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# On Solutions of Quadratic Integral Equations in Orlicz Spaces

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**Abstract.** In this paper we study the quadratic integral equation of the form

$$x(t) = g(t) + \lambda \cdot G(x)(t) \cdot \int_a^b K(t,s)f(s,x(s))\mathrm{d}s.$$

We discuss the existence of solutions for the above equation in different function spaces. We stress on the case when $f$ has non-polynomial growth which leads to solutions in Orlicz spaces. The detailed theory for a wide class of spaces is presented. Some existence theorems for a.e. monotonic solutions in Orlicz spaces are proved either for strongly nonlinear functions $f$ or for rapidly growing kernel $K$. The presented method allows us to extend the current results as well as to unify the proofs for both quadratic and non-quadratic cases.

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**Keywords.** Quadratic integral equation, monotonic solutions, Orlicz spaces, superposition operators.

## 1. Introduction

The paper is devoted to study the following quadratic integral equation

$$x(t) \;=\; g(t) + G(x)(t) \cdot \lambda \int_a^b K(t,s)f(s,x(s))\mathrm{d}s. \tag{1.1}$$

Since the quadratic problems are related with the pointwise product of two operators, it is usually solved in the space of continuous functions or in a context of Banach algebras of continuous functions. This approach has some disadvantages. Firstly, for classical equations ($G(x) = \text{const.}$) discontinuous solutions are frequently considered and such solutions are applicable. Secondly, for some quadratic problems like the Chandrasekhar equation [4,5] discontinuous solutions are expected (see some comments in [13]), then continuous solutions seem to be inadequate for integral problems and lead to several restrictions on considered functions. Moreover, we allow to consider problems in which either the growth of the function $f$ or the kernel $K$ is not

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polynomial. An operator $G$ is supposed to be continuous on a required space of solutions. The problem is modelled on some quadratic integral equations (for which $G$ is identity operator or the Nemytskii superposition operator, we are not restricted only to this case). Thus we have one more motivation: our approach allows to include also classical integral equations. For this class of integral equations there are different types of considered equations. Usually this implies some restriction for the growth of $f$ and $K$. Recall that similar investigations for quadratic integral equations relate mainly to continuous solutions. The key point is to ensure that an operator of pointwise multiplication is well defined and has some compactness properties. We prefer an approach to this problem allowing us to consider a wide class of integral equations with solutions in some spaces of discontinuous functions (growth conditions are relaxed).

Let us briefly recall a historical background and some motivations. We prefer a method, which allows us to unify classical and quadratic integral equations and to consider the same classes of solutions. We started such an approach in [13], but it was done only in that case of Banach–Orlicz algebras or in case when an intermediate space (described later) is $L^\infty$ ([12]). Here we show a detailed theory and omit such restrictions.

The new starting point is to consider some integral problems with exponential nonlinarities (see [10]) or with rapidly growing kernels ([11], for instance). In such a case the Nemytskii operator should be considered as acting on some Orlicz spaces ([11]). In our opinion, it is very convenient and we will try to keep this class of solutions for quadratic equations. This suggest an operator-oriented approach. In a class of Orlicz spaces we consider spaces associated with a kind of assumed growth for $G$ and $f$.

For a moment denote by $X$ an Orlicz space of solutions for our problem and by $F$ the Nemytskii superposition operator generated by $f$. Thus, we have $G : X \to W_1$, $F : X \to U$ and finally the linear integral operator $H$ with a kernel $K$ is acting from $U$ into $W_2$. The space $U$ is depending on some growth assumptions of $f$—not necessarily of polynomial type. In a typical case of quadratic problems the spaces $W_1$ and $W_2$ are supposed both to be the space of continuous functions and then some properties of this Banach algebra allow to solve the problem. Unfortunately, this is really restrictive assumption. We started to replace this assumption by considering $X = L^1(I)$ and $W_2 = L^\infty(I)$ ([12]). Then, we considered some Banach–Orlicz algebras ([13], but such spaces are "small" so this still leads to some restrictions on $H$.

Here we present a complete theory for such problems. In general, allowing $U$ be an Orlicz space we consider the triple of Orlicz spaces (not necessarily Banach algebras) for which the pointwise multiplication takes a pair of functions from $W_1$ and $W_2$ into $X$. For a given $U$ a correct choice of $W_1$ and $W_2$ allows us to control both acting and growth conditions for considered operators. As will be clarified later some additional assumptions on $U$ allow us to prove their boundedness and continuity. To show a detailed theory we need to consider a few different cases (for different classes of Orlicz spaces).

Since for equations of this type an approach via the Schauder fixed point theorem is not useful and the Banach contraction principle is too restrictive in many applications, we prefer to investigate the properties of operators with respect to the topology of convergence in measure. We investigate some properties of this topology on considered Orlicz spaces and then use the Darbo fixed-point theorem for proving main results.

## 2. Notation and Auxiliary Facts

Let $\mathbb{R}$ be the field of real numbers. In this paper by $I$ we will denote a compact interval $[a, b] \subset \mathbb{R}$. Assume that $(E, \|\cdot\|)$ is an arbitrary Banach space with zero element $\theta$. Denote by $B_r(x)$ the closed ball centered at $x$ and with radius $r$. The symbol $B_r$ stands for the ball $B(\theta, r)$. When necessary we will also indicate the space using the notation $B_r(E)$.

Let $S = S(I)$ denotes the set of measurable (in Lebesgue sense) functions on $I$ and let meas stands for the Lebesgue measure in $\mathbb{R}$. Identifying the functions equals almost everywhere the set $S$ becomes a complete metric space. Note that the topology of convergence in measure on $I$ is metrizable (cf. Proposition 2.14 in [14]). The compactness in such spaces we will call a "compactness in measure" and such sets have important properties when considered as subsets of some Orlicz spaces.

To make the paper self-contained we need to recall some basic notions and facts in the theory of Orlicz spaces.

Let $M$ and $N$ be complementary $N$-functions, i.e. $N(x) = \sup_{y\geq 0}(xy - M(x))$, where $N : [0, +\infty) \to [0, +\infty)$ is continuous, even and convex with $\lim_{x\to 0} \frac{N(x)}{x} = 0$, $\lim_{x\to\infty} \frac{N(x)}{x} = \infty$ and $N(x) > 0$ if $x > 0$ ($N(u) = 0 \iff u = 0$). The Orlicz class, denoted by $\mathcal{O}_M$, consists of measurable functions $x : I \to \mathbb{R}$ for which $\rho(x; M) = \int_I M(x(t))\mathrm{d}t < \infty$. We shall denote by $L_M(I)$ the Orlicz space of all measurable functions $x : I \to \mathbb{R}$ for which $\|x\|_M = \inf_{\lambda>0} \left\{\int_I M\left(\frac{x(s)}{\lambda}\right) \mathrm{d}s \leq 1\right\}$. Let $E_M(I)$ be the closure in $L_M(I)$ of the set of all bounded functions. Note that $E_M \subseteq L_M \subseteq \mathcal{O}_M$.

For Orlicz spaces we have different situations than for Lebesgue ones. The inclusion $L_M \subset L_P$ holds if, and only if, there exist positive constants $u_0$ and $a$ such that $P(u) \leq aM(u)$ for $u \geq u_0$.

An important property of $E_M$ spaces lies in the fact that this is a class of functions from $L_M$ having absolutely continuous norms. Moreover, we have $E_M = L_M = \mathcal{O}_M$ if $M$ satisfies the $\Delta_2$-condition, i.e. there exists $\omega,\ t_0 \geq 0$ such that for $t \geq t_0$, we have $M(2t) \leq \omega M(t)$.

An $N$-function $M$ is said to satisfy $\Delta'$-condition if there exists $K,\ t_0 \geq 0$ such that for $t, s \geq t_0$, we have $M(ts) \leq KM(t)M(s)$. If the $N$-function $M$ satisfies the $\Delta'$-condition, then it also satisfies $\Delta_2$-condition.

The last important class of $N$-functions consists of functions which increase more rapidly than power functions. An $N$-function $M$ is said to satisfy $\Delta_3$-condition if there exists $K,\ t_0 \geq 0$ such that for $t \geq t_0$, we have $tM(t) \leq M(Kt)$.

Sometimes, we will use more general concept of function spaces, i.e. ideal spaces. A normed space $(X, \|\cdot\|)$ of (classes of) measurable functions $x : I \to U$ ($U$ is a normed space) is called pre-ideal if for each $x \in X$ and each measurable $y : I \to U$ the relation $|y(s)| \leq |x(s)|$ (for almost all $s \in I$) implies $y \in X$ and $\|y\| \leq \|x\|$. If $X$ is also complete, it is called an ideal space (see [15]). The class of Orlicz spaces stands for an important (but not unique) example of ideal spaces. If possible, we will describe our results in terms of ideal spaces. This will indicate the possible extensions for our results.

## 3. Nonlinear Operators

In our paper we propose to reduce the considered problem to the operator form. In particular, we will investigate many properties of operators acting on different function spaces.

One of the most important operator studied in nonlinear functional analysis is the so-called superposition (or : Nemytskii) operator [16]. Assume that a function $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions, i.e. it is measurable in $t$ for any $x \in \mathbb{R}$ and continuous in $x$ for almost all $t \in I$. Then to every function $x(t)$ being measurable on $I$ we may assign the function $F(x)(t) = f(t, x(t))$, for $t \in I$. The operator $F$ in such a way is called the superposition operator generated by the function $f$. We will be interested in the case when $F$ acts between some Orlicz spaces.

A full discussion about necessary and sufficient conditions for continuity and boundedness of such a type of operators can be found in [16]. The following property will be used:

**Lemma 3.1.** *Assume that a function $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions. Then, the superposition operator $F$ transforms measurable functions into measurable functions.*

We will utilize the fact that Carathéodory mappings transforming measurable functions into the same space are sequentially continuous with respect to topology of convergence in (finite) measure.

**Lemma 3.2** ([17, Lemma 17.5] in $S$ and [18] in $L_M(I)$). *Assume that a function $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions. The superposition operator $F$ maps a sequence of functions convergent in measure into a sequences of functions convergent in measure.*

In our proofs we have to control the domain and the range of considered operators. The following lemma seems to be useful for the superposition operator:

**Lemma 3.3** ([11, Theorem 17.5]). *Assume that a function $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions. Then*

$$M_2(f(s, x)) \leq a(s) + bM_1(x),$$

*where $b \geq 0$ and $a \in L^1(I)$, if and only if the superposition operator $F$ acts from $L_{M_1}(I)$ to $L_{M_2}(I)$.*

In Orlicz spaces there is no automatic continuity of superposition operators like in $L^p$ spaces, but we have a useful result (remember, that the Orlicz space $L_M$ is ideal and if $M$ satisfies $\Delta_2$ condition it is also regular cf. [19, Theorem 1]):

**Lemma 3.4** ([15, Theorem 5.2.1]). *Let $f$ be a Carathéodory function, $X$ an ideal space, and $W$ a regular ideal space. Then the superposition operator $F : X \to W$ is continuous.*

Let us note, that in the case of functions of the form $f(t, x) = g(t)h(x)$, the superposition operator $F$ is continuous from the space of continuous functions $C(I)$ into $L_M(I)$ even when $M$ does not satisfy $\Delta_2$ condition ([19]). Since $E_M(I)$ is a regular part of an Orlicz space $L_M(I)$ (cf. [14, p.72]), in the context of Orlicz spaces, we will use the following (see also Lemma 3.3):

**Lemma 3.5.** *Let $f$ be a Carathéodory function. If the superposition operator $F$ acts from $L_{M_1}(I)$ into $E_{M_2}(I)$, then it is continuous.*

The problem of boundedness of such a type of operators in different classes of Orlicz spaces will be described in the proofs of our main results (cf. also [11] or [16]).

Two more operators will play an important role it this paper, namely the linear integral operator $H(x) = \lambda \int_a^b K(t, s)x(s)\mathrm{d}s$ and the pointwise multiplication operator. The first one is well known and all necessary results concerning the properties of such a kind of operators in Orlicz spaces can be found in [11], so here we omit the details and important results will be pointed out in the proofs of our main results.

Now, we need to describe the second one. By $U(x)(t)$ we will denote the operator of the form:

$$U(x)(t) = G(x)(t) \cdot A(x)(t),$$

where $A = H \circ F$ is a Hammerstein operator.

Generally speaking, the product of two functions $x, y \in L_M(I)$ is not in $L_M(I)$. However, if $x$ and $y$ belong to some particular Orlicz spaces, then the product $x \cdot y$ belongs to a third Orlicz space.

We need to consider the converse direction: if we are looking or solutions in a given Orlicz space $L_\varphi$, then we need to indicate the spaces $L_{\varphi_1}$ and $L_{\varphi_2}$ such that the product of functions from that spaces belong to $L_\varphi$. We are interested in finding that spaces as big as possible.

Let us note, that one can find two functions belonging to Orlicz spaces: $u \in L_U(I)$ and $v \in L_V(I)$ such that the product $uv$ does not belong to any Orlicz space (this product is not integrable). Nevertheless, we have:

**Lemma 3.6** ([11, Lemma 13.5]), [20, Theorem 10.2]. *Let $\varphi_1, \varphi_2$ and $\varphi$ are arbitrary $N$-functions. The following conditions are equivalent:*

1. *For every functions $u \in L_{\varphi_1}(I)$ and $w \in L_{\varphi_2}$, $u \cdot w \in L_\varphi(I)$.*
2. *There exists a constant $k > 0$ such that for all measurable $u, w$ on $I$ we have $\|uw\|_\varphi \leq k\|u\|_{\varphi_1}\|w\|_{\varphi_2}$.*
3. *There exists numbers $C > 0$, $u_0 \geq 0$ such that for all $s, t \geq u_0$ $\varphi\left(\frac{st}{C}\right) \leq \varphi_1(s) + \varphi_2(t)$.*
4. $\limsup_{t\to\infty} \frac{\varphi_1^{-1}(t)\varphi_2^{-1}(t)}{\varphi(t)} < \infty$.

Let us recall the following simple sufficient condition for the above statements hold true.

**Lemma 3.7** ([11, p. 223]). *If there exist complementary $N$-functions $Q_1$ and $Q_2$ such that the inequalities*

$$Q_1(\alpha u) < \varphi^{-1}[\varphi_1(u)], \quad Q_2(\alpha u) < \varphi^{-1}[\varphi_2(u)]$$

*are satisfied for large values of the argument and for certain constant $\alpha$, then for every functions $u \in L_{\varphi_1}(I)$ and $w \in L_{\varphi_2}$, $u \cdot w \in L_{\varphi}(I)$. If moreover $\varphi$ satisfies the $\Delta_2$-condition, then it is sufficient that the inequalities*

$$Q_1(\alpha u) < \varphi_1[\varphi^{-1}(u)], \quad Q_2(\alpha u) < \varphi_2[\varphi^{-1}(u)]$$

*hold.*

An interesting discussion about necessary and sufficient conditions for product operators can be found in [11,20]. We will consider the triple of spaces $(\varphi, \varphi_1, \varphi_2)$ satisfying the above property.

## 4. Monotone Functions

We are interested in finding of (almost everywhere) monotonic solutions for our problem. For the case of discontinuous functions we should describe this class of functions in considered solution spaces.

Let us recall, in metric spaces the set $U_0$ is compact if and only if each sequence from $U_0$ has a subsequence that converges in $U_0$ (i.e. sequentially compact). In particular, we need to use this simple fact in the space $S$.

Recall some consideration from [11] by adding some additional comments for Orlicz spaces. Let $X$ be a bounded subset of measurable functions. Assume that there is a family of subsets $(\Omega_c)_{0\leq c\leq b-a}$ of the interval $I$ such that meas$\Omega_c = c$ for every $c \in [0, b-a]$, and for every $x \in X$, $x(t_1) \geq x(t_2)$, $(t_1 \in \Omega_c, t_2 \notin \Omega_c)$.

It is clear, that by putting $\Omega_c = [0, c) \cup Z$ or $\Omega_c = [0, c)\setminus Z$, where $Z$ is a set with measure zero, this family contains nonincreasing functions (possibly except for a set $Z$). We will call the functions from this family "a.e. nonincreasing" functions. This is the case, when we choose a measurable and nonincreasing *function* $y$ and all functions equal a.e. to $y$ satisfy the above condition. This means that such a notion can be also considered in the space $S$. Thus, we can write that *elements* from $L_M(I)$ belong to this class of functions. Further, let $Q_r$ stands for the subset of the ball $B_r$ consisting of all functions which are a.e. nonincreasing on $I$. Functions a.e. nondecreasing are defined by a similar way.

It is known that such a family constitutes a set which is compact in measure in $S$ (cf. [17, section 19.8]). It is a little bit surprising that the proof of this property is not published anywhere. We are also interested, if the set is still compact in measure as a subset of some subspaces of $S$. In general, it is not true, but for the case of Orlicz spaces, we have the following:

**Lemma 4.1.** *Assume, that a bounded set $U$ is a subset of an Orlicz space $L_M(I)$ of real-valued functions over a bounded interval $I$ consisting only of a.e. monotonic functions. Then, this set is compact in measure in the space $L_M(I)$.*

*Proof.* Since the convergence in measure is a metric convergence, we need only to check sequential compactness. Without loss of generality, let us restrict to the case of a.e. increasing functions. We will follow some ideas from the Helly theorem. Take an arbitrary sequence $(x_n) \subset U$. Denote by $W$ the set such that the functions $x_n$ are increasing outside $W$ (a countable union of countable sets), i.e. on $J = I \setminus W$. Let $Z$ be countable dense subset of $J$. Since $\{x_n(t)\}$ is bounded on $Z$ (except perhaps the point $t = b$), by a diagonal procedure we can subtract a subsequence $(x_{n_k})$ of $(x_n)$ which is pointwise convergent (on $Z$).

Then, a limit $x = \lim_{k\to\infty} x_{n_k}$ is an increasing function on $Z$. It is known that this function can be extended to an increasing function $y$ defined on $J$ in such a way that $y$ is a limit of $(x_{n_k})$ on $Z$.

Let $t_0$ be an arbitrary internal point in $J$. Since $Z$ is dense in $J$ we are able to find two sequences $(s_n)$ and $(\tau_n)$ of points in $Z$ tending to $t_0$ such that $s_n < t_0 < \tau_n$. For any fixed $k \in \mathbb{N}$, $(x_{n_k})$ is increasing on $J$ and we have

$$x_{n_k}(s_n) < x_{n_k}(t_0) < x_{n_k}(\tau_n)$$

and passing to the limit with $k \to \infty$ we obtain

$$y(s_n) \leq \liminf_{k\to\infty} x_{n_k}(t_0) \leq \limsup_{k\to\infty} x_{n_k}(t_0) \leq y(\tau_n).$$

By passing to the limit with $n \to \infty$ we get

$$y(t_0-) \leq \liminf_{k\to\infty} x_{n_k}(t_0) \leq \limsup_{k\to\infty} x_{n_k}(t_0) \leq y(t_0+).$$

For any point of continuity of $y$ we have $y(t) = \lim_{k\to\infty} x_{n_k}(t)$. As the set $D$ of all points of discontinuity of this function is at most countable, $y$ is a.e. increasing.

Since the measure of $I$ is finite and $L_M \subset S$, the sequence $(x_n)$ contains an a.e. convergent subsequence. As its limit is a.e. finite, by the Lebesgue theorem the subsequence is also convergent in measure. Summarising: arbitrary sequence in $U$ contains a subsequence which is convergent in measure to some $y \in U$ and then this set is compact in measure. $\square$

We have also an important

**Lemma 4.2** (Lemma 4.2 in [21]). *Suppose the function $t \to f(t,x)$ is a.e. nondecreasing on a finite interval $I$ for each $x \in \mathbb{R}$ and the function $x \to f(t,x)$ is a.e. nondecreasing on $\mathbb{R}$ for any $t \in I$. Then, the superposition operator $F$ generated by $f$ transforms functions being a.e. nondecreasing on $I$ into functions having the same property.*

We will use the fact that the superposition operator takes the bounded sets compact in measure into the sets with the same property. Namely, we have

**Proposition 4.3.** *Assume that a function $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions and the function $t \to f(t,x)$ is a.e. nondecreasing on a finite interval $I$ for each $x \in \mathbb{R}$ and the function $x \to f(t,x)$ is a.e. nondecreasing on*

*$\mathbb{R}$ for any $t \in I$. Assume, that $F : L_M(I) \to E_M(I)$. Then, $F(V)$ is compact in measure for arbitrary bounded and compact in measure subset $V$ of $L_M(I)$.*

*Proof.* Let $V$ be a bounded and compact in measure subset of $L_M(I)$. By our assumption $F(V) \subset E_M(I)$. As a subset of $S$ the set $F(V)$ is compact in measure (cf. [22]). Since the topology of convergence in measure is metrizable, the compactness of the set is equivalent with the sequential compactness. By taking an arbitrary sequence $(y_n) \subset F(V)$ we get a sequence $(x_n)$ in $V$ such that $y_n = F(x_n)$. Since $(x_n) \subset V$, as follows from Lemma 3.2 $F$ transforms this sequence into the sequence convergent in measure. Thus, $(y_n)$ is compact in measure, so is $F(V)$. $\square$

Let us recall, that if $M$ satisfies $\Delta_2$-condition, then the assumption $F : L_M(I) \to E_M(I)$ is satisfied whenever $F : L_M(I) \to L_M(I)$ (see Lemma 3.3).

## 5. Measures of Noncompactness

Our operators need not be neither Lipschitz nor compact, in general. For quadratic integral equations the Darbo fixed-point theorem seems to an appropriate tool. It is based on a contraction property with respect to a measure of noncompactness. Although it seems to be known, we recall necessary notions for completeness.

By $\mathcal{M}_E$ denote the family of all nonempty and bounded subsets of $E$ and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact subsets. We denote the standard algebraic operations on sets by the symbols $k \cdot X$ and $X + Y$. If $X$ is a subset of $E$, then $\bar{X}$ and conv$X$ denote the closure and convex closure of $X$, respectively. We recall an axiomatic approach to the definition of measures of noncompactness.

**Definition 5.1.** *[23]* A mapping $\mu : \mathcal{M}_E \to [0, \infty)$ is said to be a measure of noncompactness in $E$ if it satisfies the following conditions:

(i) $\mu(X) = 0 \Rightarrow X \in \mathcal{N}_E$
(ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
(iii) $\mu(\bar{X}) = \mu(\text{conv}X) = \mu(X)$.
(iv) $\mu(\lambda X) = |\lambda| \, \mu(X)$, for $\lambda \in \mathbb{R}$.
(v) $\mu(X + Y) \leq \mu(X) + \mu(Y)$.
(vi) $\mu(X \bigcup Y) = \max\{\mu(X), \mu(Y)\}$.
(vii) If $X_n$ is a sequence of nonempty, bounded, closed subsets of $E$ such that $X_{n+1} \subset X_n$, $n = 1, 2, 3, \ldots$, and $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

An example of such a mapping is the following [23]: let $X$ be a nonempty and bounded subset of $E$. The Hausdorff measure of noncompactness $\beta_H(X)$ is defined as

$\inf\{r > 0 : \text{there exists a finite subset Y of E such that } x \subset Y + B_r\}$.

We need one more interesting notion. For any $\varepsilon > 0$, let $c$ be a measure of equiintegrability of the set $X$ in $L_M(I)$ (cf. Definition 3.9 in [14] or [24,25]):

$$c(X) = \lim_{\varepsilon \to 0} \sup_{\text{mes} D \leq \varepsilon} \sup_{x \in X} \|x \cdot \chi_D\|_{L_M(I)},$$

where $\chi_D$ denotes the characteristic function of $D$.

The following theorem clarifies the connections between different coefficients in Orlicz spaces. Since all Orlicz spaces $L_M(I)$ are regular, when $M$ satisfies $\Delta_2$ condition, then the Theorem 1 in [25] is read as follows:

**Proposition 5.2.** *Let $X$ be a nonempty, bounded and compact in measure subset of an ideal regular space $Y$. Then*

$$\beta_H(X) = c(X).$$

As a consequence, we obtain that bounded sets which are additionally compact in measure are compact in $L_M(I)$ iff they are equiintegrable in this space (i.e. have equiabsolutely continuous norms cf. [26], in particular $X \subset E_M(I)$).

An importance of such a kind of functions can be clarified using the contraction property with respect to this measure instead of compactness in the Schauder fixed-point theorem. Namely, we have the Darbo theorem ([23]):

**Theorem 5.3.** *Let $Q$ be a nonempty, bounded, closed and convex subset of $E$ and let $V : Q \to Q$ be a continuous transformation which is a contraction with respect to the measure of noncompactness $\mu$, i.e. there exists $k \in [0, 1)$ such that*

$$\mu(V(X)) \leq k\mu(X),$$

*for any nonempty subset $X$ of $E$. Then, $V$ has at least one fixed point in the set $Q$ and the set $FixV$ of all fixed points of $V$ satisfy $\mu(FixV) = 0$.*

## 6. Main Results

Denote by $B$ the operator associated with the right-hand side of the Eq. (1.1) i.e. $B(x) = g + U(x)$, where $U(x)(t) = G(x)(t) \cdot \lambda \int_a^b K(t,s)f(s,x(s))\mathrm{d}s$. Thus $B = g + G \cdot A = g + G \cdot H \circ F$.

We will try to choose the domains of operators defined above in such a way to obtain the existence of solutions in a desired Orlicz space $L_\varphi(I)$. We stress on conditions allowing us to consider strongly nonlinear operators and simultaneously to cover both quadratic and classical integral equations.

Let us note that our assumptions on $G$ are also referred to the case of standard quadratic integral equations (i.e. for $G(x)(t) = q(t) \cdot x(t)$).

We need to distinguish two different cases. This allow us to obtain more general growth conditions on $f$ (cf. [27–30] for non-quadratic equations). In every case we need to describe some assumptions on "intermediate" spaces being the images of $L_\varphi(I)$ for $G$ and $F$ ($L_{\varphi_1}(I)$ and $L_M(I)$, respectively) and the range for $H$ (i.e. $L_{\varphi_2}(I)$). This approach is based on a classical (non-quadratic) case as in [27–29]) and seems to be important in view of optimality of assumptions for every considered case. The main difference is in the part of the proof for continuity of integral operators.

### 6.1. The Case of $\varphi$ Satisfying the $\Delta'$-condition

**Theorem 6.1.** *Assume, that $\varphi, \varphi_1, \varphi_2$ are $N$-functions and that $M$ and $N$ are complementary $N$-functions. Moreover, put the following set of assumptions:*

(N1) *there exists a constant $k_1 > 0$ such that for every $u \in L_{\varphi_1}(I)$ and $w \in L_{\varphi_2}(I)$ we have $\|uw\|_{\varphi} \le k_1\|u\|_{\varphi_1}\|w\|_{\varphi_2}$,*

(C1) *$g \in E_{\varphi}(I)$ is nondecreasing a.e. on $I$,*

(C2) *$f : I\times\mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions and $f(t,x)$ is assumed to be nondecreasing with respect to both variables $t$ and $x$ separately,*

(C3) *$|f(t,x)| \le b(t) + R(|x|)$ for $t \in I$ and $x \in \mathbb{R}$, where $b \in E_N(I)$ and $R$ is nonnegative, nondecreasing, continuous function defined on $\mathbb{R}^+$,*

(C4) *Let $N$ satisfies the $\Delta'$-condition and suppose that there exist $\omega$, $\gamma$, $u_0 \ge 0$ for which*

$$N(\omega(R(u))) \le \gamma\varphi_2(u) \le \gamma M(u) \text{ for } u \ge u_0,$$

(G1) *$G : L_{\varphi}(I) \to L_{\varphi_1}(I)$ takes continuously $E_{\varphi}(I)$ into $E_{\varphi_1}(I)$ and there exists a constant $G_0 > 0$ such that $\|G(x)\|_{\varphi_1} \le G_0\|x\|_{\varphi}$ and that $G$ takes the set of all a.e. nondecreasing functions into itself,*

(K1) *$s \to K(t,s) \in L_M(I)$ for a.e. $t \in I$,*

(K2) *$K \in E_M(I^2)$ and $t \to K(t,s) \in E_{\varphi_2}(I)$ for a.e. $s \in I$ with $\|K\|_M < \frac{1}{2k_1\cdot|\lambda|\cdot G_0\cdot R(1)}$,*

(K3) *$\int_a^b K(t_1,s)\mathrm{d}s \ge \int_a^b K(t_2,s)\mathrm{d}s$ for $t_1, t_2 \in [a,b]$ with $t_1 < t_2$.*

*Then there exists a number $\rho > 0$ such that for all $\lambda \in \mathbb{R}$ with $|\lambda| < \rho$ and for all $g$ with $\|g\|_{\varphi} < 1$ there exists a solution $x \in E_{\varphi}(I)$ of (1.1) which is a.e. nondecreasing on $I$.*

*Proof.* We need to divide the proof into a few steps.

I. The operator $B$ is well defined from $L_{\varphi}(I)$ into itself and continuous on a domain depending on the considered case.
II. We will construct an invariant ball $B_r$ for $B$ in $L_{\varphi}(I)$.
III. We construct a subset $Q_r$ of this ball which contains a.e. nondecreasing functions and investigate the properties $Q_r$.
IV. We check the continuity and monotonicity properties of $B$ in $Q_r$, so $U : Q_r \to Q_r$.
V. We prove that $B$ is a contraction with respect to a measure of noncompactness.
VI. We use the Darbo fixed-point theorem to find a solution in $Q_r$.

I. First of all observe that under the assumptions (C2) and (C3) by Lemma 3.3 the superposition operator $F$ acts from $L_{\varphi}(I)$ to $L_N(I)$.

In this case we will prove that $U$ is a continuous mapping from the unit ball in $E_{\varphi}(I)$ into the space $E_{\varphi}(I)$.

Let us recall that $x \in E_{\varphi}(I)$ iff for arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x\chi_T\|_{\varphi} < \varepsilon$ for every measurable subset $T$ of $I$ with the Lebesgue measure smaller that $\delta$ (i.e. $x$ has absolutely continuous norm). First, let us observe that in view of Lemma 3.6, it is sufficient to check this property for the operator $A = H \circ F$.

Since $N$ is an $N$-function satisfying $\Delta'$-condition and by (C3), we are able to use [11, Lemma 19.1]. From this there exists a constant $C$ (not depending on the kernel) such that for any measurable subset $T$ of $I$ and $x \in L_\varphi(I)$, $\|x\|_\varphi \leq 1$ we have

$$\|A(x)\chi_T\|_{\varphi_2} \leq C\|K\chi_{T\times I}\|_M. \tag{6.1}$$

Now, by the Hölder inequality and the assumption (C2) we get

$$|K(t,s)f(s,x(s))| \leq \|K(t,s)\| \cdot |f(s,x(s))| \leq \|K(t,s)\| \cdot |\,(b(s)+R(|x(s)|))\,|$$

for $t, s \in I$. Put $k(t) = 2\|K(t,\cdot)\|_M$ for $t \in I$. As $K \in E_M(I^2)$ this function is integrable on $I$. By the assumptions (K1) and (K2) about the kernel $K$ of the operator $H$ (cf. [29]) we obtain that

$$\|A(x)(t)\| \leq k(t) \cdot (\|b\|_N + \|R(|x(\cdot)|)\|_N) \text{ for a.e. } t \in I.$$

Whence for arbitrary measurable subset $T$ of $I$ and $x \in E_\varphi(I)$

$$\|A(x)\chi_T\|_{\varphi_2} \leq \|k\chi_T\|_{\varphi_2} \cdot (\|b\|_N + \|R(|x(\cdot)|\|_N).$$

Finally if $t$ is such that $K(t,\cdot) \in E_M(I)$ and $x \in E_\varphi(I)$ we have

$$\int_T \|K(t,s)f(s,x(s))\| \mathrm{d}s \leq 2\|K(t,\cdot)\chi_T\|_M \cdot (\|b\|_N + \|R(|x(\cdot)|)\|_N)$$

for a.e. $t \in I$. From this it follows that $A$ maps $B_1(E_\varphi(I))$ into $E_{\varphi_2}(I)$.

We are in a position to prove the continuity of $A$ as a mapping from the unit ball $B_1(E_\varphi(I))$ into the space $E_{\varphi_2}(I)$. Let $x_n, x_0 \in B_1(E_\varphi(I))$ be such that $\|x_n - x_0\|_\varphi \to 0$ as $n$ tends to $\infty$. Suppose, contrary to our claim, that $A$ is not continuous and the $\|A(x_n) - A(x_0)\|_{\varphi_2}$ does not converge to zero. Then, there exists $\varepsilon > 0$ and a subsequence $(x_{n_k})$ such that

$$\|A(x_{n_k}) - A(x_0)\|_{\varphi_2} > \varepsilon \text{ for } k = 1, 2, \ldots \tag{6.2}$$

and the subsequence is a.e. convergent to $x_0$. Since $(x_n)$ is a subset of the ball the sequence $(\int_a^b \varphi(|x_n(t)|)\mathrm{d}t)$ is bounded. As the space $E_{\varphi_2}(I)$ is regular the balls are norm-closed in $L_1(I)$ so the sequence $(\int_a^b |x_n(t)|\mathrm{d}t)$ is also bounded.

Moreover, by (C3) and (C4) there exist $\omega$, $\gamma$, $u_0 > 0$, s.t. (cf. [11, p. 196])

$$\begin{aligned}
\|R(|x(\cdot)|)\|_N &= \frac{1}{\omega}\|\omega R(|x(\cdot)|)\|_N \\
&\leq \frac{1}{\omega} \inf_{r>0} \left\{ \int N(\omega R(|x(t)|)/r) dt \leq 1 \right\} \\
&\leq \frac{1}{\omega} \left( 1 + \int_a^b N(\omega R(|x(t)|))\mathrm{d}t \right) \\
&\leq \frac{1}{\omega} \left( 1 + N(\omega R(u_0)) \cdot (b-a) + \gamma \int_a^b \varphi_2(|x(t)|)\mathrm{d}t \right),
\end{aligned}$$

whenever $x \in L_\varphi(I)$ with $\|x\|_\varphi \leq 1$.

Thus

$$\int_T \|K(t,s)f(s,x_n(s))\|\mathrm{d}s \le 2\|K(t,\cdot)\chi_T\|_M \cdot (\|b\|_N + \|R(|x_n(\cdot)|)\|_N)$$
$$\le 2\|K(t,\cdot)\chi_T\|_M \cdot \Bigg(\|b\|_N \frac{1}{\omega}$$
$$\times \left[1 + N(\omega R(u_0))(b-a)\gamma \int_a^b \varphi_2(|x_n(t)|)\mathrm{d}t\right]\Bigg)$$

and then the sequence $(\|K(t,s)f(s,x_n(s))\|)$ is equiintegrable on $I$ for a.e. $t \in I$. By the continuity of $f(t,\cdot)$ we get $\lim_{k\to\infty} K(t,s)f(s,x_{n_k}(s)) = K(t,s)f(s,x_0(s))$ for a.e. $s \in I$. Now, applying the Vitali convergence theorem we obtain that

$$\lim_{k\to\infty} A(x_{n_k})(t) = A(x_0)(t) \quad \text{for a.e. } t \in I.$$

But the Eq. (6.1) implies that $A(x_{n_k})$ is a subset of $E_{\varphi_2}(I)$ and then $\lim_{k\to\infty} A(x_{n_k})(t) = A(x_0)(t)$ which contradicts the inequality (6.2). Since $A$ is continuous between indicated spaces, By our assumption (G1) the operator $G$ is continuous from $B_1(E_\varphi(I))$ into $E_{\varphi_1}(I)$ and then by (N1) the operator $U$ has the same property and then $U$ is a continuous mapping from $B_1(E_\varphi(I))$ into the space $E_\varphi(I)$. Finally, by the assumption (C1) $B$ maps $B_1(E_\varphi(I))$ into $E_\varphi(I)$ continuously.

II. We will prove the boundedness of the operator $U$, namely we will construct the invariant ball for this operator. By $B$ we will denote the right-hand side of our integral equation, i.e. $B = g + U$.

Set $r \le 1$ and let

$$\rho = \frac{1 - \|g\|_\varphi}{2k_1 \cdot C \cdot G_0 \cdot \|K\|_M}.$$

Let $x$ be an arbitrary element from $B_1(E_\varphi(I))$. Then using the above consideration, the assumption (C3), the formula (6.1) and Proposition 3.6 for sufficiently small $\lambda$ (i.e. $|\lambda| < \rho$) we obtain

$$\begin{aligned}
\|B(x)\|_\varphi &\le \|g\|_\varphi + \|Ux\|_\varphi \\
&= \|g\|_\varphi + \|G(x)\cdot A(x)\|_\varphi \\
&\le \|g\|_\varphi + k_1\|G(x)\|_{\varphi_1} \cdot \|A(x)\|_{\varphi_2} \\
&= \|g\|_\varphi + k_1|\lambda| \cdot G_0 \cdot \|x\|_\varphi \cdot \left\|\int_a^b K(\cdot,s)f(s,x(s))\mathrm{d}s\right\|_{\varphi_2} \\
&\le \|g\|_\varphi + 2k_1 \cdot |\lambda| \cdot C \cdot G_0 \cdot \|x\|_\varphi \cdot \|K\|_M \\
&\le \|g\|_\varphi + 2k_1 \cdot |\lambda| \cdot C \cdot G_0 \cdot r \cdot \|K\|_M \\
&\le \|g\|_\varphi + 2k_1 \cdot \rho \cdot C \cdot G_0 \cdot \|K\|_M \le r
\end{aligned}$$

whenever $\|x\|_\varphi \le r$.

Then we have $B : B_r(E_\varphi(I)) \to B_r(E_\varphi(I))$. Moreover, $B$ is continuous on $B_r(E_\varphi(I))$ (see the part I of the proof).

III. Let $Q_r$ stand for the subset of $B_r(E_\varphi(I))$ consisting of all functions which are a.e. nondecreasing on $I$. Similarly as claimed in [31] this set is nonempty, bounded (by $r$) and convex (direct calculation from the definition). It is also a closed set in $L_\varphi(I)$.

Indeed, let $(y_n)$ be a sequence of elements in $Q_r$ convergent in $L_\varphi(I)$ to $y$. Then, the sequence is convergent in measure and as a consequence of the Vitali convergence theorem for Orlicz spaces and of the characterization of convergence in measure (the Riesz theorem) we obtain the existence of a subsequence $(y_{n_k})$ of $(y_n)$ which converges to $y$ almost uniformly on $I$ (cf. [27]). Moreover, $y$ is still nondecreasing a.e. on $I$ which means that $y \in Q_r$ and so the set $Q_r$ is closed. Now, in view of Lemma 3.2 the set $Q_r$ is compact in measure.

IV. Now, we will show that $B$ preserve the monotonicity of functions. Take $x \in Q_r$, then $x$ is a.e. nondecreasing on $I$ and consequently $F(x)$ is also of the same type in virtue of the assumption (C2) and Lemma 4.2. Further, $A(x) = H \circ F(x)$ is a.e. nondecreasing on $I$ thanks for the assumption (K3). Since the pointwise product of a.e. monotone functions is still of the same type and by (G1), the operator $U$ is a.e. nondecreasing on $I$.

Moreover, the assumption (C1) permits us to deduce that $Bx(t) = g(t) + U(x)(t)$ is also a.e. nondecreasing on $I$. This fact together with the assertion that $B : B_r(E_\varphi(I)) \to B_r(E_\varphi(I))$ gives us that $B$ is also a self-mapping of the set $Q_r$. From the above considerations it follows that $B$ maps continuously $Q_r$ into $Q_r$.

V. We will prove that $B$ is a contraction with respect to the measure of noncompactness $\mu$. Assume that $X$ is a nonempty subset of $Q_r$ and let the fixed constant $> 0$ be arbitrary. Then for an arbitrary $x \in X$ and for a set $D \subset I$, $\text{meas}D \le \varepsilon$ we obtain

$$\begin{aligned}
\|B(x)\cdot\chi_D\|_\varphi &\le \|g\chi_D\|_\varphi + \|U(x)\cdot\chi_D\|_\varphi \\
&= \|g\chi_D\|_\varphi + \|G(x)\cdot A(x)\chi_D\|_\varphi \\
&\le \|g\chi_D\|_\varphi + k_1\|G(x)\chi_D\|_{\varphi_1}\cdot\|A(x)\cdot\chi_D\|_{\varphi_2} \\
&= \|g\chi_D\|_\varphi + k_1|\lambda|\|G(x)\chi_D\|_{\varphi_1}\|\int_D K(\cdot,s)f(s,x(s))\mathrm{d}s\|_{\varphi_2} \\
&\le \|g\chi_D\|_\varphi + k_1|\lambda|G_0\|x\chi_D\|_\varphi\|\int_D |K(\cdot,s)|(b(s)+R(|x(s)|))\mathrm{d}s\|_{\varphi_2} \\
&\le \|g\chi_D\|_\varphi + k_1|\lambda|G_0\|x\chi_D\|_\varphi 2\|K\|_M\|[b\chi_D + R(r))]\|_N \\
&\le \|g\chi_D\|_\varphi + 2k_1|\lambda|G_0\|x\chi_D\|_\varphi\|K\|_M\ [\ \|b\chi_D\|_N + R(1)].
\end{aligned}$$

Hence, taking into account that $g \in E_\varphi$, $b \in E_N$

$$\lim_{\varepsilon\to 0}\{\sup_{mes\ D\le\varepsilon}[\sup_{x\in X}\{\|g\chi_D\|_\varphi = 0\}]\} \text{ and } \lim_{\varepsilon\to 0}\{\sup_{mes\ D\le\varepsilon}[\sup_{x\in X}\{\|b\chi_D\|_N = 0\}]\}.$$

Thus, by definition of $c(x)$ and by taking the supremum over all $x \in X$ and all measurable subsets $D$ with $measD \le \varepsilon$ we get

$$c(B(X)) \le 2k_1\cdot|\lambda|\cdot G_0\cdot\|K\|_M\cdot R(1)\cdot c(X).$$

Since $X \subset Q_r$ is a nonempty, bounded and compact in measure subset of an ideal regular space $E_\varphi$, we can use Proposition 5.2 and get

$$\beta_H(B(X)) \le 2k_1\cdot|\lambda|\cdot G_0\cdot\|K\|_M\cdot R(1)\cdot\beta_H(X).$$

The inequality obtained above together with the properties of the operator $B$ and the set $Q_r$ established before and the inequality from the Assumption (K2) allows us to apply the Darbo Fixed-Point Theorem 5.3, which completes the proof. □

### 6.2. The Case of $\varphi$ Satisfying the $\Delta_3$-Condition

Let us consider the case of $N$-functions with the growth essentially more rapid than a polynomial. In fact, we will consider $N$-functions satisfying $\Delta_3$-condition. This is very large and important class, especially from an application point of view (cf. [32–35]). An extensive description of this class can be found in [34, Section 2.5]. Recall, that an $N$-function $M$ determines the properties of the Orlicz space $L_M(I)$ and then the less restrictive rate of the growth of this function implies the "worser" properties of the space. By $\vartheta$ we will denote the norm of the identity operator from $L_\varphi(I)$ into $L^1(I)$, i.e. $\sup\{\|x\|_1 : x \in B_1(L_\varphi(I))\}$. For the discussion about the existence of $\varphi$ which satisfies our conditions see [17, p. 61].

**Theorem 6.2.** *Assume, that $\varphi, \varphi_1, \varphi_2$ are $N$-functions and that $M$ and $N$ are complementary $N$-functions and that (N1), (C1), (C2), (C3), (G1), (K1) and (K3) hold true. Moreover, put the following assumptions:*

(C5) 1. *$N$ satisfies the $\Delta_3$-condition,*
2. *$K \in E_M(I^2)$ and $t \to K(t, s) \in E_{\varphi_2}(I)$ for a.e. $s \in I$,*
3. *There exist $\beta,\ u_0 > 0$ such that*

$$R(u) \le \beta \frac{M(u)}{u},\ \text{for } u \ge u_0,$$

(K4) *$\varphi_2$ is an N-function satisfying*

$$\iint\limits_{I^2} \varphi_2(M(|K(t, s)|))\, dtds < \infty$$

*and*

$$2k_1 \cdot (2 + (b - a)(1 + \varphi_2(1))) \cdot |\lambda| \cdot G_0 \cdot \|K\|_{\varphi_2 \circ M} \cdot R(r_0) < 1,$$

*where*

$$r_0 = \frac{1}{\vartheta} \left[ \frac{\omega}{2|\lambda| \cdot k_1 \cdot G_0 \cdot (2 + (b - a)(1 + \varphi_2(1))) \cdot \|K\|_{\varphi_2 \circ M}} - \|b\|_N \right].$$

*Then, there exist a number $\rho > 0$ and a number $\varpi > 0$ such that for all $\lambda \in \mathbb{R}$ with $|\lambda| < \rho$ and for all $g \in E_\varphi(I)$ with $\|g\|_\varphi < \varpi$ there exists a solution $x \in E_\varphi(I)$ of (1.1) which is a.e. nondecreasing on $I$.*

*Proof.* We will indicate only the points of the proof if they differ from the previous case.

I. In this case the operator $B$ can be considered as continuous when acting on the whole $E_\varphi(I)$.

By [17, Lemma 15.1 and Theorem 19.2] and the assumption (K4):

$$\|A(x)\chi_T\|_{\varphi_2} \le 2\cdot(2+(b-a)(1+\varphi_2(1)))\cdot\|K\cdot\chi_{T\times I}\|_{\varphi_2 \circ M} (\|b\|_N + \|R(|x(\cdot)|)\|_N) \tag{6.3}$$

for arbitrary $x \in L_\varphi(I)$ and arbitrary measurable subset $T$ of $I$.

Let us note that the assumption (C5) 3. implies that there exist constants $\omega, u_0 > 0$ and $\eta > 1$ such that $N(\omega R(u)) \leq \eta u$ for $u \geq u_0$.

Thus for $x \in L_\varphi(I)$

$$\begin{aligned}\|R(|x(\cdot)|)\|_N &\leq \frac{1}{\omega}\left(1 + \int_I N(\omega R(|x(s)|)\mathrm{d}s\right) \\ &\leq \frac{1}{\omega}\left(1 + \eta u_0(b-a) + \eta \int_I |x(s)|\mathrm{d}s\right).\end{aligned}$$

The remaining estimations can be derived as in the first main theorem and then we obtain that $A : E_\varphi(I) \to E_{\varphi_2}(I)$, so by the properties of $G$ we get $B : E_\varphi(I) \to E_\varphi(I)$.

II. Put

$$\rho = \frac{1}{2k_1 G_0(2 + (b-a)(1+\varphi_2(1)))\|K\|_{\varphi_2 \circ M}\left[\|b\|_N + \frac{1}{\omega}(1+\eta u_0(b-a))\right]}.$$

Fix $\lambda$ with $|\lambda| < \rho$.

Choose a positive number $r$ in such a way that

$$\begin{aligned}&\|g\|_\varphi + 2|\lambda| k_1 G_0(2 + (b-a)(1+\varphi_2(1)))\|K\|_{\varphi_2 \circ M} \\ &\quad \cdot r\left(\|b\|_N + \frac{1}{\omega}(1 + \eta u_0(b-a) + \eta\vartheta r)\right) \leq r. \qquad (6.4)\end{aligned}$$

As a domain for the operator $B$ we will consider the ball $B_r(E_\varphi(I))$.

Let us remark that the above inequality is of the form $a + (b + vr)cr \leq r$ with $a, b, c, v > 0$. Then, $vc > 0$ and if we assume that $bc - 1 < 0$ and that the discriminant is positive, then Viète's formulas imply that the quadratic equation has two positive solutions $r_1 < r_2$ for sufficiently small $\lambda$. By the definition of $\rho$ it is clear that our assumptions guarantee the above requirements, so there exists a positive number $r$ satisfying this inequality.

Put $C = (2 + (b-a)(1+\varphi(1)))$. Let us note, in view of the above considerations, that the assumption about the discriminant which implies that the existence of solutions for the above problem is of the form:

$$\begin{aligned}&\left[\|b\|_N + \frac{1}{\omega}(1+\eta u_0(b-a)) - \frac{1}{2|\lambda| \cdot k_1 \cdot G_0 \cdot C \cdot \|K\|_{\varphi_2 \circ M}}\right]^2 \\ &\quad \cdot |\lambda| \cdot k_1 \cdot G_0 \cdot C \cdot \|K\|_{\varphi_2 \circ M} > \frac{2\|g\|_\varphi \eta\vartheta}{\omega}\end{aligned}$$

i.e.

$$\begin{aligned}\varpi = &\left[\|b\|_N + \frac{1}{\omega}(1+\eta u_0(b-a)) - \frac{1}{2|\lambda| \cdot k_1 \cdot G_0 \cdot C \cdot \|K\|_{\varphi_2 \circ M}}\right]^2 \\ &\times \frac{|\lambda| \cdot k_1 \cdot G_0 \cdot C \cdot \omega \|K\|_{\varphi_2 \circ M}}{2\eta\vartheta}.\end{aligned}$$

For $x \in B_r(E_\varphi(I))$ we have the following estimation:

$$
\begin{aligned}
\|B(x)\|_\varphi &\le \|g\|_\varphi + \ \|Ux\|_\varphi \\
&= \|g\|_\varphi + \ \|G(x) \cdot A(x)\|_\varphi \\
&\le \|g\|_\varphi + \ k_1 \|G(x)\|_{\varphi_1} \cdot \|A(x)\|_{\varphi_2} \\
&= \|g\|_\varphi + k_1|\lambda| \|G(x)\|_{\varphi_1} \cdot \| \int_a^b K(\cdot, s) f(s, x(s)) \mathrm{d}s \|_{\varphi_2} \\
&\le \|g\|_\varphi + 2k_1 \cdot C \cdot G_0 \cdot |\lambda| \cdot \|x\|_\varphi \|K\|_{\varphi_2 \circ M} \\
&\quad \cdot \left[ \|b\|_N + \frac{1}{\omega} (1 + N(\omega R(u_0)) \cdot (b-a) \eta \int_I |x(s)| \mathrm{d}s) \right] \\
&\le \|g\|_\varphi + 2k_1 \cdot C \cdot G_0 \cdot |\lambda| \cdot \|x\|_\varphi \|K\|_{\varphi_2 \circ M} \\
&\quad \cdot \left[ \|b\|_N + \frac{1}{\omega} (1 + N(\omega R(u_0)) \cdot (b-a) \eta \|x\|_1) \right] \\
&\le \|g\|_\varphi + 2k_1 \cdot C \cdot G_0 \cdot |\lambda| \cdot \|x\|_\varphi \|K\|_{\varphi_2 \circ M} \\
&\quad \cdot \left[ \|b\|_N + \frac{1}{\omega} (1 + N(\omega R(u_0)) \cdot (b-a) \eta \vartheta \|x\|_\varphi) \right] \\
&= \|g\|_\varphi + 2rk_1 \cdot C \cdot G_0 \cdot |\lambda| \cdot \|K\|_{\varphi_2 \circ M} \\
&\quad \cdot \left[ \|b\|_N + \frac{1}{\omega} (1 + \eta u_0 (b-a) + \eta \vartheta r) \right] \\
&\le r.
\end{aligned}
$$

Then $B : B_r(E_\varphi(I)) \to B_r(E_\varphi(I))$.

Note, that the parts III. and IV. of the previous proof are similar to those from the first theorem, so we omit the details.

V. We will prove that $B$ is a contraction with respect to a measure of noncompactness. Assume that $X$ is a nonempty subset of $Q_r$ and let the fixed constant $\varepsilon > 0$ be arbitrary. Then, for an arbitrary $x \in X$ and for a set $D \subset I$, $\mathrm{meas} D \le \varepsilon$ we obtain

$$
\begin{aligned}
\|B(x) \cdot \chi_D\|_\varphi &\le \|g\chi_D\|_\varphi \ + \ \|U(x) \cdot \chi_D\|_\varphi \\
&= \|g\chi_D\|_\varphi + \ \|G(x) \cdot A(x) \chi_D\|_\varphi \\
&\le \|g\chi_D\|_\varphi + k_1 \|G(x)\chi_D\|_{\varphi_1} \cdot \|A(x) \cdot \chi_D\|_{\varphi_2} \\
&= \|g\chi_D\|_\varphi + k_1 \cdot |\lambda| \cdot G_0 \cdot \|x\chi_D\|_\varphi \| \int_D K(\cdot, s) f(s, x(s)) \mathrm{d}s \|_{\varphi_2} \\
&\le \|g\chi_D\|_\varphi + k_1 \cdot |\lambda| \cdot G_0 \cdot \|x\chi_D\|_\varphi \\
&\quad \cdot \| \int_D |K(\cdot, s)| (b(s) + R(|x(s)|)) \mathrm{d}s \|_{\varphi_2} \\
&\le \|g\chi_D\|_\varphi \ + \ k_1 \cdot |\lambda| \cdot G_0 \cdot \|x\chi_D\|_\varphi \\
&\quad \cdot \left( \left\| \int_D |K(\cdot, s)| b(s) \mathrm{d}s \right\|_{\varphi_2} + \left\| \int_D |K(\cdot, s)| R(|x(s)|)) \mathrm{d}s \right\|_{\varphi_2} \right) \\
&\le \|g\chi_D\|_\varphi + 2 \cdot C \cdot k_1 \cdot G_0 \cdot |\lambda| \cdot \|x\chi_D\|_\varphi \cdot \vartheta \cdot \|K\|_{\varphi_2 \circ M} \|b\chi_D\|_N \\
&\quad + 2 \cdot C \cdot k_1 \cdot G_0 \cdot \|x\chi_D\|_\varphi \cdot \left\| \int_D |K(\cdot, s)| R(|x(s)|) \mathrm{d}s \right\|_{\varphi_2}
\end{aligned}
$$

$$\le \|g\chi_D\|_\varphi + \ 2Ck_1G_0|\lambda| \cdot \|x\chi_D\|_\varphi \cdot \|K\|_{\varphi_2\circ M} \ [\ \|b\chi_D\|_N + R(r)\ ]$$
$$\le \|g\chi_D\|_\varphi + \ 2Ck_1G_0|\lambda| \cdot \|x\chi_D\|_\varphi \cdot \|K\|_{\varphi_2\circ M} \ [\ \|b\chi_D\|_N + R(r_0)\ ],$$

where

$$r_0 = \frac{1}{\vartheta}\left[\frac{\omega}{2|\lambda|\cdot k_1 \cdot G_0 \cdot (2+(b-a)(1+\varphi_2)) \cdot \|K\|_{\varphi_2\circ M}} - \|b\|_N\right].$$

Let us note that $r_0$ is an upper bound for solutions of (6.4).

Similarly as in the previous theorem we get

$$\beta_H(B(X)) \ \le \ 2 \cdot k_1 \cdot C \cdot G_0 \cdot |\lambda| \|K\|_{\varphi_2\circ M} \cdot R(r_0) \cdot \beta_H(X).$$

The inequality obtained above together with the properties of the operator $B$ and the set $Q_r$ established before and then the assumption (K4) allow us to apply the Theorem 5.3, which completes the proof. □

We need to stress on some aspects of our results. First of all we can observe that our solutions are not necessarily continuous as in previously investigated cases. In particular, we need not to assume that the Hammerstein operator transforms the space $C(I)$ into itself. For the examples and conditions related to Hammerstein operators in Orlicz spaces we refer the readers to [34, Chapter VI.6.1., Corollary 6 and Example 7].

We have two more information about the set of solutions: it is included in $E_\varphi(I)$ and in view of Theorem 5.3 it can be proved that this set is compact as a subset of $L_\varphi(I)$.

Let $X, Y$ be ideal spaces. A superposition operator $F : X \to Y$ is called improving if it takes bounded subsets of $X$ into the subsets of $Y$ with equiabsolutely continuous norms. The applications of such operators are based on the observation that large classes of linear integral operators

$$Hy(t) = \lambda \int_D k(t,s)y(s)\mathrm{d}s,$$

although not being compact, map sets with equiabsolutely continuous norms into precompact sets. In contrast to the classical (non-quadratic) case, for quadratic integral equations even such a nice assumption is not sufficient for using the Schauder fixed-point theorem.

Moreover, we assume in our main theorem that $G$ maps sets with equiabsolutely continuous norms into the same family, but we do not need to assume, that it is an improving operator. Conversely, any improving operator $G$ can be considered in our results. Let us note that for operators from Lebesgue spaces $L_p(I)$ into $L_r(I)$ (i.e. Orlicz spaces with $p(x) = x^p$ and $r(x) = x^r$, respectively), the characterization of improving operators is known [36]: a superposition operator $F : L_p(I) \to L_r(I)$ is improving if and only if there exists a continuous and even function $M$ satisfying $\lim_{u\to\infty} \frac{M(u)}{u} = \infty$ and such that $G(x)(t) = M(f(t, x(t)))$ is also an operator from $L_p(I)$ into $L_r(I)$ (for an appropriate growth condition of $f$ see [36]).

The aspect of applicability of our results deals also with the technique of Orlicz spaces for partial differential equations, so for an appropriate class of integral equations. In this context one can consider more singular equations than in a classical case. Motivated by previously considered equations (see [10,32,33,35] or [27,29]) we extend this method to the case of quadratic integral equations.

It should be recalled that our method of the proof can be also adapted to classical equations considered in [10,11,32,35]. For more information we refer the readers to the Chapter IX "Nonlinear PDEs and Orlicz spaces" in [37].

Finally, let us remark that our results can be applied also for Lebesgue spaces $L_p(I)$ $(p \geq 1)$ (cf. [38,39]). As mentioned above this class of spaces is also included into the class of Orlicz spaces. But even in this case we allow for $f$ or $K$ to be strongly nonlinear. The simplest case is that when $F : L_1(I) \to L_N(I)$, $H : L_N(I) \to L_p(I)$, $G : L_1(I) \to L_q(I)$ $(\frac{1}{p} + \frac{1}{q} = 1)$. Thus, we will have integrable solutions (in $L_1(I)$), but $f$ or $K$ can be strongly nonlinear. Of course, strong nonlinearity of one of them implies that we need to consider the weak one for another (cf. [11, Chapter IV. 19]).

Let us present an example of such spaces. By $N_1, N_2$ we denote complementary functions for $M_1, M_2$, respectively. Put $M_1(u) = \exp|u| - |u| - 1$ and $M_2(u) = \frac{u^2}{2} = N_2(u)$. Note, that $M_1$ satisfies the $\Delta_3$-condition. In this case $N_1(u) = (1 + |u|) \cdot \ln(1 + |u|) - |u|$. If we define an $N$-function either as $\Psi(u) = M_2[N_1(u)]$ or $\Psi(u) = N_1[M_2(u)]$, then by choosing arbitrary kernel $K$ from the space $L_\Psi(I)$ we are able to apply [11, Theorem 15.4]. Thus, $H : L_{M_1}(I) \to L_{M_2}(I)$ is continuous and we may apply our result (Theorem 6.2) for operators $G : L_1(I) \to L_q(I)$ and $F : L_1(I) \to L_{M_1}(I)$ (with natural growth conditions, see Lemma 3.3).

Let us also pay attention to the particular case of our problem with $G(x) = a(t)x(t)$:

$$x(t) = g(t) + \lambda a(t) \cdot x(t) \int_0^1 K(t,s) f(s, x(s)) \mathrm{d}s.$$

Since we are motivated by some study on quadratic integral equations, this is of our particular interest. Note, that a full description for acting and continuity conditions for $G(x) = a(t)x(t)$ can be found in [11, Theorem 18.2].



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