining θ 's both sides are equal. Under these additional assumptions, the majorization can be rewritten in the equivalent form

$$G(\theta) := \left[\cot \left(\frac{\pi}{2p} \right) \cos \left(p \left(\theta - \frac{\pi}{2} \right) \right) - 1 \right] \cos^{-p} \theta + \sin^{-p} \left(\frac{\pi}{2p} \right) \leq 0.$$

However, we have $G(\pi/2 - \pi/(2p)) = 0$ and, as we will show now, G is nonincreasing on $(\pi/2 - \pi/(2p), \pi/2)$. Since

$$G'(\theta) = p \cdot \frac{\cot(\pi/(2p)) \sin(\theta - p(\theta - \pi/2)) - \sin \theta}{\cos^{p+1} \theta},$$

the announced monotonicity of G is equivalent to saying that the numerator is nonpositive. After the substitution $\psi = \pi/2 - \theta$, the latter can be rewritten as:

$$\frac{\cos((p-1)\psi)}{\cos((p-1)\frac{\pi}{2p})} \leq \frac{\cos \psi}{\cos \frac{\pi}{2p}}, \quad \text{for } \psi \in (0, \pi/2p).$$

But recall that we work in the case $1 < p \leq 2$; therefore, it is enough to show that for any $0 \leq s \leq t \leq \pi/2$, the function

$$\xi(\alpha) = \frac{\cos(\alpha s)}{\cos(\alpha t)}, \quad \alpha \in [0, 1],$$

is nondecreasing. A direct differentiation shows that

$$\begin{aligned} \xi'(\alpha) &= \frac{-s \sin(\alpha s) \cos(\alpha t) + t \sin(\alpha t) \cos(\alpha s)}{\cos^2(\alpha t)} \\ &\geq \frac{-t \sin(\alpha s) \cos(\alpha t) + t \sin(\alpha t) \cos(\alpha s)}{\cos^2(\alpha t)} = \frac{t \sin(\alpha(t-s))}{\cos^2(\alpha t)} \geq 0. \end{aligned}$$

This proves $G' \leq 0$ on $(\pi/2 - \pi/(2p), \pi/2)$ and establishes the majorization (2.1).

- (iii) It is easy to check that U_p is of class C^1 and hence, by the symmetry, it is enough to verify the subharmonicity on $(0, \infty)^2$. Clearly, U_p is harmonic in the angle $\theta \in (\pi/2 - \pi/(2p), \pi/2)$. On the other hand, if $\theta \in (0, \pi/2 - \pi/(2p))$, we compute that

$$\begin{aligned} \Delta U_p(x, y) &= \left[\frac{\partial^2}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R} + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2} \right] U_p(x, y) \\ &= p(p-1)R^{p-2} \left[\frac{p}{p-1} - \sin^{-p} \left(\frac{\pi}{2p} \right) \cos^{p-2} \theta \right]. \end{aligned}$$

Since $1 < p \leq 2$, the expression in the square brackets is larger than

$$\frac{p}{p-1} - \sin^{-p} \left(\frac{\pi}{2p} \right) \cos^{p-2} \left(\frac{\pi}{2} - \frac{\pi}{2p} \right) = \frac{p}{p-1} - \sin^{-2} \left(\frac{\pi}{2p} \right),$$

which is nonnegative. Indeed, after the substitution $r = 1/p \in [1/2, 1)$, we get

$$\sin^2 \left(\frac{\pi}{2p} \right) - \frac{p-1}{p} = \sin^2 \left(\frac{\pi r}{2} \right) - 1 + r,$$

which is zero for $r = 1/2$ and is an increasing function of $r \in (1/2, 1)$. □

Proof of (1.6). We are ready to establish the lower bound'. Let us fix a function F as in the statement and gather some information. First, since $\text{Re } F$ belongs to $L^p(\mathbb{T})$, the restriction $F|_{\mathbb{T}}$ also has this property, by Riesz' theorem. But, clearly, we have $|U_p(x, y)| \leq c_p R^p$ for all x, y and some c_p depending only on p , so the restriction of $U_p \circ F$ to the unit circle is integrable. Finally, observe that by the third part of the above lemma, the composition $U_p \circ F$ is a subharmonic function on the unit disc. Therefore, using the first two parts of the lemma and the mean-value property, we obtain

$$U_p(X, 0) \leq U_p(F(0)) \leq \int_{\mathbb{T}} U_p \circ F(u) dm(u) \leq \|F\|_p^p - \sin^p \left(\frac{\pi}{2p} \right) \|\text{Re } F\|_p^p.$$

This is equivalent to

$$\|F\|_p \geq \left(\sin^{-p} \left(\frac{\pi}{2p} \right) (Y^p - X^p) + X^p \right)^{1/p},$$

which is precisely the desired lower bound. □ □

Sharpness. Let X be an arbitrary real number and fix $\varepsilon > 0$. Pick $\varphi_0 \in (0, \pi/2 - \pi/(2p))$ and $M < 0$ such that the angle $A = \{(x, y) : x > 0, y \geq M + x \tan \varphi_0\}$ contains the point $(X, 0)$. Let $\mu_{\partial A}^{(X,0)}$ denote the harmonic measure on ∂A with respect to the point $(X, 0)$. Clearly, we have

$$\int_{\partial A} x d\mu_{\partial A}^{(X,0)} = X.$$

We will prove that if φ_0 and M are chosen appropriately, then

$$\int_{\partial A} |x|^p d\mu_{\partial A}^{(X,0)} = Y^p \tag{2.2}$$

and

$$\int_{\partial A} (|x|^2 + |y|^2)^{p/2} d\mu_{\partial A}^{(X,0)} < \sin^{-p} \left(\frac{\pi}{2p} \right) (Y^p - |X|^p) + |X|^p + \varepsilon. \tag{2.3}$$

This will yield the claim: if we take F to be the conformal map sending \mathbb{D} onto A and 0 onto $(X, 0)$, then we will have $\operatorname{Re} F(0) = X$ and, by (2.2) and (2.3),

$$\|\operatorname{Re} F\|_{L^p(\mathbb{T})} = Y \quad \text{and} \quad \|F\|_{L^p(\mathbb{T})} < (C_p(X, Y)^p + \varepsilon)^{1/p},$$

so the sharpness will hold due to the fact that ε is arbitrary.

By symmetry and continuity, we may and do assume that $X > 0$. Let us start with (2.2). The left-hand side does not change if we translate the angle A and the point $(X, 0)$ by the vector $(0, -M)$. For the translated angle, the analysis of the harmonic measure is simpler: the function

$$H(x, y) = \frac{\cos^p \varphi_0}{\cos(p(\varphi_0 - \pi/2))} R^p \cos(p(\varphi - \pi/2))$$

is harmonic on $A + (0, -M)$ and equals $(x, y) \mapsto |x|^p$ on the boundary of this set. Hence

$$\begin{aligned} \int_{\partial A} |x|^p d\mu_A^{(X,0)} &= \int_{\partial A} |x|^p d\mu_{A+(0,-M)}^{(X,-M)} \\ &= H(X, -M) \\ &= (-M)^p \cdot \frac{\cos^p \varphi_0}{\cos(p(\varphi_0 - \pi/2))} \frac{\cos(p(\psi - \pi/2))}{\sin^p \psi}, \end{aligned}$$

where ψ is the angle corresponding to the point $(X, -M)$ (that is, $\psi = \arctan(-M/X)$). So, we can rewrite the above identity in the equivalent form

$$\begin{aligned} \int_{\partial A} |x|^p d\mu_A^{(X,0)} &= \frac{\cos^p \varphi_0}{\cos(p(\varphi_0 - \pi/2))} \\ &\times (M^2 + X^2)^{p/2} \cos(p(\arctan(-M/X) - \pi/2)). \end{aligned} \tag{2.4}$$

Now, suppose that φ_0 is close to $\pi/2 - \pi/(2p)$. The largest allowed value of M is $-X \tan \varphi_0$: then the point $(X, 0)$ lies at the boundary of A , so $\mu_A^{(X,0)} = \delta_{(X,0)}$ and hence $\int_{\partial A} |x|^p d\mu_A^{(X,0)} = X^p$. On the other hand, if we let $M \rightarrow -\infty$, then $\psi \rightarrow \pi/2$ and $\int_{\partial A} |x|^p d\mu_A^{(X,0)} \rightarrow \infty$. Finally, if we fix X and φ_0 , then the right-hand side of (2.4) is a strictly increasing function of M . Hence there is a unique number $M = M(\varphi_0)$ such that (2.2) holds. The crucial observation is that $M(\varphi)$ is bounded and $\psi \rightarrow \pi/2 - \pi/(2p)$ as $\varphi_0 \rightarrow \pi/2 - \pi/(2p)$. Indeed, when φ_0 approaches $\pi/2 - \pi/(2p)$, then we have

$$\frac{\cos^p \varphi_0}{\cos(p(\varphi_0 - \pi/2))} \rightarrow \infty,$$

so, if the last expression in (2.4) is equal to Y^p , the term $\cos(p(\psi - \pi/2))$ must tend to 0. This implies $\psi \rightarrow \pi/2 - \pi/(2p)$ and, in consequence, $M(\varphi_0) \rightarrow -X \cot(\pi/(2p))$. To deal with (2.3), note that

$$\begin{aligned} & \int_{\partial A} (|x|^2 + |y|^2)^{p/2} d\mu_{\partial A}^{(X,0)} \\ &= \int_{\partial A \cap \{|y| \leq M\}} (|x|^2 + |y|^2)^{p/2} d\mu_{\partial A}^{(X,0)} + \int_{\partial A \cap \{|y| \geq M\}} (|x|^2 + |y|^2)^{p/2} d\mu_{\partial A}^{(X,0)}. \end{aligned}$$

As $\varphi_0 \rightarrow \pi/2 - \pi/(2p)$, the first integral tends to X^p , since the measures $\mu_{\partial A}^{(X,0)}$ converge weakly to $\delta_{(X,0)}$. To deal with the second integral, observe that $|y| \leq |x| \tan \varphi_0$ when $(x, y) \in \partial A$ and $|y| \geq M$. Consequently

$$\begin{aligned} & \int_{\partial A \cap \{|y| \geq M\}} (|x|^2 + |y|^2)^{p/2} d\mu_{\partial A}^{(X,0)} \\ & \leq \cos^{-p} \varphi_0 \int_{\partial A \cap \{|y| \geq M\}} |x|^p d\mu_{\partial A}^{(X,0)} \\ & = \cos^{-p} \varphi_0 \left[\int_{\partial A} |x|^p d\mu_{\partial A}^{(X,0)} - \int_{\partial A \cap \{|y| < M\}} |x|^p d\mu_{\partial A}^{(X,0)} \right] \\ & \rightarrow \sin^{-p} \left(\frac{\pi}{2p} \right) [Y^p - X^p]. \end{aligned}$$

Combining all the above facts, we obtain

$$\liminf_{\varphi_0 \rightarrow \pi/2 - \pi/(2p)} \int_{\partial A} (|x|^2 + |y|^2)^{p/2} d\mu_{\partial A}^{(X,0)} = C_p(X, Y),$$

which yields (2.3). This completes the proof. □

3. The Case $2 \leq p < \infty$

Recall the number ϕ_p defined in (1.5) and introduce the parameters

$$c_p = \frac{\sin(p\phi_p)}{\cos^{p-1} \phi_p \sin((p-1)\phi_p)}$$

and

$$\alpha_p = -\frac{\sin \phi_p}{\sin((p-1)\phi_p)}.$$

Consider the function $U_p : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$, given by

$$U_p(x, y) = \begin{cases} \alpha_p R^p \cos(p\theta) & \text{if } |\theta| \leq \phi_p, \\ R^p - c_p x^p & \text{if } |\theta| > \phi_p, \end{cases}$$

where, as previously, we have used the polar coordinates. Let us extend U_p to the whole \mathbb{R}^2 , setting $U_p(x, y) = U_p(-x, -y)$ for all $x, y \in \mathbb{R}$. As in the previous section, first we study some elementary properties of this special function.

Lemma 3.1. (i) We have $U_p(x, y) \geq U_p(x, 0)$ for all $x, y \in \mathbb{R}$.

(ii) For all $x, y \in \mathbb{R}$ we have the majorization

$$U_p(x, y) \leq R^p - c_p|x|^p. \tag{3.1}$$

(iii) The function U_p is subharmonic.

Proof. (i) It suffices to show the inequality $\frac{\partial}{\partial y}U_p(x, y) \geq 0$ for $x, y > 0$.

This is evident if $\theta \geq \phi_p$, so let us assume that $\theta \in (0, \phi_p)$. A direct differentiation gives

$$\frac{\partial}{\partial y}U_p(x, y) = -p\alpha_p R^{p-1} \sin((p-1)\theta) > 0,$$

as needed.

(ii) By symmetry, we may assume that $x, y > 0$. Clearly, it suffices to verify the majorization for $\theta \in (0, \phi_p)$. We rewrite the bound in the equivalent form

$$G(\theta) = \frac{\alpha_p \cos p\theta - 1}{\cos^p \theta} \leq -c_p.$$

Both sides are equal when $\theta = \phi_p$, so it is enough to prove that G is nondecreasing. We derive that $G'(\theta)$ equals

$$\begin{aligned} & \frac{p}{\cos^{p+1} \theta} [-\alpha_p \sin((p-1)\theta) - \sin \theta] \\ &= \frac{p \sin \phi_p}{\cos^{p+1} \theta} \left[\frac{\sin((p-1)\theta)}{\sin((p-1)\phi_p)} - \frac{\sin \theta}{\sin \phi_p} \right]. \end{aligned}$$

Since $p-1 \geq 1$ and $(p-1)\theta \leq (p-1)\phi_p \leq \pi/2$, we will be done if we prove that for any fixed $0 < s < t$, the function

$$\xi(\alpha) = \frac{\sin \alpha s}{\sin \alpha t}$$

is nondecreasing on $(0, \pi/(2y))$. We compute that

$$\xi'(\alpha) = \frac{\sin \alpha s}{\alpha \sin \alpha t} [\alpha s \cot \alpha s - \alpha t \cot \alpha t]$$

and note that the function $\zeta(u) = u \cot u$ is decreasing on $(0, \pi/2)$: $\zeta'(u) = (2 \sin^2 u)^{-1}(\sin 2u - 2u) \leq 0$. This implies that $\xi' \leq 0$ and the majorization follows.

(iii) The function U_p is of class C^1 on the plane, and it is harmonic on the set $\{|\theta| < \phi_p\}$. Consequently, it suffices to check that the Laplacian of U_p is nonnegative on $\{|\theta| > \phi_p\}$. We derive that on this set, we have

$$\begin{aligned} \Delta U_p(x, y) &= p(p-1)R^{p-2} \left[\frac{p}{p-1} - c_p \cos^{p-2} \theta \right] \\ &\geq p(p-1)R^{p-2} \left[\frac{p}{p-1} - c_p \cos^{p-2} \phi_p \right] \\ &= p(p-1)R^{p-2} \left[\frac{p}{p-1} - \frac{\sin(p\phi_p)}{\cos \phi_p \sin((p-1)\phi_p)} \right]. \end{aligned}$$

Note that

$$\lim_{r \rightarrow 0} \left[\frac{p}{p-1} - \frac{\sin(pr)}{\cos r \sin((p-1)r)} \right] = 0.$$

Therefore, we will be done if we show that the function

$$\xi(r) = \frac{\sin(pr)}{\cos r \sin((p-1)r)}, \quad r \in (0, \pi/(2p)),$$

is nonincreasing. A direct differentiation shows that

$$\xi'(r) = \frac{\sin [2(p-1)r] - (p-1) \sin 2r}{2 \cos^2 r \sin^2((p-1)r)}.$$

If we denote the numerator by $\zeta(r)$, we see that $\zeta(0) = 0$ and

$$\zeta'(r) = 2(p-1) \{ \cos [2(p-1)r] - \cos 2r \} \leq 0 \quad \text{for } r \in (0, \pi/(2p)),$$

since the cosine function is decreasing on $(0, (p-1)\pi/p)$. Thus, ζ is nonpositive on $(0, \pi/(2p))$ and hence ξ is nonincreasing. This proves the claim. □

Equipped with the above lemma, we turn our attention to Theorem 1.1.

Proof. Proof of (1.6) The reasoning is the same as in the case $1 < p \leq 2$: we obtain

$$U_p(x, 0) \leq U_p(F(0)) \leq \int_{\mathbb{T}} U_p \circ F(u) dm(u) \leq \|F\|_p^p - c_p \| \operatorname{Re} F \|_p^p,$$

or, equivalently,

$$\|F\|_p^p \geq c_p Y^p + \alpha_p |X|^p.$$

Now, by the definition of c_p, α_p and the identity (1.5), the latter estimate is equivalent to

$$\|F\|_p \geq \frac{Y}{\cos \phi_p} = C_p(X, Y),$$

which is the claim. □

Sharpness. Here the reasoning is a bit simpler than in the case $1 < p \leq 2$. Consider the angle $A = \{(x, y) : x > 0, |\theta| \leq \phi_p\}$ and let μ be the harmonic measure on ∂A with respect to the point $(X, 0)$. The restriction of the function U_p to the set A is harmonic and $U_p(x, y) = x^p(\cos^{-p} \phi_p - c_p)$ for $(x, y) \in \partial A$. Consequently, by the mean-value property, we see that

$$\int_{\partial A} x d\mu = X \quad \text{and} \quad \int_{\partial A} x^p d\mu = \frac{U_p(X, 0)}{\cos^{-p} \phi_p - c_p} = \frac{\alpha_p X^p}{\cos^{-p} \phi_p - c_p} = Y^p.$$

Consequently,

$$\int_{\partial A} (x^2 + y^2)^{p/2} d\mu = \int_{\partial A} (x^2 + x^2 \tan^2 \phi_p)^{p/2} d\mu = \frac{Y^p}{\cos^p \phi_p}.$$

Therefore, if F is the univalent mapping which sends \mathbb{D} onto A and 0 onto $(X, 0)$, then $\operatorname{Re} F(0) = X$, $\|\operatorname{Re} F\|_{L^p(\mathbb{T})} = Y$ and

$$\|F\|_{L^p(\mathbb{T})} = \left(\int_{\partial A} (x^2 + y^2)^{p/2} d\mu \right)^{1/p} = \frac{Y}{\cos \phi_p}.$$

So, the lower bound (1.6) is attained and the proof is complete.

4. On the Search of the Special Functions

In this section, we will explain briefly some informal argumentation which leads to the discovery of the special functions U_p used above. We will focus on the case $2 \leq p < \infty$, for $1 < p < 2$ the reasoning is similar. For the sake of clarity, let us start with the general idea behind the proof of Theorem 1.1. Given $2 \leq p < \infty$ and a constant $\beta > 0$, one searches for the optimal (i.e., the largest) constant $\gamma(p, \beta)$ such that

$$\|F\|_{L^p(\mathbb{T})}^p \geq \beta \|\operatorname{Re} F\|_{L^p(\mathbb{T})}^p + \gamma(p, \beta) |\operatorname{Re} F(0)|^p \tag{4.1}$$

for all holomorphic functions F on the unit disc. This clearly gives some initial insight into (1.6): having analyzed (4.1), we see that

$$\|F\|_{L^p(\mathbb{T})} \geq \sup \left\{ (\beta Y^p + \gamma(p, \beta) |X|^p)^{1/p} : \beta > 0 \right\}. \tag{4.2}$$

Is this bound optimal? To answer this question, *suppose* that for each p and β , there is an extremizer: a nonzero function $F = F^{p,\beta}$ for which both sides are equal. Clearly, for any p and β such an object is not unique: the inequality (4.1) is homogeneous of order p , so if we multiply an extremizer by a constant, we again obtain an extremizer. It is evident how to proceed: we take the number β for which the supremum in (4.2) is attained, consider the extremizer of (4.1), scaled so that $\operatorname{Re} F^{p,\beta}(0) = X$, and verify that it satisfies $\|\operatorname{Re} F\|_p = Y$. So, we see that the problem boils down to a thorough analysis of the inequality (4.1).

Next, the reasoning presented in the papers [2, 6, 16] links the validity of the estimate (4.1) with the existence of special functions on the plane. Namely, given p and β , we search for a largest subharmonic function $U_{p,\beta}$ on \mathbb{R}^2 , satisfying the majorization

$$U_{p,\beta}(x, y) \leq (y^2 + x^2)^{p/2} - \beta |x|^p. \tag{4.3}$$

Then, as we have already seen in the previous sections, the mean-value property yields:

$$\|F\|_{L^p(\mathbb{T})}^p \geq \beta \|\operatorname{Re} F\|_{L^p(\mathbb{T})}^p + U_{p,\beta}(F(0)).$$

Hence, one is forced to take $\gamma(p, \beta) = \inf_{x,y} U_{p,\beta}(x, y) / |x|^p$.

How to find $U_{p,\beta}$? Such a function, if it exists, must satisfy the symmetry condition $U_{p,\beta}(x, y) = U_{p,\beta}(x, -y) = U_{p,\beta}(-x, y)$ for all $x, y \in \mathbb{R}$; indeed, if this did not hold, we could replace $U_{p,\beta}$ with a larger subharmonic function

$$(x, y) \mapsto \max \{U_{p,\beta}(x, y), U_{p,\beta}(-x, y), U_{p,\beta}(x, -y), U_{p,\beta}(-x, -y)\},$$

for which the majorization (4.3) is still valid. A similar argument shows that $U_{p,\beta}$ must be homogeneous of order p : otherwise $U_{p,\beta}$ would be strictly majorized by the subharmonic function $\sup_{\lambda>0} \lambda^p U_{p,\beta}(\cdot/\lambda, \cdot/\lambda)$. So, $U_{p,\beta}$ can be written in polar coordinates as:

$$U_{p,\beta}(x, y) = R^p g_{p,\beta}(\theta),$$

for some function $g_{p,\beta}$ to be found. Now, we assume that $U_{p,\beta}$ is of class C^2 ; despite the fact that the function we obtain at the end does not have this regularity, it will facilitate our further considerations. A closer look at the papers [2, 6, 16] suggests that there is a number $\kappa(p, \beta) > 0$ such that either

- (i) $U_{p,\beta}$ is harmonic on the set $\{(x, y) : |y| < \kappa(p, \beta)|x|\}$, and $U_{p,\beta}(x, y) = (y^2 + x^2)^{p/2} - \beta|x|^p$ on $\{(x, y) : |y| \geq \kappa(p, \beta)|x|\}$, or
- (ii) $U_{p,\beta}$ is harmonic on the set $\{(x, y) : |x| < \kappa(p, \beta)|y|\}$, and $U_{p,\beta}(x, y) = (y^2 + x^2)^{p/2} - \beta|x|^p$ on $\{(x, y) : |x| \geq \kappa(p, \beta)|y|\}$.

So, we have two possibilities to check. Suppose that (i) holds true and takes $\phi(p, \beta) \in (0, \pi/2)$ such that $\kappa(p, \beta) = \tan \phi(p, \beta)$. A direct calculation shows that $\Delta U_{p,\beta} = 0$ if and only if $g''(\theta) + p^2 g(\theta) = 0$, so we see that

$$g(\theta) = a_1 \cos(p\theta) + a_2 \sin(p\theta), \quad \theta \in [-\phi(p, \beta), \phi(p, \beta)],$$

for some unknown constants a_1, a_2 . Since $U_{p,\beta}(x, y) = U_{p,\beta}(x, -y)$, we conclude that $a_2 = 0$. To derive a_1 , we use the fact that $U_{p,\beta}$ is of class C^1 and see what happens on the common boundary of the sets $\{(x, y) : |y| > \kappa(p, \beta)|x|\}$ and $\{(x, y) : |y| \leq \kappa(p, \beta)|x|\}$. By the continuity of $U_{p,\beta}$, we get that

$$a_1 \cos(p\phi(p, \beta)) = 1 - \beta \cos^p \phi(p, \beta),$$

while the comparison of the partial derivatives yields

$$-a_1 \sin(p\phi(p, \beta)) = \beta \cos^{p-1} \phi(p, \beta) \sin \phi(p, \beta).$$

The system of these two equations can be easily solved: we get

$$\beta = \frac{\sin(p\phi(p, \beta))}{\cos^{p-1} \phi(p, \beta) \sin((p-1)\phi(p, \beta))}$$

and

$$a_1 = -\frac{\sin \phi(p, \beta)}{\sin((p-1)\phi(p, \beta))}.$$

This suggests to use the number $\phi = \phi(p, \beta) \in (0, \pi/2)$ as a “free” parameter (instead of β). We obtain the candidate for the special function $U_{p,\beta}$; it can be checked, with similar calculations as those presented in Sect. 3, that it enjoys all the required properties; furthermore

$$\gamma(p, \beta) = \inf_{x,y>0} \frac{U_{p,\beta}(x, y)}{|x|^p} = a_1 = -\frac{\sin \phi(p, \beta)}{\sin((p-1)\phi(p, \beta))}.$$

It remains to check that the right-hand side of (4.2) is precisely the constant $C_p(X, Y)$; when we take $\phi(p, \beta) = \phi_p$ (defined in (1.5)), then $U_{p,\beta}$ is precisely the function U_p used above.

Finally, let us say a few words about the search for the appropriate extremizers F in (4.1). A look at the above proof immediately gives three conditions on F . First, we must have equality in the majorization (4.3), i.e.,

$$U_{p,\beta}(F(e^{i\theta})) = |F(e^{i\theta})|^p - \beta |\operatorname{Re} F(e^{i\theta})|^p$$

for almost all $\theta \in [-\pi, \pi)$. That is, F must send the unit circle into the set $\{(x, y) : |x| \geq \kappa(p, \beta)|y|\}$. The second condition is that the mean value property, applied to the subharmonic function $U_{p,\beta} \circ F$, must return equality: this suggests that F must send the open unit disc into $\{(x, y) : x > 0, |y| < \kappa(p, \beta)|x|\}$, inside which $U_{p,\beta}$ is harmonic. Finally, we must have $F(0) = X$; this will guarantee the equality $U_{p,\beta}(F(0)) = \gamma(p, \beta)|X|^p$. Let us combine the three observations: we see that a natural choice for F is a conformal mapping of \mathbb{D} onto the angle $A = \{(x, y) : x > 0, |y| \leq \kappa(p, \beta)x\}$, sending $0 \in \mathbb{D}$ onto $(X, 0) \in A$. One can check that this is indeed the right choice.

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