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A Family of Measures of Noncompactness in the Space $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and its Application to Some Nonlinear Volterra Integral Equation

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Abstract. The aim of this paper is to study a new family of measures of noncompactness in the space $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ consisting of all real functions locally integrable on $\mathbb{R}_+$, equipped with a suitable topology. As an example of applications of the technique associated with that family of measures of noncompactness, we study the existence of solutions of a nonlinear Volterra integral equation in the space $L^1_{\mathrm{loc}}(\mathbb{R}_+)$. The obtained result generalizes several ones obtained earlier with help of other methods.

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1. Introduction

Recently, there appeared a lot of papers [1,2,5,6,9–12,15,16] in which solvability of various integral equations in the Banach space $L^1(\mathbb{R}_+)$ is considered with help of measures of noncompactness introduced by Appell and De Pascale in [3] and by Banaś and Knap in [8]. On the other hand, the technique related to measures of noncompactness was not applied in the space $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ consisting of all functions locally integrable on $\mathbb{R}_+$.

In this paper, fulfilling this gap, we define two topological structures on the space $L^1_{\mathrm{loc}}(\mathbb{R}_+)$: the Fréchet metric topology $\mathcal{T}^F_{\mathrm{loc}}$ given by a sequence of seminorms and the topology $\mathcal{T}^w_{\mathrm{loc}}$ generated by the family of projections $\pi_T : L^1_{\mathrm{loc}}(\mathbb{R}_+) \to L^1[0,T], T \geq 0$, where the spaces $L^1[0,T]$ are furnished with weak topologies. Next, applying the weak measure of noncompactness in $L^1[0,T]$ introduced by Appell and De Pascale in [3], we define the family of measures of noncompactness $\{\mu_T\}_{T\geq 0}$ in $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ with the topology $\mathcal{T}^w_{\mathrm{loc}}$ and we investigate the basic properties of $\{\mu_T\}_{T\geq 0}$. As an example of applications of this method, we give a theorem on the existence of solutions in $L^1_{\text{loc}}(\mathbb{R}_+)$ of the nonlinear Volterra integral equation of the form

$$x(t) = f\left(t, \int_0^t v(t,s,x(s))\mathrm{d}s\right).$$

The efficiency of the technique related to the family of measures of noncompactness $\{\mu_T\}_{T\geq 0}$ manifests itself in this way that our result generalizes some others on the solvability in $L^1(\mathbb{R}_+)$ obtained with help of other methods.

2. Notation and Auxiliary Facts

Let $m(A)$ denote the Lebesgue measure of a Lebesgue measurable subset $A \subset \mathbb{R}_+ = [0,\infty)$. For subset $A \subset [0,T]$ of a fixed interval $[0,T]$, we will write

$$A' := [0,T]\backslash A.$$

Further denote by $L^1[0,T]$ the space of all real functions defined and Lebesgue integrable on the set $[0,T]$.

If $0 \leq S \leq T$ then the symbol π_S^T stands for the operator of restriction

$$\pi_S^T : L^1[0,T] \to L^1[0,S],\ \pi_S^T(x) := x|_{[0,S]}, \tag{2.1}$$

i.e., $\pi_S^T(x)$ is the restriction of the function $x \in L^1[0,T]$ to the interval $[0,S]$.

Denote by $L^1_{\text{loc}}(\mathbb{R}_+)$ (shortly L^1_{loc}) the space of all real measurable functions $x : \mathbb{R}_+ \to \mathbb{R}$ and locally Lebesgue integrable on $\mathbb{R}_+$, i.e., $||x||_T < \infty$ for $T \geq 0$, where

$$||x||_T := \int_0^T |x(t)|\mathrm{d}t \quad \text{for } x \in L^1_{\text{loc}}(\mathbb{R}_+). \tag{2.2}$$

Further, we denote by π_T the operator of restriction

$$\pi_T : L^1_{\text{loc}} \to L^1[0,T],\ \pi_T(x) := x|_{[0,T]}, \tag{2.3}$$

i.e., $\pi_T(x)$ is the restriction of the function $x \in L^1_{\text{loc}}$ to the interval $[0,T]$.

In the space L^1_{loc} we will consider two topologies. The first of them is the Fréchet metric topology $\mathcal{T}^F_{\text{loc}}$.

Definition 2.1. The metrizable topology induced in L^1_{loc} by the family of seminorms (2.2), i.e., the topology defined by the distance

$$d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{||x-y||_n}{1+||x-y||_n},$$

or equivalently

$$d_1(x,y) := \sup\{2^{-T}||x-y||_T : T \geq 0\},$$

will be called *Fréchet metric topology* in L^1_{loc} and it will be denoted by $\mathcal{T}^F_{\text{loc}}$.

The convergence and compactness in the topology $\mathcal{T}^F_{\text{loc}}$ are characterized in the following proposition.

Proposition 2.2. 1. *A sequence* $(x_n) \subset L^1_{\text{loc}}$ *is convergent to* $x \in L^1_{\text{loc}}$ *in the topology* $\mathcal{T}^F_{\text{loc}}$ *if and only if* $\lim_{n\to\infty} ||x - x_n||_T = 0$ *for* $T \geq 0$.
2. *A set* $X \subset L^1_{\text{loc}}$ *is relatively compact in the topology* $\mathcal{T}^F_{\text{loc}}$ *if and only if* $\pi_T(X)$ *is relatively compact in the Banach space* $L^1[0,T]$ *for* $T \geq 0$.

If X is a subset of L^1_{loc}, we write $\overline{X}^F$ and conv$X$ in order to denote the closure of X in the topology $\mathcal{T}^F_{\text{loc}}$ and the convex of X, respectively.

The second topology $\mathcal{T}^w_{\text{loc}}$ in L^1_{loc}, the so-called topology generated by the family of mappings $\{\pi_T\}_{T\geq 0}$, will be introduced in Sect. 3. In order to investigate the basic properties of the topology $\mathcal{T}^w_{\text{loc}}$, we recall some facts concerned with limits of the inverse systems (see [13]).

Suppose that to every σ in a set Σ directed by the relation $\leq$ corresponds a topological space X_σ, and that for any $\sigma, \rho \in \Sigma$ satisfying $\rho \leq \sigma$, a continuous mapping $\hat{\pi}^\sigma_\rho : X_\sigma \to X_\rho$ is defined; suppose further that $\hat{\pi}^\rho_\tau \hat{\pi}^\sigma_\rho = \hat{\pi}^\sigma_\tau$ for any $\sigma, \rho, \tau \in \Sigma$ satisfying $\tau \leq \rho \leq \sigma$ and that $\hat{\pi}^\sigma_\sigma = \text{id}_{X_\sigma}$ for every $\sigma \in \Sigma$. In this situation, we say that the family $\mathbf{S} = \{X_\sigma, \hat{\pi}^\sigma_\rho, \Sigma\}$ is an *inverse system of the spaces* X_σ; the mappings $\hat{\pi}^\sigma_\rho$ are called *bonding mappings* of the inverse system $\mathbf{S}$.

Let $\mathbf{S} = \{X_\sigma, \hat{\pi}^\sigma_\rho, \Sigma\}$ be an inverse system; an element $\{x_\sigma\}$ of the Cartesian product $\Pi_{\sigma\in\Sigma} X_\sigma$ is called a *thread* of $\mathbf{S}$ if $\hat{\pi}^\sigma_\rho(x_\sigma) = x_\rho$ for any $\sigma, \rho \in \Sigma$ satisfying $\rho \leq \sigma$, and the subspace of $\Pi_{\sigma\in\Sigma} X_\sigma$ consisting of all threads of $\mathbf{S}$ is called the *limit of the inverse system* $\mathbf{S} = \{X_\sigma, \hat{\pi}^\sigma_\rho, \Sigma\}$ and is denoted by $\varprojlim \mathbf{S}$ or by $\varprojlim\{X_\sigma, \hat{\pi}^\sigma_\rho, \Sigma\}$.

Let $\mathbf{S} = \{X_\sigma, \hat{\pi}^\sigma_\rho, \Sigma\}$ be an inverse system of topological spaces and let $X = \varprojlim \mathbf{S}$. For every $\sigma \in \Sigma$ a continuous mapping $\hat{\pi}_\sigma = p_\sigma|_X : X \to X_\sigma$, where $p_\sigma : \Pi_{\sigma\in\Sigma} X_\sigma \to X_\sigma$ is the projection, is defined; it is called the *projection of the limit of* $\mathbf{S}$ *to* X_σ. Clearly, for any $\sigma, \rho \in \Sigma$ such that $\rho \leq \sigma$, the projections $\hat{\pi}_\sigma$ and $\hat{\pi}_\rho$ satisfy the equality $\hat{\pi}_\rho = \hat{\pi}^\sigma_\rho \hat{\pi}_\sigma$.

For our further purposes, we need the following lemmas.

Lemma 2.3. [13] *The limit of an inverse system* $\mathbf{S} = \{X_\sigma, \hat{\pi}^\sigma_\rho, \Sigma\}$ *of non-empty compact spaces is compact and non-empty.*

Lemma 2.4. [13] *For every subspace* A *of the limit* $\varprojlim \mathbf{S}$ *of an inverse system* $\mathbf{S} = \{X_\sigma, \hat{\pi}^\sigma_\rho, \Sigma\}$, *the family* $\mathbf{S_A} = \{A_\sigma, \tilde{\pi}^\sigma_\rho, \Sigma\}$, *where* $A_\sigma = \overline{\hat{\pi}_\sigma(A)}$ *and* $\tilde{\pi}^\sigma_\rho := \hat{\pi}^\sigma_\rho|_{A_\sigma}$, *is an inverse system and* $\overline{A} = \varprojlim \mathbf{S_A}$.

Lemma 2.5. *For all subspaces* $A_n, n = 1, 2, \ldots$ *of the limit* $\varprojlim \mathbf{S}$ *of an inverse system* $\mathbf{S} = \{X_\sigma, \hat{\pi}^\sigma_\rho, \Sigma\}$, *the family* $\mathbf{S_A} = \{A_\sigma, \tilde{\pi}^\sigma_\rho, \Sigma\}$, *where* $A_\sigma = \bigcap_{n=1}^\infty \overline{\hat{\pi}_\sigma(A_n)}$ *and* $\tilde{\pi}^\sigma_\rho = \hat{\pi}^\sigma_\rho|_{A_\sigma}$, *is an inverse system and* $\bigcap_{n=1}^\infty \overline{A_n} = \varprojlim \mathbf{S_A}$.

Proof. (Based on the proof of Proposition 2.5.6. [13]) At the beginning let us observe that if f is a continuous mapping between two topological spaces then

$$f\left(\bigcap_{n=1}^\infty \overline{B_n}\right) \subset \bigcap_{n=1}^\infty \overline{f(B_n)} \tag{2.4}$$

for all subsets $B_n, n = 1, 2, \dots$ contained in the domain of f.

Let us denote $X := \underleftarrow{\lim}\{X_\sigma, \hat{\pi}^\sigma_\rho, \Sigma\}$. As $\hat{\pi}_\rho(x) = \hat{\pi}^\sigma_\rho \hat{\pi}_\sigma(x)$ for $x \in X$ and $\rho \le \sigma$, we have from (2.4)

$$\tilde{\pi}^\sigma_\rho(A_\sigma) = \hat{\pi}^\sigma_\rho(A_\sigma) = \hat{\pi}^\sigma_\rho\left(\bigcap_{n=1}^{\infty} \overline{\hat{\pi}_\sigma(A_n)}\right) \subset \bigcap_{n=1}^{\infty} \overline{\hat{\pi}^\sigma_\rho \hat{\pi}_\sigma(A_n)} = \bigcap_{n=1}^{\infty} \overline{\hat{\pi}_\rho(A_n)} = A_\rho,$$

which proves that $\mathbf{S_A}$ is an inverse system.

Now, using (2.4) we get

$$\hat{\pi}_\sigma\left(\bigcap_{n=1}^{\infty} \overline{A_n}\right) \subset \bigcap_{n=1}^{\infty} \overline{\hat{\pi}_\sigma(A_n)} = A_\sigma.$$

Hence,

$$\bigcap_{n=1}^{\infty} \overline{A_n} \subset \underleftarrow{\lim}\, \mathbf{S_A}. \tag{2.5}$$

Now let us take any point $x = \{x_\sigma\} \in \underleftarrow{\lim}\, \mathbf{S_A}$.

By Proposition 2.5.5 [13], the family of all sets $\hat{\pi}_\sigma^{-1}(U_\sigma)$, where U_σ is a neighbourhood of x_σ in the space X_σ, is a base for X at the point x. For every member $\hat{\pi}_\sigma^{-1}(U_\sigma)$ of that base we have

$$x_\sigma \in A_\sigma \cap U_\sigma = \bigcap_{n=1}^{\infty} (\overline{\hat{\pi}_\sigma(A_n)} \cap U_\sigma),$$

so that $x_\sigma \in \overline{\hat{\pi}_\sigma(A_n)} \cap U_\sigma$ for $n = 1, 2, \dots$. This implies that $\hat{\pi}_\sigma(A_n) \cap U_\sigma \ne \emptyset$ for $n = 1, 2, \dots$, i.e., $A_n \cap \hat{\pi}_\sigma^{-1}(U_\sigma) \ne \emptyset$ for $n = 1, 2, \dots$. Hence $x = \{x_\sigma\} \in \overline{A_n}$ for $n = 1, 2, \dots$ and therefore $x = \{x_\sigma\} \in \bigcap_{n=1}^{\infty} \overline{A_n}$, i.e.,

$$\bigcap_{n=1}^{\infty} \overline{A_n} \supset \underleftarrow{\lim}\, \mathbf{S_A}.$$

Linking this inclusion with (2.5), we finish proof. □

In further parts of this paper we will apply the weak measure of noncompactness $c(\cdot)$ in $L^1[0, T]$ which was introduced by Appell and De Pascale in [3]. The measure $c(\cdot)$ is defined on bounded (in norm) subsets of $L^1[0, T]$ by the formula

$$c(X) := \lim_{\varepsilon \to 0+} \left\{ \sup_{x \in X} \left\{ \sup \left\{ \int_D |x(t)| \mathrm{d}t : D \subset [0, T], m(D) \le \varepsilon \right\} \right\} \right\}. \tag{2.6}$$

The fundamental properties of this measure are contained in the lemma given below.

Lemma 2.6. *The weak measure of noncompactness $c(\cdot)$ satisfies the following conditions:*

1. $c(X) = 0$ *iff* X *is bounded weakly relatively compact in* $L^1[0, T]$.
2. $X \subset Y \Rightarrow c(X) \le c(Y)$.

3. $c(X) = c(\overline{X}^w) = c(\overline{\text{conv}X})$, *where* $\overline{X}^w$ *is the weak closure of* X *and* $\overline{\text{conv}X}$ *is the closed convex hull (with respect to the norm topology) of a set* X.
4. $c(\lambda X) = |\lambda| c(X)$.
5. $c(X+Y) \le c(X) + c(Y)$.
6. *If* $\{X_n\}$ *is a sequence of nonempty, bounded (in norm), closed (in weak topology) subsets of* $L^1[0,T]$ *and* $X_1 \supset X_2 \supset \cdots$ *with* $\lim_{n\to\infty} c(X_n) = 0$, *then the intersection set* $X_\infty := \bigcap_{n=1}^\infty X_n$ *is nonempty and weakly compact in* $L^1[0,T]$.

3. The Topology $\mathcal{T}^w_{\text{loc}}$ and the Family of Measures of Noncompactness

In this section, we will define the next topology $\mathcal{T}^w_{\text{loc}}$ in L^1_{loc}. Moreover, we will introduce the family of measures of noncompactness $\{\mu_T\}_{T\ge 0}$ in L^1_{loc} with the topology $\mathcal{T}^w_{\text{loc}}$ and we will investigate its properties.

Definition 3.1. The topology in the space in L^1_{loc} generated by the family of projections $\{\pi_T\}_{T\ge 0}$ with weak topologies in the spaces $L^1[0,T]$ for $T \ge 0$ will be denoted by $\mathcal{T}^w_{\text{loc}}$. In other words, $\mathcal{T}^w_{\text{loc}}$ is the weakest topology in L^1_{loc} in which each of the mappings $\pi_T : L^1_{\text{loc}} \to L^1[0,T]$ [defined in (2.3)] for $T \ge 0$ is continuous, where the space $L^1[0,T]$ is equipped with the weak topology for $T \ge 0$.

It means that the base for $\mathcal{T}^w_{\text{loc}}$ is the collection of all sets of the form $\pi_T^{-1}(G)$, where $T \ge 0$ and G is an arbitrary open subset of $L^1[0,T]$ with weak topology. In other words, $\mathcal{T}^w_{\text{loc}}$ is the topology in L^1_{loc} such that it induces weak topology in the spaces $L^1[0,T], T \ge 0$.

The closure of a subset $X \subset L^1_{\text{loc}}$ in the topology $\mathcal{T}^w_{\text{loc}}$ will be denoted by $\overline{X}^w$.

The space L^1_{loc} with the topology $\mathcal{T}^w_{\text{loc}}$ defined in this way is homeomorphic to the limit of the inverse system $\varprojlim\{L^1[0,T], \pi^T_S, \mathbb{R}_+\}$, where we assume that the spaces $L^1[0,T]$ are equipped with the weak topologies. There is the natural topological homeomorphism $h : L^1_{\text{loc}} \to \varprojlim\{L^1[0,T], \pi^T_S, \mathbb{R}_+\}$ between these spaces, given by formula

$$h(x) := \{\pi_T(x)\}_{T\ge 0} \quad \text{for } x \in L^1_{\text{loc}}.$$

Therefore,

$$\pi_T(X) = \hat{\pi}_T(hX) \quad \text{for } X \subset L^1_{\text{loc}}, \tag{3.1}$$

where $\hat{\pi}_T$ is the projection of the limit $\varprojlim\{L^1[0,T], \pi^T_S, \mathbb{R}_+\}$ to $L^1[0,T]$.

Now we are going to introduce a family of measures of noncompactness in L^1_{loc} with the topology $\mathcal{T}^w_{\text{loc}}$.

We will say that a subset $X \subset L^1_{\text{loc}}$ is *bounded*, if X is bounded for every pseudonorm $||\cdot||_T$, $T \ge 0$. Moreover, we denote by $\mathfrak{M}_{L^1_{\text{loc}}}$ the family of all nonempty and bounded subsets of L^1_{loc} and by $\mathfrak{N}_{L^1_{\text{loc}}}$ its subfamily consisting of all relatively compact sets in the topology $\mathcal{T}^w_{\text{loc}}$. It appears that it is more effective to operate with a family of measures of noncompactness in L^1_{loc} instead of one measure.

Definition 3.2. The family of mappings $\{\mu_T\}_{T\geq 0}$, $\mu_T : \mathfrak{M}_{L^1_{\text{loc}}} \to \mathbb{R}_+$, defined by formula

$$\mu_T(X) := \lim_{\varepsilon \to 0+} \left\{ \sup_{x\in X} \left\{ \sup \left\{ \int_D |x(t)|\mathrm{d}t : D \subset [0,T], m(D) \leq \varepsilon \right\} \right\} \right\} \tag{3.2}$$

is said to be *the family of measures of noncompactness* in L^1_{loc} with topology $\mathcal{T}^w_{\text{loc}}$.

In other words,

$$\mu_T(X) = c(\pi_T(X)), \tag{3.3}$$

where $c(\cdot)$ is defined in (2.6).

The following theorem, being the main result of this section, describes the fundamental properties of the family $\{\mu_T\}_{T\geq 0}$.

Theorem 3.3. *The family of measures of noncompactness* $\{\mu_T\}_{T\geq 0}, \mu_T : \mathfrak{M}_{L^1_{\text{loc}}} \to \mathbb{R}_+$ *defined by* (3.2)*, fulfils conditions:*

1. *The family* $\ker\{\mu_T\} := \{X \in \mathfrak{M}_{L^1_{\text{loc}}} : \mu_T(X) = 0$ *for* $T \geq 0\}$ *is nonempty and* $\ker\{\mu_T\} = \mathfrak{N}_{L^1_{\text{loc}}}$.
2. $X \subset Y \Rightarrow \mu_T(X) \leq \mu_T(Y)$ *for* $T \geq 0$.
3. $\mu_T(X) = \mu_T(\overline{X}^w) = \mu_T(\overline{\text{conv}X}^F)$ *for* $T \geq 0$*, where* $\overline{X}^w$ *denotes the closure of a set* X *in the topology* $\mathcal{T}^w_{\text{loc}}$ *and* $\overline{\text{conv}X}^F$ *is the closure of a convex hull of the set* X *with respect to the topology* $\mathcal{T}^F_{\text{loc}}$.
4. $\mu_T(\lambda X) = |\lambda|\mu_T(X)$ *for* $T \geq 0$.
5. $\mu_T(X + Y) \leq \mu_T(X) + \mu_T(Y)$ *for* $T \geq 0$.
6. *If* $\{X_n\}$ *is a sequence of nonempty, bounded* (*in* L^1_{loc})*, closed* (*in the topology* $\mathcal{T}^w_{\text{loc}}$) *subsets of* L^1_{loc} *and* $X_1 \supset X_2 \supset \cdots$ *with* $\lim_{n\to\infty} \mu_T(X_n) = 0$ *for* $T \geq 0$*, then the intersection set* $X_\infty := \bigcap_{n=1}^\infty X_n$ *is nonempty and compact in* $\mathcal{T}^w_{\text{loc}}$.

Remark 3.4. In the proof given below, the symbol $\overline{X}$ stands for the closure of a subset $X \subset \underleftarrow{\lim}\{L^1[0,T], \pi^T_S, \mathbb{R}_+\}$, whilst $\overline{X}^w$ denotes the closure of $X \subset L^1[0,T]$ or $X \subset L^1_{\text{loc}}$ in the weak topology of $L^1[0,T]$ or in the topology $\mathcal{T}^w_{\text{loc}}$, respectively.

Proof. (of Theorem 3.3.) To prove 1, we first assume that $X \in \ker\{\mu_T\}$. It means that $\mu_T(X) = c(\pi_T(X)) = 0$ and therefore $\overline{\pi_T(X)}^w$ is weakly compact in $L^1[0,T]$ for $T \geq 0$. Hence, in view of (3.1) we obtain that $\overline{\hat{\pi}_T(h(X))}^w$ is weakly compact in $L^1[0,T]$ for $T \geq 0$. Applying Lemma 2.4 we get

$$\overline{h(X)} = \underleftarrow{\lim}\left\{\overline{\hat{\pi}_T(h(X))}^w, \pi^T_S, \mathbb{R}_+\right\}.$$

Using Lemma 2.3 and above-established facts, we derive that $\underleftarrow{\lim}\{\overline{\hat{\pi}_T(h(X))}^w, \pi^T_S, \mathbb{R}_+\}$ is compact and therefore $\overline{h(X)} = h(\overline{X}^w)$ is also compact. Since h is a homeomorphism then $\overline{X}^w$ is also compact in $\mathcal{T}^w_{\text{loc}}$ and $X \in \mathfrak{N}_{L^1_{\text{loc}}}$.

Now let us assume that $X \in \mathfrak{N}_{L^1_{\mathrm{loc}}}$. Then $\overline{X}^w$ is compact in $\mathcal{T}^w_{\mathrm{loc}}$ and therefore $\pi_T(\overline{X}^w)$ is weakly compact in $L^1[0,T]$, $T \geq 0$. Hence, in virtue of Lemma 2.6 we obtain:

$$\mu_T(X) = c(\pi_T(X)) \leq c\left(\pi_T(\overline{X}^w)\right) = 0, \quad T \geq 0,$$

i.e., $X \in \ker\{\mu_T\}$.

The properties 2, 4, 5 are easy consequence of analogous properties of the measure $c(\cdot)$ given in Lemma 2.6.

Now we prove 3. Let us fix $X \subset L^1_{\mathrm{loc}}$. Using Lemma 2.6 3 and inclusion $\pi_T(\overline{X}^w) \subset \overline{\pi_T(X)}^w$ [see (2.4) for $f = \pi_T, B_n := X, n = 1, 2, \ldots$] we get

$$c(\pi_T(X)) = c(\overline{\pi_T(X)}^w) \geq c(\pi_T(\overline{X}^w)) \geq c(\pi_T(X)).$$

Hence, $c(\pi_T(X)) = c(\pi_T(\overline{X}^w))$ and by (3.3) we get

$$\mu_T(X) = \mu_T(\overline{X}^w).$$

Next, keeping in mind inclusion $\pi_T(\overline{Y}^F) \subset \overline{\pi_T(Y)}$ for $Y \in L^1_{\mathrm{loc}}$, the equality $\pi_T(\mathrm{conv}(X)) = \mathrm{conv}(\pi_T(X))$ for $X \subset L^1_{\mathrm{loc}}$ and applying Lemma 2.6 3, we derive that

$$\begin{aligned} c(\pi_T(X)) &\leq c(\pi_T(\mathrm{conv} X)) \leq c(\pi_T(\overline{\mathrm{conv} X}^F)) \\ &\leq c(\overline{\pi_T(\mathrm{conv} X)}) = c(\overline{\mathrm{conv}(\pi_T(X))}) = c(\pi_T(X)). \end{aligned}$$

In view of (3.3), this means that $\mu_T(X) = \mu_T(\overline{\mathrm{conv} X}^F)$ for $T \geq 0$.

To prove 6, assume that $X_n \in \mathfrak{M}_{L^1_{\mathrm{loc}}}, X_n = \overline{X_n}^w$, $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty} \mu_T(X_n) = 0$ for $T \geq 0$. Applying Lemma 2.5 we get

$$\bigcap_{n=1}^{\infty} \overline{h(X_n)} = \underleftarrow{\lim} \left\{ \bigcap_{n=1}^{\infty} \overline{\hat{\pi}_T(h(X_n))}^w, \pi_S^T, \mathbb{R}_+ \right\}.$$

Using (3.1) we derive

$$\bigcap_{n=1}^{\infty} \overline{h(X_n)} = \underleftarrow{\lim} \left\{ \bigcap_{n=1}^{\infty} \overline{\pi_T(X_n)}^w, \pi_S^T, \mathbb{R}_+ \right\}. \tag{3.4}$$

In view of Lemma 2.6 3, we have

$$\lim_{n\to\infty} c(\overline{\pi_T(X_n)}^w) = \lim_{n\to\infty} c(\pi_T(X_n)) = \lim_{n\to\infty} \mu_T(X_n) = 0.$$

Then, in virtue of Lemma 2.6 6, we obtain that $\bigcap_{n=1}^{\infty} \overline{\pi_T(X_n)}^w$ is nonempty and weakly compact in $L^1[0,T]$ for $T \geq 0$. Keeping in mind Lemma 2.3 and (3.4) we get that $\bigcap_{n=1}^{\infty} \overline{h(X_n)}$ is nonempty and compact. Hence and since h is homeomorphic, we get

$$\bigcap_{n=1}^{\infty} \overline{h(X_n)} = \bigcap_{n=1}^{\infty} h(\overline{X_n}^w) = h\left(\bigcap_{n=1}^{\infty} \overline{X_n}^w\right),$$

i.e., $\bigcap_{n=1}^{\infty} \overline{X_n}^w = \bigcap_{n=1}^{\infty} X_n = X_\infty$ is nonempty and compact in L^1_{loc}. □

As an immediate consequence of Theorem 3.3 1, we have the following corollary.

Corollary 3.5. *A subset $X \subset L^1_{\mathrm{loc}}$ is relatively compact in the topology $\mathcal{T}^w_{\mathrm{loc}}$ if and only if $\pi_T(X)$ is relatively weakly compact in $L^1[0,T]$ for $T > 0$.*

Lemma 3.6. [14] *A subset X of $L^1([0,T])$ is relatively compact if and only if X is relatively weakly compact and every sequence in X has an a.e. convergent subsequence.*

We call a subset $X \subset L^1[0,T]$ *a.e. equicontinuous* on $[0,T]$, if for each $\varepsilon > 0$ there is a closed subset $D_\varepsilon \subset [0,T]$ such that $m(D'_\varepsilon) \leq \varepsilon$ and the restriction of all functions from X to the set D_ε form an equicontinuous family of functions.

A subset $X \subset L^1[0,T]$ is said to be *countable a.e. equicontinuous* on $[0,T]$ if every sequence $\{x_n\} \subset X$ has a subsequence $\{\overline{x}_n\}$, which is a.e. equicontinuous on $[0,T]$.

The following lemma is based on ideas from [5,6].

Lemma 3.7. *Assume that a subset $X \subset L^1[0,T]$ is bounded (in norm). Then X is relatively compact if and only if X is relatively weakly compact and countable a.e. equicontinuous on $[0,T]$.*

Proof. We first suppose that X is a relatively compact subset of the Banach space $L^1[0,T]$. Since the set X is also weakly relatively compact, then it suffices to show that X is countable a.e. equicontinuous. Let us choose arbitrary sequence $\{x_n\} \subset X$ and $\varepsilon > 0$. From Lemma 3.6, Egorov's and Lusin's theorems it follows that there are a closed subset $D_\varepsilon \subset [0,T]$ and a subsequence $\{\overline{x}_n\}$ of the sequence $\{x_n\}$ such that $m(D'_\varepsilon) < \varepsilon$, $\overline{x}_n$ are continuous on D_ε for $n = 1, 2, \ldots$ and the sequence $\{\overline{x}_n\}$ is uniformly convergent on D_ε. This fact, together with Ascoli–Arzéla theorem, implies that $\{\overline{x}_n\}$ is equicontinuous on D_ε, i.e., X is countable a.e. equicontinuous.

Now we assume that a subset $X \subset L^1[0,T]$ is weakly relatively compact and countable a.e. equicontinuous. Let us fix arbitrary sequence $\{x_n\} \subset X$. The countable a.e. equicontinuity of X, Ascoli–Arzéla theorem and the diagonal method yield that there is a subsequence $\{\overline{x}_n\}$ of $\{x_n\}$ such that $\{\overline{x}_n\}$ is a.e. convergent in $[0,T]$. Applying Lemma 3.6 we infer that X is relatively compact in $L^1[0,T]$. □

As an immediate consequence of above lemma and Proposition 2.2, we have the following lemma.

Lemma 3.8. *Assume that a subset $X \subset L^1_{\mathrm{loc}}$ is bounded. Then X is relatively compact in the topology $\mathcal{T}^F_{\mathrm{loc}}$ if and only if X is relatively compact in the topology $\mathcal{T}^w_{\mathrm{loc}}$ and X is countable a.e. equicontinuous on bounded subintervals of $\mathbb{R}_+$.*

Lemma 3.9. [7] *Assume that (S, d) is a complete metric space. Let X, X_n, $n = 1, 2, \ldots$ be nonempty subsets of S such that X_n are relatively compact for $n = 1, 2, \ldots$ and*

$$\lim_{n\to\infty} D(X, X_n) = 0$$

where $D(\cdot, \cdot)$ denotes the Hausdorff distance between sets. Then the set X is also relatively compact.

4. Example of Applications of the Family of Measures of Noncompactness $\{\mu_T\}_{T\geq 0}$

To demonstrate the applicability of the family of measures of noncompactness $\{\mu_T\}_{T\geq 0}$ in the space L^1_{loc} with the topology $\mathcal{T}^w_{\mathrm{loc}}$, we give in this section an application to certain nonlinear Volterra integral equation of the form

$$x(t) = f\left(t, \int_0^t v(t, s, x(s))\mathrm{d}s\right), \quad t \in \mathbb{R}_+, \tag{4.1}$$

where we look for solutions of Eq. (4.1) in the space L^1_{loc}.
For further purposes, we collect a few auxiliary facts.

Let us assume that $J \subset \mathbb{R}$ is a given measurable subset and S is a metric space. We say that a function $f(t, x) = f : J \times S \to \mathbb{R}$ satisfies Carathéodory conditions if it is measurable in t for any $x \in S$ and is continuous in x for almost all $t \in J$. The theorem of Scorza Dragoni given bellow explains the structure of functions satisfying Carathéodory conditions.

Theorem 4.1. *Let $f : J \times S \to \mathbb{R}$ be a function satisfying Carathéodory conditions. Then, for each $\varepsilon > 0$ there exists a closed subset D_ε of the set J such that $m(J\backslash D_\varepsilon) \leq \varepsilon$ and $f|_{D_\varepsilon\times S}$ is continuous.*

Now, let us assume that $I \subset \mathbb{R}_+$ is a given interval, bounded or not. In what follows we will always assume that a function $f : I \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies Carathéodory conditions. Then, to every real function $x = x(t)$ which is measurable on I, we may assign the function $(Fx)(t) = f(t, x(t)), t \in I$. It is well known that the function Fx is also measurable on I. The operator F defined in such a way is said to be the superposition (or Nemytskii) operator generated by the function f. Using some facts from [4] we can prove the following theorem.

Theorem 4.2. *The operator F generated by the function $f(t, x)$ maps continuously the space L^1_{loc} with the Fréchet topology $\mathcal{T}^F_{\mathrm{loc}}$ into itself if and only if $|f(t, x)| \leq a(t) + b(t)|x|$ for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}_+$, where a is nonnegative function from L^1_{loc} and b is measurable and locally essentially bounded nonnegative function defined on $\mathbb{R}_+$.*

Finally, we proceed by recalling some facts concerning the linear Volterra integral operator in L^1_{loc}.

To begin denote by Δ the triangle $\Delta = \{(t,s) : 0 \leq s \leq t\}$ and assume that $k : \Delta \to \mathbb{R}_+$ is a given function which is measurable with respect to both variables. Next, for an arbitrary function $x \in L^1_{\mathrm{loc}}$ let us put

$$(Kx)(t) := \int_0^t k(t,s)x(s)\mathrm{d}s, \quad t \in \mathbb{R}_+.$$

The operator K defined in such a way is the well-known linear Volterra integral operator. The lemma given below contains a fundamental property of the Volterra integral operator (see [17]).

Lemma 4.3. *If the Volterra integral operator K transforms the space L^1_{loc} into itself, then it is continuous in the Fréchet metric topology $\mathcal{T}^F_{\mathrm{loc}}$.*

For such operator K, let us put $K_T := K|_{L^1[0,T]}$, i.e., K_T is the restriction of the operator K to the Banach space $L^1[0,T]$. The operator $K_T : L^1[0,T] \to L^1[0,T]$ is continuous (cf. [14,17]), and the norm of the linear operator K_T we denote by $||K_T||$ for $T \geq 0$.

Equation (4.1) will be investigated under the following assumptions:

(i) The function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions and there exist functions: a measurable locally essentially bounded $b : \mathbb{R}_+ \to \mathbb{R}_+$ and $a \in L^1_{\mathrm{loc}}$ such that

$$|f(t,x)| \leq a(t) + b(t)|x|$$

for $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$.

(ii) The function $v(t,s,x) = v : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions, i.e., the function $t \mapsto v(t,s,x)$ is measurable on $\mathbb{R}_+$ for all $(s,x) \in \mathbb{R}_+ \times \mathbb{R}$ and the function $(s,x) \mapsto v(t,s,x)$ is continuous on the set $\mathbb{R}_+ \times \mathbb{R}$ for each a.e. $t \in \mathbb{R}_+$.

(iii)

$$|v(t,s,x)| \leq k(t,s)(a_1(s) + b_1(s)|x|)$$

for $(t,s,x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$, where $b_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is locally essentially bounded function and $a_1 \in L^1_{\mathrm{loc}}$. Moreover, we assume that the function $k(t,s) = k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies Carathéodory conditions and is such that the linear Volterra integral operator K generated by the function $k(t,s)$, that is,

$$(Kx)(t) := \int_0^t k(t,s)x(s)\mathrm{d}s \quad t \geq 0,$$

transforms the space L^1_{loc} into itself.

(iv)

$$\overline{b}(T)\overline{b}_1(T)||K_T|| < 1$$

for $T \geq 0$, where

$$\overline{b}(T) := \operatorname*{ess\,sup}_{t \in [0,T]} b(t), \; \overline{b}_1(T) := \operatorname*{ess\,sup}_{t \in [0,T]} b_1(t).$$

The existence result is contained in the following theorem.

Theorem 4.4. *Under assumptions* (i)–(iv) *Eq.* (4.1) *has at least one solution* $x = x(t)$ *which belongs to the space* $L^1_{\rm loc}$. *Moreover*

$$|x(t)| \leq \frac{||a||_t + \overline{b}(t)||a_1||_t||K_t||}{1 - \overline{b}(t)\overline{b}_1(t)||K_t||}, \quad \text{for a.e.} \quad t \geq 0. \tag{4.2}$$

Proof. Consider the operator $G : L^1_{\rm loc} \to L^1_{\rm loc}$ defined by the right side of Eq. (4.1). Observe that G can be written as the product

$$G = FV$$

where F is Nemytskii operator and the operator $V : L^1_{\rm loc} \to L^1_{\rm loc}$ is given by formula

$$(Vx)(t) := \int_0^t v(t, s, x(s))\mathrm{d}s, \quad t \geq 0.$$

Let us fix arbitrary $x \in L^1_{\rm loc}$ and $t \in \mathbb{R}_+$. Then, in view of (i)–(iii) we have

$$||Gx||_t \leq \int_0^t \left| f\left(s, \int_0^s v(s, \tau, x(\tau))\mathrm{d}\tau\right)\right| \mathrm{d}s \leq ||a||_t + \overline{b}(t)||a_1||_t||K_t|| + \overline{b}(t)\overline{b}_1(t)||K_t||||x||_t.$$

This means that

$$||Gx||_t \leq A(t) + B(t)||x||_t, \tag{4.3}$$

where

$$A(t) := ||a||_t + \overline{b}(t)||a_1||_t||K_t||, \; B(t) := \overline{b}(t)\overline{b}_1(t)||K_t||.$$

Now we define the nonnegative, nondecreasing function $r : \mathbb{R}_+ \to [0, \infty)$ and the subset B of $L^1_{\rm loc}$ as follows:

$$r(t) := \frac{A(t)}{1 - B(t)}, \tag{4.4}$$

$$B := \{x \in L^1_{\rm loc} : ||x||_t \leq r(t) \text{ for } t \geq 0\}.$$

The set B is convex and bounded in $L^1_{\rm loc}$. Condition (4.3) ensures that G transforms B into itself, i.e.,

$$G : B \to B. \tag{4.5}$$

Moreover, the set

$$\pi_T(B) = \{x \in L^1[0, T] : ||x||_t \leq r(t) \quad \text{for } t \in [0, T]\}$$

is convex and closed in the Banach space $L^1[0, T]$, so it is weakly closed in $L^1[0, T]$ for $T \geq 0$. Since

$$B = \bigcap_{T \geq 0} \pi_T^{-1}(\pi_T(B)),$$

then B is closed in the topology $\mathcal{T}^w_{\mathrm{loc}}$, i.e.,

$$\overline{B}^w = B. \tag{4.6}$$

Next, keeping in mind the inequality from assumption (iii) and applying the so-called majorant principle (cf. [14,17]), we infer that the operator $V|_{L^1[0,T]} : L^1[0,T] \to L^1[0,T]$ is continuous. This fact together with Proposition 2.2 1, implies that (we omit details of this reasoning)

$$V : L^1_{\mathrm{loc}} \to L^1_{\mathrm{loc}} \text{ is continuous in the topology } \mathcal{T}^F_{\mathrm{loc}}. \tag{4.7}$$

Further, let us take a nonempty subset X of the set B and an arbitrary number $T \geq 0$. Fix a number $\varepsilon > 0$ and a nonempty subset D of $[0,T]$ such that D is measurable and $m(D) \leq \varepsilon$. Then, for arbitrarily fixed $x \in X$ we obtain:

$$\begin{aligned}\int_D |(Gx)(t)|\mathrm{d}t &\leq \int_D a(t)\mathrm{d}t + \bar{b}(T)\int_D \left|\int_0^t v(t,s,x(s))\mathrm{d}s\right|\mathrm{d}t \\ &\leq \int_D a(t)\mathrm{d}t + \bar{b}(T)\int_D (Ka_1)(t)\mathrm{d}t + \bar{b}(T)\bar{b}_1(T)||K_T||\int_D |x(t)|\mathrm{d}t.\end{aligned}$$

Keeping in mind absolute continuity of integrals of functions a and Ka_1 and letting $\varepsilon \to 0+$ we have

$$\mu_T(GX) \leq \bar{b}(T)\bar{b}_1(T)||K_T||\mu_T(X). \tag{4.8}$$

In the sequel let us put $B_0 = B, B_n = \overline{\mathrm{conv}G(B_{n-1})}^w, n = 1,2,\ldots,$ where $\mathrm{conv}G(B_{n-1})$ denotes the convex hull of the $G(B_{n-1})$. Notice that in view of (4.5) and (4.6) we have $B_n \subset B_{n-1}$ for all $n \in \mathbb{N}$. Moreover, all sets belonging to this sequence are convex and closed in $\mathcal{T}^w_{\mathrm{loc}}$. Apart from this we have

$$\mu_T(B_n) \leq (\bar{b}(T)\bar{b}_1(T)||K_T||)^n\mu_T(B), \quad T \geq 0,$$

which, together with (iv) and $\mu_T(B) < \infty$, yields that $\lim_{n\to\infty}\mu_T(B_n) = 0$ for $T \geq 0$. Hence, taking into account 6, in Theorem 3.3 we deduce that the set $B_\infty := \bigcap_{n=1}^\infty B_n$ is nonempty, convex, and compact in $\mathcal{T}^w_{\mathrm{loc}}$. This means that

$$\lim_{\varepsilon\to 0+}\left\{\sup_{x\in B_\infty}\left\{\sup\left\{\int_D |x(t)|dt : D \subset [0,T], m(D) \leq \varepsilon\right\}\right\}\right\} = 0, \quad T \geq 0. \tag{4.9}$$

Further, we consider the sequence of the operators $P_n : B \to B$, $n = 1,2,\ldots$ defined by formula

$$(P_nx)(t) := \begin{cases} x(t) & \text{when} \quad |x(t)| \leq n, \\ n\cdot\mathrm{sgn}(x(t)) & \text{when} \quad |x(t)| > n. \end{cases}$$

We show that VB_∞ is relatively compact in the space L^1_{loc} with the topology $\mathcal{T}^F_{\mathrm{loc}}$. To this end we apply Lemma 3.9. In order to prove that

$\lim_{n\to\infty} D(VB_\infty, VP_nB_\infty) = 0$ let us fix a number $T > 0$ and let us put

$$A_{x,n} := \{t \in [0,T] : x(t) \neq (P_nx)(t)\} \quad \text{for } x \in B_\infty, n \in \mathbb{N}.$$

Since all functions of the set B_∞ are uniformly bounded in $||\cdot||_T$ then

$$\lim_{n\to\infty} \sup\{m(A_{x,n}) : x \in B_\infty\} = 0. \tag{4.10}$$

Let us fix $x \in B_\infty$. Keeping in mind that $|(P_nx)(s)| \leq |x(s)|$ we get

$$\begin{aligned} ||Vx - VP_nx||_T &\leq \int_0^T \int_{A_{x,n}} k(t,s)(a_1(s) + b_1(s)|x(s)|)\mathrm{d}s\mathrm{d}t \\ &\quad + \int_0^T \int_{A_{x,n}} k(t,s)(a_1(s) + b_1(s)|(P_nx)(s)|)\mathrm{d}s\mathrm{d}t \\ &\leq 2||K_T|| \int_{A_{x,n}} (a_1(s) + b_1(s)|x(s)|)\mathrm{d}s. \end{aligned}$$

Linking (4.10), absolute continuity of integrals of function a_1 and (4.9) we derive that

$$\lim_{n\to\infty} \sup_{x\in B_\infty} ||Vx - VP_nx||_T = 0 \quad \text{for } T \geq 0.$$

This equality together with inclusion $VP_nB_\infty \subset VB_\infty$ imply that

$$\lim_{n\to\infty} D(VB_\infty, VP_nB_\infty) = 0, \tag{4.11}$$

where $D(\cdot,\cdot)$ denotes the Hausdorff distance in the Fréchet metric space $(L^1_{\rm loc}, d)$. Now we show that VP_nB_∞ is relatively compact in the topology $\mathcal{T}^F_{\rm loc}$. In order to do this we apply Lemma 3.8. Essentially the same reasoning as in (4.8) guarantees that $\mu_T(VP_nB_\infty) = 0$ for $T \geq 0$. This, together with Corollary 3.5 implies that VP_nB_∞ is relatively compact in the topology $\mathcal{T}^w_{\rm loc}$. This set is also bounded. So, it suffices to show that the set VP_nB_∞ is countably a.e. equicontinuous on bounded subsets. Let us fix a numbers $T > 0$ and $\varepsilon > 0$. In view of Luzin theorem, we can find a closed subset D_ε of the interval $[0,T]$ such that $m(D_\varepsilon^{'}) \leq \varepsilon$, the functions $v|_{D_\varepsilon\times\mathbb{R}_+\times\mathbb{R}}$ and $k|_{D_\varepsilon\times\mathbb{R}_+}$ are continuous. Now let us take arbitrarily $t_1, t_2 \in D_\varepsilon, t_1 \leq t_2$. Then, for $x \in B_\infty$ we have

$$\begin{aligned} &|(VP_nx)(t_2) - (VP_nx)(t_1)| \\ &\quad \leq T\omega^T(v, |t_2 - t_1|) + \overline{k} \int_{t_1}^{t_2} a_1(s)\mathrm{d}s + \overline{k}\, \overline{b}_1(T) \int_{t_1}^{t_2} |x(s)|\mathrm{d}s, \end{aligned}$$

where $\omega^T(v,\cdot)$ denotes the modulus of continuity of the function v on the set $D_\varepsilon \times [0,T] \times [-n,n]$ and $\overline{k} := \max\{k(t,s) : (t,s) \in D_\varepsilon \times [0,T]\}$.

Keeping in mind uniform continuity of the function v on the compact set $D_\varepsilon \times [0,T] \times [-n,n]$, absolute continuity of the integral of the function a_1 and (4.9), we obtain that $|(VP_nx)(t_2) - (VP_nx)(t_1)|$ is arbitrarily small provided the number $t_2 - t_1$ is small enough. This proves that the set VP_nB_∞ is countably a.e. equicontinuous on bounded intervals. Linking all established facts and applying Lemma 3.8, we conclude that VP_nB_∞ is relatively compact in $\mathcal{T}^F_{\text{loc}}$. Using these facts, (4.11) and Lemma 3.9 we derive that VB_∞ is also relatively compact in $\mathcal{T}^F_{\text{loc}}$. Taking into account the continuity of $F : L^1_{\text{loc}} \to L^1_{\text{loc}}$ (see Theorem 4.2), we get that $GB_\infty = FVB_\infty$ is also relatively compact in $\mathcal{T}^F_{\text{loc}}$.

In the last step of the proof let us consider the set $Y = \overline{\text{conv}GB_\infty}^F$. Obviously Y is convex and compact subset of the space L^1_{loc} with the topology $\mathcal{T}^F_{\text{loc}}$ and in view of (4.7) and Theorem 4.2, the operator $G = FV$ transforms continuously the set Y into itself. Thus, applying the classical Tichonov fixed point principle we conclude that there is a solution $x \in B_\infty \subset L^1_{\text{loc}}$ of Eq. (4.1). Moreover, linking the definition of the set B and (4.4) we obtain estimation (4.2). This completes the proof of our theorem. □

Remark 4.5. Let us observe that our Theorem 4.4 generalizes Theorems 4.1 from [5,6] on existence of solutions in $L^1(\mathbb{R}_+)$. Indeed, Eq. (4.1) is more general than those considered in [5,6]. Moreover, the assumptions of Theorems in [5,6] imply our assumptions (i)–(iv). Then, in virtue of Theorem 4.4 there is a solution $x \in L^1_{\text{loc}}$ satisfying (4.2). Since those assumptions in [5,6] yield that the right side of (4.2) belongs to $L^1(\mathbb{R}_+)$ then $x \in L^1(\mathbb{R}_+)$.

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