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Laplace Type Multipliers for Laguerre Expansions of Hermite Type

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Abstract. We investigate Laplace transform type and Laplace-Stieltjes type multipliers associated to the multi–dimensional Laguerre function expansions of Hermite type. We prove that, under the assumption $\alpha_i \geq -1/2,\ \alpha_i \notin (-1/2, 1/2)$, these operators are Calderón-Zygmund operators. Consequently, their mapping properties follow by the general theory.

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1. Introduction

The study of multipliers for various Laguerre systems began with the paper of Długosz [3]. In [17] Stempak and Trebels studied multipliers for Laguerre expansions of convolution type. Recent papers dealing with Laplace transform type multipliers for the same Laguerre system are the articles by Drelichman, Durán, de Nápoli [4], and Szarek [18]. These types of multipliers, again for Laguerre expansions of convolution type, were also studied by Nowak and Szarek in [12]. In [13] Sasso treated the topic in the Laguerre polynomials setting. Laplace transform type multipliers have been also considered for continuous orthogonal systems, see for instance Betancor, Martínez and Rodríguez-Mesa [1].

In this article we study Laplace transform type and Laplace-Stieltjes type multipliers associated with Laguerre function expansions of Hermite type (see Section 2 for the definitions). Laplace transform type multipliers are given by $m_\kappa(x) = x\int_0^\infty e^{-xt}\kappa(t)\,dt$ and have their roots in Stein's monograph [14, p. 58, 121]. Laplace-Stieltjes type multipliers, $m_\mu(x) = \int_0^\infty e^{-xt}\,d\mu(t)$, are defined according to [4] (this definition has been also used in [12] and [18]). To treat Laplace transform type multipliers we use methods developed in [10], supported by an adaptation of the technicalities from [21]. In this context the paper is a generalization of the results obtained by the author in [20]. Note that the (unweighted) L^p-boundedness, $1 < p < \infty$, of the multiplier operator $m_\kappa(\mathcal{L}_\alpha)$ follows from the refinement of Stein's general Littlewood-Paley theory for semigroups (see [14, Corollary 3, p. 121]) due to Coifman, Rochberg and Weiss [2]. The assumption needed here is the L^p contractivity of $e^{-t\mathcal{L}_\alpha}$, which is true precisely for $\alpha \in (\{-1/2\} \cup [1/2, \infty))^d$, see [9]. To treat Laplace-Stieltjes type multipliers we use some pointwise estimates for the heat-semigroup kernel, see Lemma 2.4.

The main result of our paper is Theorem 2.5. We prove it assuming $\alpha \in (\{-1/2\} \cup [1/2, \infty))^d$. However, with the sole exception of the smoothness conditions, all the partial results of our paper are valid under the weaker assumption $\alpha \in [-1/2, \infty)^d$. Since the techniques we use break down if $\alpha_k \in (-1/2, 1/2)$, for some $k = 1, \dots, d$, we do not know if the smoothness estimates (or even its weaker variants) hold in this case. This is in contrast with analogous smoothness estimates in the case of Laguerre function expansions of convolution type, which are true for all $\alpha \in (-1, \infty)^d$, see [12, Theorem 3.1]. The difference between these two cases of orthogonal expansions, as enlightened in [9], is the fact that the heat semigroup for Laguerre function expansions of convolution type is L^p contractive for all $\alpha \in (-1, \infty)^d$, whereas, in our case we have L^p contractivity only for $\alpha \in (\{-1/2\} \cup [1/2, \infty))^d$. Note that the lack of L^p contractivity if some $\alpha_k \in (-1/2, 1/2)$, $k = 1, \dots, d$, prevents us from using the general theory for the multiplier m_κ in this case. Therefore, if $\alpha_k \in (-1/2, 1/2)$, for some, $k = 1, \dots, d$, we do not know not only if the operators $m_\kappa(\mathcal{L}_\alpha)$ are Calderón-Zygmund operators, but also if they are bounded on L^p, $p \in (1, \infty) \setminus \{2\}$. Having Theorem 2.5 and Proposition 2.6, we are able to use the general Calderón-Zygmund theory. Thus, in Corollary 2.7, we obtain that both $m_\kappa(\mathcal{L}_\alpha)$ and $m_\mu(\mathcal{L}_\alpha)$, are bounded from $L^1(w)$ to $L^{1,\infty}(w)$, and on all $L^p(w)$, $1 < p < \infty$, spaces (with weights w from the A_p Muckenhoupt class).

The paper is organized as follows. Section 2 contains the setup, the definitions of both Laplace transform type and Laplace-Stieltjes type multipliers, basic lemmata and the statement of the main theorem. In particular we give the definitions of the Calderón-Zygmund kernels $K^\alpha_\kappa(x, y)$ and $K^\alpha_\mu(x, y)$ associated in the sense of the Calderón-Zygmund theory to $m_\kappa(\mathcal{L}_\alpha)$ and $m_\mu(\mathcal{L}_\alpha)$, respectively, see Proposition 2.6. Section 3 is devoted to the proof of the main theorem. In subsection 3.1 we justify the growth and smoothness conditions for the kernel $K^\alpha_\kappa(x, y)$, by referring to analogous proofs from [21]. Therefore we omit most of details. In subsection 3.2 we show the growth and smoothness estimates for the kernel $K^\alpha_\mu(x, y)$.

Throughout the paper we use a fairly standard notation with all symbols referring to $\mathbb{R}^d_+ = (0, \infty)^d$. Thus $A_p = A_p(\mathbb{R}^d_+)$ stands for the Muckenhoupt class of A_p weights, $L^p(w) = L^p(\mathbb{R}^d_+, w(x)dx)$ denotes the weighted L^p space (w being a non-negative weight on $\mathbb{R}^d_+$); we simply write L^p if $w \equiv 1$. By $\langle f, g\rangle$ we mean the canonical inner product in L^2. The symbol ∇_x represents the gradient operator with respect to the x variable. The notation $X \lesssim Y$ will be used to indicate that $X \le CY$ with a positive constant C independent of significant quantities. We write $X \approx Y$ when $X \lesssim Y$ and $Y \lesssim X$. We shall also make a frequent use, often without mentioning it in relevant places, of the fact that for any $A > 0$ and $a \ge 0$,

$$\sup_{t>0} t^a e^{-At} = C_{a,A} < \infty.$$

2. Preliminaries

Since the setting and majority of the notation we use are the same as in [20] and [21] we shall be brief. Let $\varphi_k^\alpha(x) = \varphi_{k_1}^{\alpha_1}(x_1) \cdot \ldots \cdot \varphi_{k_d}^{\alpha_d}(x_d)$ be the system of d-dimensional Laguerre functions of Hermite type (as in [19, 6.4.12]), with $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in (-1, \infty)^d$. Each φ_k^α is an eigenfunction of the operator

$$L_\alpha = -\Delta + |x|^2 + \sum_{i=1}^d \frac{1}{x_i^2}\left(\alpha_i^2 - \frac{1}{4}\right)$$

corresponding to the eigenvalue $\lambda_{|k|}^\alpha = 4|k| + 2|\alpha| + 2d$; here Δ is the Laplacian restricted to $\mathbb{R}_+^d$, $|\alpha| = \alpha_1 + \ldots + \alpha_d$ and $|k| = k_1 + \ldots + k_d$ is the length of k. Moreover, $\{\varphi_k^\alpha : k \in \mathbb{N}^d\}$ is an orthonormal basis in L^2.

Let $\mathcal{L}_\alpha$ be the self–adjoint extension of L_α for which the spectral decomposition is given by φ_k^α and let $\{T_t^\alpha\}$ denote the heat-diffusion semigroup $T_t^\alpha = e^{-t\mathcal{L}_\alpha}$. Then for $f \in L^2$

$$T_t^\alpha f(x) = \int_{\mathbb{R}_+^d} \mathcal{G}_t^\alpha(x, y) f(y)\, dy, \qquad x \in \mathbb{R}_+^d,$$

with

$$\mathcal{G}_t^\alpha(x, y) = \sum_{n=0}^\infty e^{-t\lambda_n^\alpha} \sum_{|k|=n} \varphi_k^\alpha(x)\varphi_k^\alpha(y) \tag{2.1}$$

$$= (\sinh 2t)^{-d} \exp\left(-\frac{1}{2}\coth 2t\left(|x|^2 + |y|^2\right)\right) \prod_{i=1}^d \sqrt{x_i y_i}\, I_{\alpha_i}\left(\frac{x_i y_i}{\sinh 2t}\right),$$

where I_ν, $\nu > -1$, is the modified Bessel function of the first kind and order ν. It is known, that $I_\nu(z)$, as a function of $z > 0$, is real, positive, smooth and satisfies

$$\frac{d}{dz} I_\nu(z) = \frac{\nu}{z} I_\nu(z) + I_{\nu+1}(z). \tag{2.2}$$

We shall use the standard asymptotics,

$$I_\nu(z) \approx z^\nu, \quad z \to 0^+, \qquad I_\nu(z) \approx z^{-\frac{1}{2}} e^z, \quad z \to \infty, \tag{2.3}$$

and the following lemmata.

Lemma 2.1. *(See [7]). Let $\nu \geq -1/2$. Then*

$$0 < I_\nu(z) - I_{\nu+1}(z) < \frac{2(\nu+1)}{z} I_{\nu+1}(z), \qquad z > 0.$$

Lemma 2.2. *(See [21]). Let $\nu \geq -1/2$, then*

$$z\left[2I_{\nu+1}(z) - I_{\nu+2}(z) - I_\nu(z)\right] - I_{\nu+1}(z) = O(z^{-3/2}e^z), \quad z \to \infty.$$

Lemma 2.3. *(See [15].) Given $a > 1$, we have*

$$\int_0^1 u^{-a} \exp\left(-\frac{C|x-y|^2}{u}\right) du \lesssim |x-y|^{-2a+2}, \qquad x, y \in \mathbb{R}^d_+.$$

The following Lemma shows that the heat kernel $\mathcal{G}^\alpha_t(x,y)$ is dominated up to a multiplicative constant by the heat kernel

$$G_t(x,y) = (2\pi)^{-d/2}(\sinh 2t)^{-d/2} \exp\left(-\frac{1}{4\tanh t}|x-y|^2 - \frac{\tanh t}{4}|x+y|^2\right)$$

corresponding to the harmonic oscillator.

Lemma 2.4. *Given $\alpha \in [-1/2,\infty)^d$, there exists a constant C_α such that*

$$\mathcal{G}^\alpha_t(x,y) \leq C_\alpha\, G_t(x,y), \qquad t > 0, \quad x, y \in \mathbb{R}^d_+.$$

Proof. It is perhaps noteworthy that, for $\alpha \in [1/2,\infty)^d$ we can take $C_\alpha = 1$, while for $\alpha = \alpha_0 = (-1/2, \cdots, -1/2)$ we can take $C_\alpha = 2^d$. The proof of this statement makes use of the monotonicity of the function $[1/2,\infty) \ni \nu \to I_\nu(z)$ (here $z \in (0,\infty)$ is fixed), see [10, Proposition 2.1]. Since we are not interested in the precise value of C_α we use the standard asymptotics for the Bessel function (2.3), obtaining the inequalities

$$I_{\alpha_i}\left(\frac{x_i y_i}{\sinh 2t}\right) \lesssim (x_i y_i)^{-1/2}(\sinh 2t)^{1/2} \exp\left(\frac{x_i y_i}{\sinh 2t}\right), \qquad i = 1, \ldots, d.$$

The above in turn implies $\mathcal{G}^\alpha_t(x,y) \lesssim (\sinh 2t)^{-d/2} e^{-T}$, where we set $T = \frac{1}{2}\coth 2t(|x|^2 + |y|^2) - \sum_{i=1}^d \frac{x_i y_i}{\sinh 2t}$. Now, an elementary computation shows that $T = \frac{1}{4\tanh t}|x-y|^2 + \frac{\tanh t}{4}|x+y|^2$, hence the lemma follows. □

According to [14] we call a function $m : (0,\infty) \to \mathbb{C}$ a *Laplace transform type multiplier* if it is of the form

$$m(x) = m_\kappa(x) = x \int_0^\infty e^{-xt}\kappa(t)\, dt, \tag{2.4}$$

with κ being a bounded measurable function on $(0,\infty)$. It should be noted, that m defined as above, satisfies Mihlin's conditions of any order, that is, $|x^j m^{(j)}(x)| \leq C_j$, $j = 0,1,2,\ldots$ (in particular m is bounded). Remarkable special cases of Laplace transform type multipliers include the imaginary powers and functions of the form $m_\lambda(x) = \frac{x}{x+\lambda}$, for $\mathrm{Re}(\lambda) > 0$. The imaginary powers, $m_\gamma(x) = x^{-i\gamma}$, $\gamma \in \mathbb{R}$, $\gamma \neq 0$, are Laplace transform type multipliers corresponding to $\kappa(t) = \frac{1}{\Gamma(1+i\gamma)} t^{i\gamma}$. L^p boundedness properties of the imaginary power operators in the context of our paper have been studied by the author in [20]. Thus the present paper includes results of [20] as special cases. The other example, $m_\lambda(x)$, is a Laplace transform type multiplier corresponding to $\kappa_\lambda(t) = e^{-\lambda t}$. Since $m_\lambda(x) = 1 - \frac{\lambda}{x+\lambda}$, the resulting operator is the sum $I + \lambda(-\lambda - \mathcal{L}_\alpha)^{-1}$, of the identity operator I and a constant λ times the resolvent operator $(-\lambda - \mathcal{L}_\alpha)^{-1}$.

Assume now that $\alpha \in [-1/2, \infty)^d$. We call a function $m : (0, \infty) \to \mathbb{C}$ a *Laplace-Stieltjes type multiplier* if it is of the form

$$m(x) = m_\mu(x) = \int_0^\infty e^{-xt}\, d\mu(t), \tag{2.5}$$

where μ is a complex Borel measure on $(0, \infty)$ with total variation $|\mu|$ satisfying the condtition $\int_0^\infty e^{-td} d|\mu|(t) < \infty$. The latter assumption is a technical one, well suited for our setting. It implies in particular that $m(x)$ is bounded on the halfline $[d, \infty)$ (hence, in view of $\alpha \in [-1/2, \infty)^d$, also on the spectrum $\sigma(\mathcal{L}_\alpha) = \{\lambda_n^\alpha : n \in \mathbb{N}\}$). Here, remarkable special cases of Laplace-Stieltjes type multiplier operators include the Laguerre fractional integral operators I^σ, $\sigma > 0$, and the resolvent operators $R_\lambda = (-\lambda - \mathcal{L}_\alpha)^{-1}$, for $\mathrm{Re}(\lambda) > -d$. The operators $I^\sigma = (\mathcal{L}_\alpha)^{-\sigma}$, correspond to the choice of $m_\sigma(x) = x^{-\sigma}$, and $d\mu_\sigma(t) = \frac{1}{\Gamma(\sigma)} t^{\sigma-1}\, dt$. $L^p - L^q$ boundedness properties of these operators have been studied by Nowak and Stempak in [11]. Thus Theorem 2.5 of this paper contains an enhancement of [11, Theorem 3.1] in the case $p = q$. The resolvent operators correspond to the choice of $m_\lambda(x) = -\frac{1}{x+\lambda}$ and $d\mu_\lambda(t) = -e^{-\lambda t}\, dt$.

It should be noted that in many cases the two definitions are comparable up to a constant. Namely, if we assume for example that κ is bounded and continuously differentiable, $\lim_{t\to 0+} \kappa(t) = \kappa(0)$ exists, and κ' is integrable then

$$x \int_0^\infty e^{-xt} \kappa(t)\, dt = \kappa(0) + \int_0^\infty e^{-xt} \kappa'(t)\, dt.$$

The left hand side of the above equation is a Laplace transform type multiplier m_κ of the function κ, while the right hand side is a constant plus a Laplace-Stieltjes type multiplier m_μ of the measure μ with the density $\kappa'(t)$. On the level of multiplier operators the above equation says that, $m_\kappa(\mathcal{L}_\alpha) - m_\mu(\mathcal{L}_\alpha) = \kappa(0)I$, where I is the identity operator. From now on, whenever we refer to both Laplace transform type and Laplace-Stieltjes type multipliers, for short we use the phrase 'Laplace type multiplier'.

For a Laplace type multiplier m by the spectral theorem we have

$$m(\mathcal{L}_\alpha) f = \sum_{k \in \mathbb{N}^d} m\big(4|k| + 2|\alpha| + 2d\big) \langle f, \varphi_k^\alpha \rangle\, \varphi_k^\alpha, \qquad f \in L^2. \tag{2.6}$$

Since in both definitions (2.4) and (2.5) the function m is bounded on the spectrum of $\mathcal{L}_\alpha$, $m(\mathcal{L}_\alpha)$ is a bounded operator on L^2. Motivated by the fact that (at least formally) we have

$$m_\kappa(\mathcal{L}_\alpha)f(x) = \mathcal{L}_\alpha \int_0^\infty e^{-t\mathcal{L}_\alpha} f(x)\,\kappa(t)\,dt = \int_0^\infty -\frac{d}{dt}T_t^\alpha f(x)\,\kappa(t)\,dt$$
$$= \int_{\mathbb{R}_+^d}\Big(-\int_0^\infty \frac{d}{dt}\mathcal{G}_t^\alpha(x,y)\kappa(t)\,dt\Big) f(y)\,dy,$$

we *define* the kernel of the Laplace transform type multiplier m_κ as

$$K^\alpha(x,y) = K_\kappa^\alpha(x,y) = -\int_0^\infty \frac{d}{dt}\mathcal{G}_t^\alpha(x,y)\kappa(t)\,dt. \tag{2.7}$$

Analogously, the formal computations

$$m_\mu(\mathcal{L}_\alpha)f(x) = \int_0^\infty e^{-t\mathcal{L}_\alpha} f(x)\,d\mu(t) = \int_{\mathbb{R}_+^d}\Big(\int_0^\infty \mathcal{G}_t^\alpha(x,y)d\mu(t)\Big) f(y)\,dy,$$

lead us to *define* the kernel of the Laplace-Stieltjes type multiplier m_μ as

$$K^\alpha(x,y) = K_\mu^\alpha(x,y) = \int_0^\infty \mathcal{G}_t^\alpha(x,y)d\mu(t). \tag{2.8}$$

From the estimates that follow it can be deduced that the definitions (2.7), (2.8) are valid for $x \neq y$. The main result of our paper is the following.

Theorem 2.5. *Let* $\alpha \in (\{-1/2\}\cup[1/2,\infty))^d$ *and* m *be a Laplace type multiplier given either by* (2.4) *or by* (2.5). *Then the kernel* $K^\alpha(x,y)$ *given either by* (2.7) *or by* (2.8), *respectively, satisfies the growth condition*

$$|K^\alpha(x,y)| \lesssim |x-y|^{-d}, \qquad x,y \in \mathbb{R}_+^d, \tag{2.9}$$

and the smoothness condition

$$|\nabla_x K^\alpha(x,y)| + |\nabla_y K^\alpha(x,y)| \lesssim |x-y|^{-d-1}, \qquad x,y \in \mathbb{R}_+^d. \tag{2.10}$$

The restriction $\alpha \in [-1/2,\infty)^d$ is to some extent natural, see [10, p. 407] for additional comments. For some comments on the further restriction $\alpha \in (\{-1/2\}\cup[1/2,\infty))^d$ see the third paragraph of Section 1 of the present paper. Methods from [16] can be easily adapted to prove the following.

Proposition 2.6. *Let* $\alpha \in [-1/2,\infty)^d$ *and* m *be a Laplace type multiplier given either by* (2.4) *or by* (2.5). *Then the multiplier operator defined by* (2.6), *is associated with the kernel* $K^\alpha(x,y)$ *given either by* (2.7) *or by* (2.8), *respectively, in the sense that for any two functions* $f,g \in C_c^\infty\left(\mathbb{R}_+^d\right)$ *with disjoint supports we have*

$$\langle m(\mathcal{L}_\alpha)f,g\rangle = \int_{\mathbb{R}_+^d}\int_{\mathbb{R}_+^d} K^\alpha(x,y)f(y)\overline{g(x)}\,dy\,dx.$$

Proof. In the case of the multiplier m_κ we need to slightly modify the reasoning from [20]. It is enough to observe that the relevant proof in [20] is also valid if we replace therein $t^{-i\gamma}$ by the bounded function $\kappa(t)$. In the case of the multiplier m_μ, looking in detail at the argument used in the proof of [16, Proposition 4.2], together with some pointwise estimates for the Laugerre functions φ_k^α, see for instance [8, Section 5], shows that in order to repeat that argument in the present situation it is enough to verify that

$$\int_{\mathbb{R}_+^d}\int_{\mathbb{R}_+^d}\int_0^\infty \left|\mathcal{G}_t^\alpha(x,y)f(y)\overline{g(x)}\right|\,d|\mu|(t)\,dy\,dx<\infty.$$

The above follows from the proof of the growth condition for the kernel $K_\mu^\alpha(x,y)$, see Section 3.2. □

By the general Calderón-Zygmund theory, see for instance [5], combining Theorem 2.5, Proposition 2.6 and the fact that $m(\mathcal{L}_\alpha)$ is bounded on L^2, we also get the following.

Corollary 2.7. *Let $\alpha\in(\{-1/2\}\cup[1/2,\infty))^d$ and m be a Laplace type multiplier given either by (2.4) or by (2.5). Then the multiplier operator $m(\mathcal{L}_\alpha)$, defined initially on L^2 by (2.6), extends uniquely to a bounded operator on $L^p(w)$, $1<p<\infty$, $w\in A_p$, and to a bounded operator from $L^1(w)$ to $L^{1,\infty}(w)$, $w\in A_1$.*

3. Proof of Theorem 2.5

3.1. The case of $K_\kappa^\alpha(x,y)$

Since the structure of the proofs and the proofs themselves are similar to those from [21], we shall be brief. We use the change of variable

$$t=t(u)=\frac{1}{2}\log\frac{1+u}{1-u}. \tag{3.1}$$

It seems that the proof would be less complicated without using the above, however since in [21] we relied deeply on (3.1) and here we concentrate on adapting the reasoning from the latter paper, for the sake of readers convenience, we maintain using (3.1). Till the end of this subsection, the following notation from [21] will be used:

$$A=A(u)=\frac{1+u^2}{4u},\quad B=B(u)=\frac{1-u^2}{2u},\quad Z_i=Z_i(u,x_i,y_i)=Bx_iy_i,$$

$$\sqrt{xy}=\prod_{i=1}^d\sqrt{x_iy_i},\qquad I_\alpha(Z)=\prod_{i=1}^d I_{\alpha_i}(Z_i),$$

$$D_\nu=D_\nu(x_1,y_1,u)=-2Ax_1I_\nu(Z_1)+By_1I_{\nu+1}(Z_1)+\frac{\nu+1/2}{x_1}I_\nu(Z_1),$$

$$W=\sum_{i=1}^d Z_i-A(|x|^2+|y|^2),\ W_k=W-Z_k,\ U_k=x_k^2+y_k^2+\sum_{i\neq k}(x_i-y_i)^2.$$

Further, in several places we will need to split D_ν; as it can be easily seen

$$D_\nu=M_1^\nu+M_2^\nu+M_3^\nu+\frac{\nu+1/2}{x_1}I_\nu(Z_1),$$

where

$$M_1^\nu = (-x_1 + y_1)\frac{1}{2u} I_\nu(Z_1), \qquad M_2^\nu = -(x_1+y_1)\frac{u}{2} I_\nu(Z_1),$$
$$M_3^\nu = By_1(I_{\nu+1}(Z_1) - I_\nu(Z_1)). \tag{3.2}$$

In the proof of the standard estimates for the kernel $K^\alpha(x,y)$ the following lemma justified in [21] will be used.

Lemma 3.1. *Let $u \in (0,1)$, $\alpha \in [-1/2,\infty)^d$, $i = 1,\dots,d$. We have*

(A) $\log(\frac{1+u}{1-u}) \approx u$ *for* $u \in (0,1/2)$, $\log(\frac{1+u}{1-u}) \lesssim (1-u)^{-1/2}$ *for* $u \in (1/2,1)$; *consequently,* $\log\frac{1+u}{1-u} \lesssim u\,(1-u)^{-1/2}$.

(B) $A \lesssim u^{-1}$, $B \lesssim u^{-1}$; *moreover, if* $u \in (1/2,1)$, *then* $B \le 3/4$ *and* $-A \le -1/2$.

(C) $Z_i \approx (1-u)u^{-1}x_iy_i$; *in particular for* $u \in (0,1/2)$, $Z_i \approx \frac{x_iy_i}{u}$, *while for* $u \in (1/2,1)$, $Z_i \approx (1-u)x_iy_i$.

(D) $W = -\frac{1}{4u}|x-y|^2 - \frac{u}{4}|x+y|^2 \le -\frac{1}{4u}|x-y|^2$.

(E) *If* $u \in (1/2,1)$, *then* $W \le -\frac{1}{8}|x+y|^2 - \frac{1}{4}|x-y|^2$.

(F) $W_k \le -\frac{1}{4u}U_k \le -\frac{1}{4u}|x-y|^2$.

(G) *If* $Z_1 < 1$, *then* $I_{\alpha_1}(Z_1) \lesssim (\frac{1-u}{u})^{\alpha_1}(x_1y_1)^{\alpha_1} \lesssim u^{1/2}(1-u)^{-1/2}(x_1y_1)^{-1/2}$.

(H) $I_{\alpha_i}(Z_i) \lesssim (x_iy_i)^{-1/2}u^{1/2}(1-u)^{-1/2}e^{Z_i}$.

(I) $|I_{\alpha_i+1}(Z_i) - I_{\alpha_i}(Z_i)| \lesssim (x_iy_i)^{-1}u(1-u)^{-1}I_{\alpha_i+1}(Z_i) \lesssim (x_iy_i)^{-3/2} \times u^{3/2}(1-u)^{-3/2}e^{Z_i}$.

We will first justify the growth estimate (2.9), under the assumption $\alpha \in [-1/2,\infty)^d$. We follow the outline of the proof of the growth condition for the kernel $K(x,y)$ from [21, pp. 56–58]. Using (3.1) we see that it suffices to show that $\int_0^1 |K_u(x,y)|\,du \lesssim |x-y|^{-d}$, where

$$K_u(x,y) = (1-u^2)^{-1}\kappa(t)\frac{d}{dt}\mathcal{G}_t^\alpha(x,y)\Big|_{t=\frac{1}{2}\log\frac{1+u}{1-u}},$$

so that $K^\alpha(x,y) = \int_0^1 K_u(x,y)\,du$. Differentiating (2.1), with the aid of (2.2), we see that $\frac{d}{dt}\mathcal{G}_t^\alpha(x,y)$ equals

$$-2d\cosh 2t(\sinh 2t)^{-d-1}e^{-\frac{1}{2}\coth 2t\left(|x|^2+|y|^2\right)}\prod_{i=1}^d \sqrt{x_iy_i}\,I_{\alpha_i}\left(\frac{x_iy_i}{\sinh 2t}\right)$$
$$+\left(|x|^2+|y|^2\right)(\sinh 2t)^{-d-2}e^{-\frac{1}{2}\coth 2t\left(|x|^2+|y|^2\right)}\prod_{i=1}^d \sqrt{x_iy_i}\,I_{\alpha_i}\left(\frac{x_iy_i}{\sinh 2t}\right)$$
$$-2\cosh 2t(\sinh 2t)^{-d-2}e^{-\frac{1}{2}\coth 2t\left(|x|^2+|y|^2\right)}S_t^\alpha(x,y)\prod_{i=1}^d \sqrt{x_iy_i},$$

with

$$S_t^\alpha(x,y) = \sum_{j=1}^d \left[\alpha_j \sinh 2t\ I_{\alpha_j}\left(\frac{x_jy_j}{\sinh 2t}\right) + x_jy_jI_{\alpha_j+1}\left(\frac{x_jy_j}{\sinh 2t}\right)\right]\prod_{i\neq j} I_{\alpha_i}\left(\frac{x_iy_i}{\sinh 2t}\right).$$

After some rearrangement of terms we write $\cosh 2t\, S_t^\alpha(x,y)$ as

$$\begin{aligned}&|\alpha|\cosh 2t \sinh 2t \prod_{i=1}^d I_{\alpha_i}\left(\frac{x_iy_i}{\sinh 2t}\right) + \langle x,y\rangle \prod_{i=1}^d I_{\alpha_i}\left(\frac{x_iy_i}{\sinh 2t}\right)\\ &+\sum_{j=1}^d x_jy_j\left[I_{\alpha_j+1}\left(\frac{x_jy_j}{\sinh 2t}\right) - I_{\alpha_j}\left(\frac{x_jy_j}{\sinh 2t}\right)\right]\prod_{i\neq j} I_{\alpha_i}\left(\frac{x_iy_i}{\sinh 2t}\right)\\ &+(\cosh 2t-1)\sum_{j=1}^d x_jy_jI_{\alpha_j+1}\left(\frac{x_jy_j}{\sinh 2t}\right)\prod_{i\neq j} I_{\alpha_i}\left(\frac{x_iy_i}{\sinh 2t}\right).\end{aligned}$$

Using the above expression for $\cosh 2t\, S_t^\alpha(x,y)$, together with the fact that the change of variables (3.1) transforms $\sinh 2t$ into $\frac{2u}{1-u^2}$ and $\cosh 2t$ into $\frac{1-u^2}{1+u^2}$, we decompose $K_u(x,y) = \sum_{i=1}^4 K_i$, where

$$\begin{aligned}K_1 &= -2(d+|\alpha|)\widetilde{\kappa}(u)(2u)^{-d-1}(1-u^2)^{d-1}(1+u^2)\sqrt{xy}\, e^{-A(|x|^2+|y|^2)}\, I_\alpha(Z),\\ K_2 &= |x-y|^2\widetilde{\kappa}(u)(2u)^{-d-2}(1-u^2)^{d+1}\sqrt{xy}\, e^{-A(|x|^2+|y|^2)}\, I_\alpha(Z),\\ K_3 &= 2\widetilde{\kappa}(u)(2u)^{-d-2}(1-u^2)^{d+1}\sqrt{xy}\, e^{-A(|x|^2+|y|^2)}\\ &\qquad\times\sum_{j=1}^d x_jy_j\left[I_{\alpha_j}(Z_j) - I_{\alpha_j+1}(Z_j)\right]\prod_{i\neq j} I_{\alpha_i}(Z_i),\\ K_4 &= -\widetilde{\kappa}(u)(2u)^{-d}(1-u^2)^d\sqrt{xy}\, e^{-A(|x|^2+|y|^2)}\sum_{j=1}^d x_jy_jI_{\alpha+e_j}(Z),\end{aligned} \tag{3.3}$$

with $\widetilde{\kappa}(u) = \kappa(\tanh t)$. Therefore, to prove the growth condition for the kernel $K^\alpha(x,y)$ it suffices to show that

$$\int_0^1 |K_i|\, du \lesssim |x-y|^{-d}, \qquad i=1,\dots,4. \tag{3.4}$$

Now, to avoid collision of symbols, we denote the expressions K_i, $i=1,\dots,4$, appearing in the proof of the growth condition for the kernel $K(x,y)$ in [21, p. 57] by $\overline{K}_i$, $i=1,\dots,4$. Then obviously, $K_i = \widetilde{\kappa}(u)(1-u^2)^{-1/2}\overline{K}_i$. Since $\widetilde{\kappa}$ is bounded we may focus on showing that $\int_0^1 |\overline{K}_i|(1-u^2)^{-1/2}\, du \lesssim |x-y|^{-d}$, $i=1,\dots,4$. This may be accomplished by following the proof of the growth condition for the kernel $K(x,y)$ from [21], with the aid of Lemma 3.1. Details and appropriate modifications are left to the reader.

Now we pass to the justification of the smoothness condition (2.10) for the kernel $K^\alpha(x,y)$. From the symmetry in x and y of the kernel and the fact that none of the variables $x_1, \ldots, x_d$ is distinguished we see that in order to prove (2.10) it suffices to show that

$$\left| \int_0^1 \partial_{x_1} K_u(x,y)\, du \right| \lesssim |x-y|^{-d-1},$$

(differentiation under the integral sign is implicitly justified by the estimates that follow). Using (2.2) we see that $\partial_{x_1} \left(I_\nu(Z_1) e^{-A(|x|^2+|y|^2)} \sqrt{x_1 y_1} \right) = D_\nu e^{-A(|x|^2+|y|^2)} \sqrt{x_1 y_1}$. Hence, $\partial_{x_1} K_u(x,y) = \sum_{k=1}^9 J_k^\alpha = \sum_{k=1}^9 J_k^\alpha(x,y,u)$, where

$$J_1^\alpha = -\, 2(d+|\alpha|)\widetilde{\kappa}(u)(2u)^{-d-1}(1-u^2)^{d-1}(1+u^2)\sqrt{xy}e^{-A(|x|^2+|y|^2)} D_{\alpha_1} \times \prod_{i=2}^d I_{\alpha_i}(Z_i),$$

comes from differentiating K_1,

$$J_2^\alpha = 2(x_1-y_1)\widetilde{\kappa}(u)(2u)^{-d-2}(1-u^2)^{d+1}\sqrt{xy}e^{-A(|x|^2+|y|^2)} I_\alpha(Z),$$

$$J_3^\alpha = |x-y|^2\widetilde{\kappa}(u)(2u)^{-d-2}(1-u^2)^{d+1}\sqrt{xy}e^{-A(|x|^2+|y|^2)} D_{\alpha_1} \prod_{i=2}^d I_{\alpha_i}(Z_i),$$

come from differentiating K_2,

$$J_4^\alpha = 2\widetilde{\kappa}(u)(2u)^{-d-2}(1-u^2)^{d+1}\sqrt{xy}e^{-A(|x|^2+|y|^2)} \times (D_{\alpha_1} - D_{\alpha_1+1})\, x_1 y_1 \prod_{i=2}^d I_{\alpha_i}(Z_i),$$

$$J_5^\alpha = 2\widetilde{\kappa}(u)(2u)^{-d-2}(1-u^2)^{d+1} y_1 \sqrt{xy}e^{-A(|x|^2+|y|^2)} \times (I_{\alpha_1}(Z_1) - I_{\alpha_1+1}(Z_1)) \prod_{i=2}^d I_{\alpha_i}(Z_i),$$

$$J_6^\alpha = 2\widetilde{\kappa}(u)(2u)^{-d-2}(1-u^2)^{d+1}\sqrt{xy}e^{-A(|x|^2+|y|^2)} \times D_{\alpha_1} \sum_{j=2}^d x_j y_j (I_{\alpha_j}(Z_j) - I_{\alpha_j+1}(Z_j)) \prod_{i\neq 1,j} I_{\alpha_i}(Z_i),$$

come from differentiating K_3, and finally

$$J_7^\alpha = -\widetilde{\kappa}(u)(2u)^{-d}(1-u^2)^d y_1 \sqrt{xy}e^{-A(|x|^2+|y|^2)} I_{\alpha+e_1}(Z),$$

$$J_8^\alpha = -\widetilde{\kappa}(u)(2u)^{-d}(1-u^2)^d x_1 y_1 \sqrt{xy}e^{-A(|x|^2+|y|^2)} D_{\alpha_1+1} \prod_{i=2}^d I_{\alpha_i}(Z_i),$$

$$J_9^\alpha = -\widetilde{\kappa}(u)(2u)^{-d}(1-u^2)^d \sqrt{xy}e^{-A(|x|^2+|y|^2)} \times D_{\alpha_1+1} \sum_{j=2}^d x_j y_j I_{\alpha_j+1}(Z_j) \prod_{i\neq j,1}^d I_{\alpha_i}(Z_i),$$

come from differentiating K_4. Therefore, the proof of the smoothness condition for the kernel $K^\alpha(x,y)$ will follow, if we show that, for $k=1,\dots,9$,

$$\int_0^1 \left|J_k^\alpha(x,y,u)\right| du \lesssim |x-y|^{-d-1}. \tag{3.5}$$

As previously, to avoid collision of symbols, we denote the expressions J_k^α, $k=1,\dots,9$, appearing in the proof of the smoothness condition for the kernel $K(x,y)$ in [21, pp. 58–59] by $\overline{J_k^\alpha}$, $k=1,\dots,9$. Then, clearly $J_k^\alpha = \kappa(u)(1-u^2)^{-1/2}\,\overline{J_k^\alpha}$ and since κ is bounded our task reduces to showing that $\int_0^1 (1-u^2)^{-1/2}|\overline{J_k^\alpha}(x,y,u)|\,du \lesssim |x-y|^{-d-1}$. This may be obtained by using Lemma 3.1 and following (step by step) the scheme of the analogous proof from [21, pp. 58–66]. As in the latter paper, when $Z_1>1$, we also need to use the splitting (3.2). In the present paper the whole task is a bit simpler (at least in notation) than in [21], since the kernel $K^\alpha(x,y)$ is not vector–valued, therefore we can use the gradient condition and do not need to use the mean value theorem. The most subtle part is for $k=4$, when we also need to use Lemma 2.2. We omit the details, however for the sake of completeness, we state the analogues of [21, Lemmata 4.2, 4.3, 4.4] in our context.

Lemma 3.2. *Let $\nu=-1/2$ or $\nu\in[1/2,\infty)$, and $b\ge 0$. Then*

$$E_{\nu,b} = (|x|+|y|)^b \int_{1/2}^1 (1-u)^{d/2-1/2}\left|D_\nu\sqrt{x_1y_1}e^{W_1}\right| du \lesssim |x-y|^{-d-1}.$$

Lemma 3.3. *Let $\nu=-1/2$ or $\nu\in[1/2,\infty)$, and $a\ge 0$, $b\ge 0$. Then,*

$$R = R_{\nu,a,b} = (x_1+y_1)^b \int_0^{1/2} u^{-d/2-a}\left|D_\nu\sqrt{x_1y_1}e^{W_1}\right| du \lesssim |x-y|^{-d-2a+2-b}.$$

Lemma 3.4. *Let $a\ge 0$, $b\ge 0$. Then,*

$$\begin{aligned} X = X_{a,b} = X_{a,b}(x,y) &= \int_{(0,1/2)\cap\{Z_1>1\}} \frac{1}{(x_1)^b} u^{-d/2-a} e^{-\frac{C}{u}|x-y|^2}\, du \\ &\lesssim |x-y|^{-d-2a+2-b}. \end{aligned}$$

The proofs of the above, are all similar to the proofs of [21, Lemmata 4.2, 4.3, 4.4], therefore we omit them.

3.2. The case of $K_\mu^\alpha(x,y)$

This time we will not use the change of the variable (3.1). As previously, we start with justifying the growth estimate (2.9). By assumptions made on the measure μ, in order to justify (2.9) it is enough to check that

$$\mathcal{G}_t^\alpha(x,y) \lesssim e^{-td}|x-y|^{-d}, \qquad t>0, \quad x,y\in\mathbb{R}_+^d. \tag{3.6}$$

This is easy since by Lemma 2.4 we have

$$\mathcal{G}_t^\alpha(x,y) \lesssim |x-y|^{-d}\left(\frac{\tanh t}{\sinh 2t}\right)^{d/2}\left(\frac{|x-y|^2}{\tanh t}\right)^{d/2}\exp\left(-\frac{|x-y|^2}{4\tanh t}\right)$$

and then (3.6) follows.

Proving the smoothness estimate (2.10) for the kernel $K_\mu^\alpha(x,y)$ note that once again the symmetry reasons and the fact that none of the variables $x_1, \dots, x_d$ is distinguished, reduce our task to showing that

$$\left| \int_0^\infty \partial_{x_1} \mathcal{G}_t^\alpha(x,y)\, d\mu(t) \right| \lesssim |x-y|^{-d-1}$$

(differentiation under the integral sign is implicitly justified by the estimates that follow). Again, it is enough to check the pointwise estimate

$$|\partial_{x_1} \mathcal{G}_t^\alpha(x,y)| \lesssim e^{-td}|x-y|^{-d-1}, \qquad t>0, \quad x,y \in \mathbb{R}^d_+. \tag{3.7}$$

With an aid of (2.2) a computation shows that

$$\begin{aligned} \partial_{x_1} \mathcal{G}_t^\alpha &= \left(-x_1 \coth 2t + \frac{\alpha_1 + 1/2}{x_1} \right) \mathcal{G}_t^\alpha + \frac{y_1}{\sinh 2t} \mathcal{G}_t^{\alpha+e_1} \qquad (3.8)\\ &= -\frac{1}{2} \left((x_1 - y_1) \coth t + (x_1+y_1) \tanh t - \frac{\alpha_1 + 1/2}{x_1} \right) \mathcal{G}_t^\alpha \\ &\quad + \frac{y_1}{\sinh 2t} \left(\mathcal{G}_t^{\alpha+e_1} - \mathcal{G}_t^\alpha \right). \end{aligned}$$

The use of the first identity above and the fact that $\mathcal{G}_t^{\alpha+e_1} \le \mathcal{G}_t^\alpha$ (which follows from Lemma 2.1) allow to write

$$|\partial_{x_1} \mathcal{G}_t^\alpha(x,y)| \lesssim \left((x_1+y_1) \coth 2t + \frac{\alpha_1 + 1/2}{x_1} \right) \mathcal{G}_t^\alpha(x,y). \tag{3.9}$$

In the proof of (3.7) we shall consider the cases $t \in (0,1)$ and $t \in [1,\infty)$ separately.

Assume first that $t \ge 1$. The inequality

$$\frac{1}{x_1} \mathcal{G}_t^\alpha(x,y) \lesssim y_1 G_t(x,y), \qquad t \ge 1, \quad x,y \in \mathbb{R}^d_+, \tag{3.10}$$

valid when $\alpha_1 \ge 1/2$ and which will be proved momentarily, together with Lemma 2.4, gives from (3.9)

$$|\partial_{x_1} \mathcal{G}_t^\alpha(x,y)| \lesssim (x_1+y_1) G_t(x,y), \qquad t \ge 1, \quad x,y \in \mathbb{R}^d_+,$$

if either $\alpha_1 = -1/2$ or $\alpha_1 \ge 1/2$. Verifying (3.7) for $t \ge 1$ we now simply write (using the explicit form of $G_t(x,y)$)

$$\begin{aligned} &|\partial_{x_1} \mathcal{G}_t^\alpha(x,y)| \lesssim e^{-td}(x_1+y_1) \exp\left(-\frac{\tanh 1}{4} |x+y|^2 \right) \exp\left(-\frac{1}{4}|x-y|^2 \right) \\ &\lesssim e^{-td}|x-y|^{-d-1}. \end{aligned}$$

Coming back to (3.10) note that it is a consequence of

$$I_{\alpha_1}(z) \lesssim z I_{-1/2}(z) \lesssim \sinh 2t\, z I_{-1/2}(z), \qquad t \ge 1, \quad z>0,$$

which follows from the asymptotics (2.3).

Assume now that $t<1$. We will split the reasoning according to the subcases $\frac{x_1 y_1}{\sinh 2t} < 1$ and $\frac{x_1 y_1}{\sinh 2t} \ge 1$. In the first subcase it will be justified momentarily that, for $\alpha_1 \ge 1/2$, if $0<t<1$,

$$\frac{1}{x_1} \mathcal{G}_t^\alpha(x,y) \lesssim \frac{y_1}{\sinh 2t} G_t\left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}} \right) \exp\left(-\frac{1}{4} \coth(2t)(x_1^2 + y_1^2) \right), \tag{3.11}$$

and this together with

$$\mathcal{G}_t^\alpha(x,y) \lesssim G_t\left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right) \exp\left(-\frac{1}{4}\coth(2t)(x_1^2+y_1^2)\right)$$

will give from (3.9)

$$|\partial_{x_1}\mathcal{G}_t^\alpha(x,y)| \lesssim \frac{x_1+y_1}{t}\, G_t\left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right) \exp\left(-\frac{1}{4}\coth(2t)(x_1^2+y_1^2)\right).$$

Verifying (3.7) for $0<t<1$ and under the assumption $\frac{x_1 y_1}{\sinh 2t} < 1$ we write

$$\begin{aligned}
&|\partial_{x_1}\mathcal{G}_t^\alpha(x,y)| \\
&\lesssim \frac{x_1+y_1}{\sqrt{t}} \exp\left(-\frac{1}{4}\coth(2t)(x_1^2+y_1^2)\right) t^{-(d+1)/2} \exp\left(-\frac{|x-y|^2}{8\tanh t}\right) \\
&\lesssim |x-y|^{-d-1}.
\end{aligned}$$

Proving (3.11) we note that by the asymptotics (2.3) the assumption $\alpha_1 \geq 1/2$ implies

$$I_{\alpha_1}\left(\frac{x_1y_1}{\sinh 2t}\right) \lesssim \left(\frac{x_1y_1}{\sinh 2t}\right)^{1/2}, \qquad \frac{x_1y_1}{\sinh 2t} < 1.$$

Then we use the product structure of $\mathcal{G}_t^\alpha(x,y)$, applying Lemma 2.4 to its $d-1$ dimensional factor $\mathcal{G}_t^{(\alpha_2,\ldots,\alpha_d)}(x^1,y^1)$ (notation: for $x=(x_1,\ldots,x_d)$, $x^1=(x_2,\ldots,x_d)$, $\hat{x}=(0,x_2,\ldots,x_d)$), which together with some properties of $G_t(x,y)$ allows us to bound $\mathcal{G}_t^\alpha(x,y)$ by

$$\begin{aligned}
&\sqrt{x_1y_1}\left(\frac{x_1y_1}{\sinh 2t}\right)^{1/2} \exp\left(-\frac{1}{2}\coth(2t)(x_1^2+y_1^2)\right)(\sinh 2t)^{-1/2} G_t(\hat{x},\hat{y}) \\
&\lesssim \frac{x_1y_1}{\sinh 2t} G_t\left(\frac{\hat{x}}{\sqrt{2}}, \frac{\hat{y}}{\sqrt{2}}\right) \exp\left(-\frac{1}{2}\coth(2t)(x_1^2+y_1^2)\right) \\
&\lesssim \frac{x_1y_1}{\sinh 2t} G_t\left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right) \exp\left(-\frac{1}{4}\coth(2t)(x_1^2+y_1^2)\right).
\end{aligned}$$

In the second subcase, $\frac{x_1y_1}{\sinh 2t} \geq 1$, first we use Lemma 2.1 and the fact that $\mathcal{G}_t^{\alpha+e_1} \leq \mathcal{G}_t^\alpha$ to get $|\frac{y_1}{\sinh 2t}\left(\mathcal{G}_t^{\alpha+e_1} - \mathcal{G}_t^\alpha\right)| \lesssim \frac{1}{x_1}\mathcal{G}_t^\alpha(x,y)$. Then, if $x,y\in\mathbb{R}_+^d$, $0<t<1$, the inequality

$$\frac{1}{x_1}\mathcal{G}_t^\alpha(x,y) \lesssim t^{-1/2}\mathcal{G}_t^\alpha(x,y) + |x_1-y_1|t^{-1}\mathcal{G}_t^\alpha(x,y), \tag{3.12}$$

which will be shown in a moment to hold for $\frac{x_1y_1}{\sinh 2t} \geq 1$, together with Lemma 2.4, will give by the second identity in the splitting (3.8) (remember that $0<t<1$)

$$\begin{aligned}
&|\partial_{x_1}\mathcal{G}_t^\alpha(x,y)| \lesssim \big(|x_1-y_1|\,t^{-1} + t^{-1/2} + (x_1+y_1)t\big) G_t(x,y) \\
&\lesssim \left(\frac{|x-y|}{\sqrt{t}} \exp\left(-\frac{|x-y|^2}{8\tanh t}\right) + 1 + (x_1+y_1)\sqrt{t}\exp\left(-\frac{\tanh t\,|x+y|^2}{4}\right)\right) \\
&\quad \times t^{-(d+1)/2}\exp\left(-\frac{|x-y|^2}{8\tanh t}\right) \lesssim |x-y|^{-d-1}.
\end{aligned}$$

Justifying (3.12), we consider two possibilities; x_1 comparable with y_1, i.e. $x_1/2 \le y_1 \le 2x_1$, and x_1 non–comparable with y_1, i.e. $x_1 > 2y_1$ or $x_1 < y_1/2$. If x_1 is comparable with y_1, then since $\frac{x_1 y_1}{\sinh 2t} \ge 1$ and $0 < t < 1$, it follows from the splitting $\frac{1}{x_1} = (\frac{y_1}{x_1})^{1/2}(x_1 y_1)^{-1/2}$ that $\frac{1}{x_1}\mathcal{G}_t^\alpha(x,y) \lesssim t^{-1/2}\mathcal{G}_t^\alpha(x,y)$. On the other hand, if x_1 and y_1 are not comparable, then $(x_1+y_1) \approx |x_1-y_1|$, and since $\frac{x_1 y_1}{\sinh 2t} \ge 1$, we get $\frac{1}{x_1} \le (x_1+y_1)(x_1y_1)^{-1} \lesssim |x_1-y_1|(\sinh 2t)^{-1}$, so that $\frac{1}{x_1}\mathcal{G}_t^\alpha(x,y) \lesssim |x_1-y_1|t^{-1}\mathcal{G}_t^\alpha(x,y)$. This finishes proving (3.7) for $0 < t < 1$ in the case when $\frac{x_1 y_1}{\sinh 2t} \ge 1$ and thus completes the proof of (3.7).

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