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Generalized Partial-Slice Monogenic Functions: A Synthesis of Two Function Theories

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Abstract. In this paper, we review the notion of generalized partial-slice monogenic functions that was introduced by the authors in Xu and Sabadini (Generalized partial-slice monogenic functions, arXiv:2309.03698, 2023). The class of these functions includes both the theory of monogenic functions and of slice monogenic functions over Clifford algebras and it is obtained via a synthesis operator which combines a generalized Cauchy–Riemann operator with an operator acting on slices. Besides recalling the fundamental features, we provide a notion of $*$-product based on the CK-extension and discuss the smoothness of generalized partial-slice functions.

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1. Introduction

Classical Clifford analysis serves as a function theory that investigates null solutions of generalized Cauchy–Riemann systems and that generalizes holomorphic functions of one complex variable to a higher dimensional case. These systems include nullsolutions of the Weyl or Dirac systems, referred to as monogenic functions, defined on domains of Euclidean space $\mathbb{R}^{n+1}$ or $\mathbb{R}^n$ and with values in the real Clifford algebra $\mathbb{R}_n$. The interest in functions in the kernel of a generalized Cauchy–Riemann operator, extending beyond complex

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holomorphic functions, emerged around the late nineteenth century. Various references can be found e.g. in the book by Colombo, Sabadini, and Struppa [5]. Quaternions correspond to the particular case $n = 2$ and the exploration of quaternionic-valued functions in the kernel of the so-called Cauchy–Fueter operator (called Cauchy–Fueter regular or regular, for short) dates back to the thirties, with foundational works by Fueter [7] and Moisil [11].

Monogenic functions, studied extensively in the literature (see e.g. [1, 2, 6, 10]), present a challenge compared to complex analysis: polynomials, series, and even powers of the paravector variable are not monogenic in Clifford analysis. Although series expansions using homogeneous monogenic polynomials are attainable, certain features, like defining an exponential function with standard properties or applications to operator theory, may be hindered. To address this, different approaches have been explored.

The theory of slice regular functions of a quaternionic variable emerged in 2006, initiated by Gentili and Struppa [8]. This theory encompasses power series of the form $f(q) = \sum_{n=0}^{\infty} q^n a_n$, where q and a_n are quaternions. In 2009, Colombo, Sabadini, and Struppa generalized this idea to functions defined on domains of Euclidean space $\mathbb{R}^{n+1}$, identified with paravectors, and with values in Clifford algebras $\mathbb{R}_n$, naming them slice monogenic functions [3]. Despite being introduced relatively recently, slice monogenic functions have become a well-developed theory, especially due to their applications in operator theory. Indeed, with this class of functions one may define functions of N-tuples of operators, not necessarily commuting among themselves. This fact alone makes this function theory meaningful. The literature on this topic is extensive, and we refer the interested reader to [4] for more information.

In the paper [14] we introduced a new class of functions that encompasses classical monogenic functions and slice monogenic functions as special cases. In comparison to slice regular and to slice monogenic functions, this new theory exhibits all the expected properties such as the identity theorem, the Representation Formula, Cauchy integral formula with a generalized partial-slice monogenic Cauchy kernel, maximum modulus principle, Taylor series, and Laurent series expansion formulas. In [15] we proved, among other results, a version of the Fueter–Sce theorem for generalized partial-slice monogenic functions, providing a method to construct monogenic functions in higher dimensions from monogenic functions in lower dimensions.

The celebrated Fueter theorem for quaternionic valued-functions was generalized by Sce to functions defined on open sets in the Euclidean space $\mathbb{R}^{n+1}$ for odd $n \in \mathbb{N}$ and with values in a Clifford algebra [12] (see [5] for the English translation). Sce's result asserts in particular that, given a holomorphic function $f_0(z) = u(a, b) + iv(a, b)$, $a, b \in \mathbb{R}, z = a + ib \in \{a + ib \in \mathbb{C} : b > 0\}$, the so-called induced function

$$\overrightarrow{f_0}(x) = u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|)$$

satisfies

$$D\Delta^{\frac{n-1}{2}} \overrightarrow{f_0}(x) = 0,$$

namely $\Delta^{\frac{n-1}{2}}\overrightarrow{f_0}$ is monogenic. Here $x = x_0 + \underline{x} = \sum_{m=0}^{n} x_m e_m \in \mathbb{R}^{n+1}$, $\{e_1, e_2, \ldots, e_n\}$ are generators of $\mathbb{R}_n$, Δ is the Laplacian in $\mathbb{R}^{n+1}$, and $D = \frac{\partial}{\partial x_0} + \sum_{m=1}^{n} e_m \frac{\partial}{\partial x_m}$. The crucial fact is that the pair (u, v) satisfies the Cauchy–Riemann system and if one considers functions u, v having values in the Clifford algebra $\mathbb{R}_n$, we obtain that the induced function $\overrightarrow{f_0}(x)$ is, in modern terms, slice monogenic. In a similar way, we introduce generalized partial-slice monogenic functions in an alternative way, by imposing that the pair (u, v) satisfies a suitable system of differential equations, see Definition 4.2. A crucial fact in this approach is the slice (or axial) nature of which we study in more detail in this article.

We now detail the organization of this paper and highlight the main results in each section. In Sect. 2 we review some basic materials on monogenic functions and on generalized partial-slice monogenic functions taken from [14, 15]. In Sect. 3 we recall the homogeneous polynomials which are generalized partial-slice monogenic and introduce the $*$-product discussing first the case of functions which can be expanded in Taylor series on balls. In Sect. 4 we discuss the smoothness and plane waves for generalized partial-slice functions.

2. Preliminaries

2.1. Clifford Algebras

Denote by $\mathfrak{B} = \{e_1, e_2, \ldots, e_n\}$ a standard orthonormal basis for the n-dimensional real Euclidean space $\mathbb{R}^n$. The real Clifford algebra $\mathbb{R}_n$, is generated by the basis $\mathfrak{B}$ assuming the relations

$$e_i e_j + e_j e_i = -2\delta_{ij}, \quad 1 \le i, j \le n,$$

where δ_{ij} is the Kronecker symbol. An element in the Clifford algebra $\mathbb{R}_n$ is of the form

$$a = \sum_A a_A e_A, \quad a_A \in \mathbb{R},$$

where

$$e_A = e_{j_1} e_{j_2} \cdots e_{j_r},$$

and $A = \{j_1, j_2, \ldots, j_r\} \subseteq \{1, 2, \ldots, n\}$ with $1 \le j_1 < j_2 < \cdots < j_r \le n$, $e_\emptyset = e_0 = 1$.

The Euclidean space $\mathbb{R}^{n+1}$ is identified with the subset of the Clifford algebra $\mathbb{R}_n$ consisting of the so-called paravectors via the map

$$(x_0, x_1, \ldots, x_n) \longmapsto x = x_0 + \underline{x} = \sum_{i=0}^{n} e_i x_i.$$

For a paravector $x \neq 0$, its norm is given by $|x| = (x\overline{x})^{1/2}$, its conjugate by $\bar{x} = x_0 - \sum_{i=1}^{n} e_i x_i$ and so its inverse is given by $x^{-1} = \overline{x}|x|^{-2}$.

2.2. Monogenic and Slice Monogenic Functions

As we mentioned in the introduction two widely studied classes of functions are those of monogenic and of slice monogenic functions which we recall below.

Definition 2.1. (*Monogenic function*) Let Ω be a domain (i.e., an open and connected set) in $\mathbb{R}^{n+1}$ and let $f : \Omega \to \mathbb{R}_n$ be a function with continuous partial derivatives. The function $f = \sum_A e_A f_A$ is called (left) monogenic in Ω if it satisfies the generalized Cauchy–Riemann equation

$$Df(x) = \sum_{i=0}^{n} e_i \frac{\partial f}{\partial x_i}(x) = \sum_{i=0}^{n} \sum_A e_i e_A \frac{\partial f_A}{\partial x_i}(x) = 0, \quad x \in \Omega.$$

The operator D is called Weyl operator or generalized Cauchy–Riemann operator.

Similarly, the function f is called right monogenic in Ω if

$$f(x)D = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(x) e_i = \sum_{i=0}^{n} \sum_A e_A e_i \frac{\partial f_A}{\partial x_i}(x) = 0, \quad x \in \Omega.$$

Note that all monogenic functions are real analytic and harmonic in $(n+1)$ variables, since D factorizes the Laplacian.

In $\mathbb{R}^{n+1}$ one can define a so-called book structure, namely the Euclidean space can be seen as union of complex spaces. This structure is based on the fact that every non-real paravector in $\mathbb{R}^{n+1}$ can be written in the form $x = x_0 + r\omega$ with $r = |\underline{x}| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$, where the element ω is uniquely determined and behaves as a classical imaginary unit, that is

$$\omega \in S^{n-1} = \left\{x \in \mathbb{R}^{n+1} : x^2 = -1\right\}.$$

When x is real, then $r = 0$ and for every $\omega \in S^{n-1}$ one can write $x = x + \omega \cdot 0$. For any fixed $\omega \in S^{n-1}$, we can form the complex plane $\mathbb{C}_\omega = \mathbb{R} \oplus \omega\mathbb{R}$.

With these notations we can now introduce the definition of slice monogenic function, see [3,4].

Definition 2.2. (*Slice monogenic function*) Let Ω be a domain in $\mathbb{R}^{n+1}$. A function $f : \Omega \to \mathbb{R}_n$ is called (left) slice monogenic if, for all $\omega \in S^{n-1}$, its restriction f_ω to $\Omega_\omega = \Omega \cap (\mathbb{R} \oplus \omega\mathbb{R}) \subseteq \mathbb{R}^2$ is holomorphic, i.e., it has continuous partial derivatives and satisfies

$$(\partial_{x_0} + \omega\partial_r) f_\omega(x_0 + r\omega) = 0$$

for all $x_0 + r\omega \in \Omega_\omega$.

2.3. Generalized Partial-Slice Monogenic Functions

In the paper [14], we introduced a new class of functions which contains both monogenic and slice monogenic functions as particular cases. We recall here the basics on these functions. Let p and q be a nonnegative and a positive integer, respectively, and consider $f : \Omega \longrightarrow \mathbb{R}_{p+q}$ where $\Omega \subset \mathbb{R}^{p+q+1}$ is a domain.

As customary, an element $\boldsymbol{x}$ in the Euclidean space $\mathbb{R}^{p+q+1}$ will be identified with a paravector in $\mathbb{R}_{p+q}$, and in addition we shall consider the (fixed) splitting of $\mathbb{R}^{p+q+1}$ as $\mathbb{R}^{p+1} \oplus \mathbb{R}^q$ so that we write a paravector $\boldsymbol{x} \in \mathbb{R}^{p+q+1}$ as

$$\boldsymbol{x} = \boldsymbol{x}_p + \underline{\boldsymbol{x}}_q \in \mathbb{R}^{p+1} \oplus \mathbb{R}^q, \quad \boldsymbol{x}_p = \sum_{i=0}^{p} x_i e_i, \ \underline{\boldsymbol{x}}_q = \sum_{i=p+1}^{p+q} x_i e_i.$$

Similarly, the generalized Cauchy–Riemann operator is split as

$$D_{\boldsymbol{x}} = D_{\boldsymbol{x}_p} + D_{\underline{\boldsymbol{x}}_q}, \quad D_{\boldsymbol{x}_p} = \sum_{i=0}^{p} e_i \partial_{x_i}, \quad D_{\underline{\boldsymbol{x}}_q} = \sum_{i=p+1}^{p+q} e_i \partial_{x_i}. \tag{1}$$

Denote by $\mathbb{S}$ the sphere of unit 1-vectors in $\mathbb{R}^q$, whose elements $(x_{p+1}, \ldots, x_{p+q})$ are identified with $\underline{\boldsymbol{x}}_q = \sum_{i=p+1}^{p+q} x_i e_i$, i.e.

$$\mathbb{S} = \big\{\underline{\boldsymbol{x}}_q : \underline{\boldsymbol{x}}_q^2 = -1\big\} = \big\{\underline{\boldsymbol{x}}_q = \sum_{i=p+1}^{p+q} x_i e_i : \sum_{i=p+1}^{p+q} x_i^2 = 1\big\}.$$

Note that, for $\underline{\boldsymbol{x}}_q \neq 0$, there exists a uniquely $r \in \mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and $\underline{\omega} \in \mathbb{S}$, such that $\underline{\boldsymbol{x}}_q = r\underline{\omega}$, i.e.

$$r = |\underline{\boldsymbol{x}}_q|, \quad \underline{\omega} = \frac{\underline{\boldsymbol{x}}_q}{|\underline{\boldsymbol{x}}_q|}.$$

When $\underline{\boldsymbol{x}}_q = 0$ (or $r = 0$), $\underline{\omega}$ is not uniquely defined, in fact for every $\underline{\omega} \in \mathbb{S}$ we have $\boldsymbol{x} = \boldsymbol{x}_p + \underline{\omega} \cdot 0$.

The upper half-space $\mathrm{H}_{\underline{\omega}}$ in $\mathbb{R}^{p+2}$ associated with $\underline{\omega} \in \mathbb{S}$ is defined by

$$\mathrm{H}_{\underline{\omega}} = \{\boldsymbol{x}_p + r\underline{\omega} : \boldsymbol{x}_p \in \mathbb{R}^{p+1}, r \geq 0\},$$

so that in this case, the analog of the book structures is given by higher dimensional slices. In fact, it holds that

$$\mathbb{R}^{p+q+1} = \bigcup_{\underline{\omega} \in \mathbb{S}} \mathrm{H}_{\underline{\omega}},$$

and

$$\mathbb{R}^{p+1} = \bigcap_{\underline{\omega} \in \mathbb{S}} \mathrm{H}_{\underline{\omega}}.$$

Recalling the notation in formula (1), we now introduce the definition of generalized partial-slice monogenic functions as follows.

Definition 2.3. Let Ω be a domain in $\mathbb{R}^{p+q+1}$. A function $f : \Omega \to \mathbb{R}_{p+q}$ is called (left) generalized partial-slice monogenic of type (p, q) if, for all $\underline{\omega} \in \mathbb{S}$, its restriction $f_{\underline{\omega}}$ to $\Omega_{\underline{\omega}} \subseteq \mathbb{R}^{p+2}$ has continuous partial derivatives and satisfies

$$D_{\underline{\omega}} f_{\underline{\omega}}(\boldsymbol{x}) := (D_{\boldsymbol{x}_p} + \underline{\omega}\partial_r) f_{\underline{\omega}}(\boldsymbol{x}_p + r\underline{\omega}) = 0,$$

for all $\boldsymbol{x} = \boldsymbol{x}_p + r\underline{\omega} \in \Omega_{\underline{\omega}} := \Omega \cap (\mathbb{R}^{p+1} \oplus \underline{\omega}\mathbb{R}) \subseteq \mathbb{R}^{p+2}$. Similarly, f is called right generalized partial-slice monogenic of type (p, q) if, under the hypothesis above, f satisfies

$$f_{\underline{\omega}}(\boldsymbol{x}) D_{\underline{\omega}} := f_{\underline{\omega}}(\boldsymbol{x}_p + r\underline{\omega}) D_{\boldsymbol{x}_p} + \partial_r f_{\underline{\omega}}(\boldsymbol{x}_p + r\underline{\omega})\underline{\omega} = 0,$$

for all $\boldsymbol{x} = \boldsymbol{x}_p + r\underline{\omega} \in \Omega_{\underline{\omega}}$.

We denote by $\mathcal{GSM}^L(\Omega)$ (resp. $\mathcal{GSM}^R(\Omega)$) the right (resp. left) $\mathbb{R}_{p+q}$-module of all left (resp. right) generalized partial-slice monogenic functions of type (p,q) in Ω. For short, we shall denote by $\mathcal{GSM}(\Omega)$ the Clifford module $\mathcal{GSM}^L(\Omega)$ when there is no confusion.

Assumption. To ease the notation, in the sequel we shall omit to specify the type (p,q) since we assume to fix the splitting.

The following remark is very important and explains in which sense this new function theory is a synthesis of the two aforementioned theories.

Remark 2.4. When $(p,q)=(n-1,1)$, the notion of generalized partial-slice monogenic functions in Definition 2.3 coincides with the notion of classical monogenic functions defined in $\Omega\subseteq\mathbb{R}^{n+1}$ with values in the Clifford algebra $\mathbb{R}_n$; see Definition 2.1.

When $(p,q)=(0,n)$, Definition 2.3 coincides with the notion of slice monogenic functions in Definition 2.2 defined in $\Omega\subseteq\mathbb{R}^{n+1}$ and with values in the Clifford algebra $\mathbb{R}_n$.

In [14] we present various examples, here we recall just one that is crucial to obtain a Cauchy formula.

Example 2.5. Let $\Omega=\mathbb{R}^{p+q+1}\backslash\{0\}$ and set

$$E(\boldsymbol{x}):=\frac{1}{\sigma_{p+1}}\frac{\overline{\boldsymbol{x}}}{|\boldsymbol{x}|^{p+2}},\quad \boldsymbol{x}\in\Omega,$$

where $\sigma_{p+1}=2\frac{\Gamma^{p+2}(\frac{1}{2})}{\Gamma(\frac{p+2}{2})}$ is the surface area of the unit ball in $\mathbb{R}^{p+2}$.

Then $E(\boldsymbol{x})\in\mathcal{GSM}^L(\Omega)\cap\mathcal{GSM}^R(\Omega)$.

Remark 2.6. Recall that the Cauchy kernel for monogenic functions defined on paravectors $\mathbb{R}^{p+1}$ is the function

$$\mathsf{E}(x)=\frac{1}{\sigma_p}\frac{\overline{x}}{|x|^{p+1}},\quad x=\sum_{i=0}^{p}x_ie_i\in\mathbb{R}^{p+1}\backslash\{0\}.$$

Compared with the classical monogenic Cauchy kernel, the function in Example 2.5 has freedom in the dimension q. The new function class of generalized partial-slice monogenic functions not only includes the two classes of classical monogenic and of slice monogenic functions, but it is much larger than their synthesis.

We recall some useful results which are typical for slice monogenic functions and that can be extended, with suitable changes, to generalized partial-slice monogenic functions.

Lemma 2.7. (Splitting lemma) *Let $\Omega\subset\mathbb{R}^{p+q+1}$ be a domain and $f:\Omega\to\mathbb{R}_{p+q}$ be a generalized partial-slice monogenic function. For every arbitrary, but fixed $\underline{\omega}\in\mathbb{S}$, also denoted by I_{p+1}, let $I_1,I_2,\ldots,I_p,I_{p+2},\ldots,I_{p+q}$ be a completion to a basis of $\mathbb{R}_{p+q}$ such that $I_rI_s+I_sI_r=-2\delta_{rs}$, $r,s=1,\ldots,p+q$, $I_r\in\mathbb{R}^p$, $r=1,\ldots,p$, $I_r\in\mathbb{R}^q$, $r=p+2,\ldots,p+q$. Then there exist 2^{q-1} monogenic functions $F_A:\Omega_{\underline{\omega}}\to\mathbb{R}_{p+1}=\mathrm{Alg}\{I_1,\ldots,I_p,I_{p+1}=\underline{\omega}\}$ such that*

$$f_{\underline{\omega}}(\boldsymbol{x}_p+r\underline{\omega})=\sum_{A=\{i_1,\ldots,i_s\}\subset\{p+2,\ldots,p+q\}}F_A(\boldsymbol{x}_p+r\underline{\omega})I_A,\quad \boldsymbol{x}_p+r\underline{\omega}\in\Omega_{\underline{\omega}},$$

where $I_A = I_{i_1} \cdots I_{i_s}, A = \{i_1, \dots, i_s\} \subset \{p+2, \dots, p+q\}$ *with* $i_1 < \cdots < i_s$, *and* $I_\emptyset = 1$ *when* $A = \emptyset$.

To proceed and state the Representation Formula, we need the following.

Definition 2.8. Let Ω be a domain in $\mathbb{R}^{p+q+1}$.

1. Ω is called slice domain if $\Omega \cap \mathbb{R}^{p+1} \neq \emptyset$ and $\Omega_{\underline{\omega}}$ is a domain in $\mathbb{R}^{p+2}$ for every $\underline{\omega} \in \mathbb{S}$.

2. Ω is called partially symmetric with respect to $\mathbb{R}^{p+1}$ (p-symmetric for short) if, for $\boldsymbol{x}_p \in \mathbb{R}^{p+1}, r \in \mathbb{R}^+$, and $\underline{\omega} \in \mathbb{S}$,

$$\boldsymbol{x} = \boldsymbol{x}_p + r\underline{\omega} \in \Omega \Longrightarrow [\boldsymbol{x}] := \boldsymbol{x}_p + r\mathbb{S} = \{\boldsymbol{x}_p + r\underline{\omega}, \quad \underline{\omega} \in \mathbb{S}\} \subseteq \Omega.$$

Theorem 2.9. (Representation Formula) *Let* $\Omega \subset \mathbb{R}^{p+q+1}$ *be a p-symmetric slice domain and* $f : \Omega \to \mathbb{R}_{p+q}$ *be a generalized partial-slice monogenic function. Then, for any* $\underline{\omega} \in \mathbb{S}$ *and for* $\boldsymbol{x}_p + r\underline{\omega} \in \Omega$,

$$f(\boldsymbol{x}_p + r\underline{\omega}) = \frac{1}{2}(f(\boldsymbol{x}_p + r\underline{\eta}) + f(\boldsymbol{x}_p - r\underline{\eta})) + \frac{1}{2}\underline{\omega}\underline{\eta}(f(\boldsymbol{x}_p - r\underline{\eta}) - f(\boldsymbol{x}_p + r\underline{\eta})), \tag{2}$$

for any $\underline{\eta} \in \mathbb{S}$.

Moreover, the following two functions do not depend on $\underline{\eta}$:

$$F_1(\boldsymbol{x}_p, r) = \frac{1}{2}(f(\boldsymbol{x}_p + r\underline{\eta}) + f(\boldsymbol{x}_p - r\underline{\eta})),$$
$$F_2(\boldsymbol{x}_p, r) = \frac{1}{2}\underline{\eta}(f(\boldsymbol{x}_p - r\underline{\eta}) - f(\boldsymbol{x}_p + r\underline{\eta})).$$

As a corollary of the Representation Formula, we state the following extension theorem.

Theorem 2.10. (Extension theorem) *Let* $\Omega \subset \mathbb{R}^{p+q+1}$ *be a p-symmetric slice domain. Let* $f_{\underline{\eta}} : \Omega_{\underline{\eta}} \to \mathbb{R}_{p+q}$ *be a function with continuous partial derivatives and satisfying*

$$(D_{\boldsymbol{x}_p} + \underline{\eta}\partial_r) f_{\underline{\eta}}(\boldsymbol{x}_p + r\underline{\eta}) = 0, \qquad \boldsymbol{x}_p + r\underline{\eta} \in \Omega_{\underline{\eta}},$$

for a given $\underline{\eta} \in \mathbb{S}$. *Then, for any* $\boldsymbol{x}_p + \underline{\boldsymbol{x}}_q = \boldsymbol{x}_p + r\underline{\omega} \in \Omega$, *the function defined by*

$$\begin{aligned} \mathrm{ext}(f_{\underline{\eta}})(\boldsymbol{x}_p + r\underline{\omega}) \; :=& \frac{1}{2}(f(\boldsymbol{x}_p + r\underline{\eta}) + f(\boldsymbol{x}_p - r\underline{\eta})) \\ &+ \frac{1}{2}\underline{\omega}\underline{\eta}(f(\boldsymbol{x}_p - r\underline{\eta}) - f(\boldsymbol{x}_p + r\underline{\eta})) \end{aligned} \tag{3}$$

is the unique generalized partial-slice monogenic extension of $f_{\underline{\eta}}$ *to the whole* Ω.

Note that there is another extension theorem [15, Theorem 4.2], which is based on the idea of the CK-extension, thus more similar to the extension for monogenic functions.

Let Ω_0 be a domain in $\mathbb{R}^{p+1}$. The set Ω_0^* is defined by

$$\Omega_0^* = \{\boldsymbol{x}_p + \underline{\boldsymbol{x}}_q : \boldsymbol{x}_p \in \Omega_0, \underline{\boldsymbol{x}}_q \in \mathbb{R}^q\}.$$

Theorem 2.11. *Let Ω_0 be a domain in $\mathbb{R}^{p+1}$ and $f_0 : \Omega_0 \to \mathbb{R}_{p+q}$ be a real analytic function. Then the function given by*

$$CK[f_0](\boldsymbol{x}) = \exp(\underline{\boldsymbol{x}}_q D_{\boldsymbol{x}_p}) f_0(\boldsymbol{x}_p) = \sum_{k=0}^{+\infty} \frac{1}{k!} (\underline{\boldsymbol{x}}_q D_{\boldsymbol{x}_p})^k f_0(\boldsymbol{x}_p), \quad \boldsymbol{x} = \boldsymbol{x}_p + \underline{\boldsymbol{x}}_q,$$

is a generalized partial-slice monogenic function f^* defined in $\Omega \subseteq \Omega_0^*$ with $\Omega_0 \subset \Omega$ and $f^*(\boldsymbol{x}_p) = f_0(\boldsymbol{x}_p)$.

The function f^* is the so-called (left) slice Cauchy–Kovalevskaya extension f_0 to the p-symmetric slice domain $\Omega \subseteq \Omega_0^*$ with $\Omega_0 \subset \Omega$.

In the next section we shall discuss more properties which are more typical of the standard monogenic case.

3. Generalized Partial-Slice Monogenic Polynomials and ∗-Product

Let $\mathrm{k} = (k_0, k_1, \ldots, k_p) \in \mathbb{N}^{p+1}$ be a multi-index and set $k = |\mathrm{k}| = k_0 + k_1 + \cdots + k_p$, $\mathrm{k}! = k_0! k_1! \cdots k_p!$. Consider $\boldsymbol{x}' = (x_0, x_1, \ldots, x_p, r) = (\boldsymbol{x}_p, r) \in \mathbb{R}^{p+2}$, $\boldsymbol{x}_p^{\mathrm{k}} = x_0^{k_0} x_1^{k_1} \ldots x_p^{k_p}$. Let $\underline{\eta}$ be an element in in $\mathbb{S}$. The so-called left Fueter variables were defined in [14] as

$$z_\ell = x_\ell + r\underline{\eta} e_\ell, \ \ \ell = 0, 1, \ldots, p,$$

while the right Fueter variables are defined as

$$z_\ell^R = x_\ell + r e_\ell \underline{\eta}, \ \ \ell = 0, 1, \ldots, p.$$

An easy calculation shows that

$$\left(\sum_{i=0}^{p} e_i \partial_{x_i} + \underline{\eta} \partial_r\right) z_\ell = z_\ell^R \left(\sum_{i=0}^{p} e_i \partial_{x_i} + \underline{\eta} \partial_r\right) = 0. \tag{4}$$

Definition 3.1. Let $(j_1, j_2, \ldots, j_k)$ to be an alignment of integers with $0 \leq j_\ell \leq p$ for any $\ell = 0, \ldots, k$ and assume that the number of 0 in the alignment is k_0, the number of 1 is k_1 and the number of p is k_p, where $k_0 + k_1 + \ldots + k_p = k$. Define

$$\mathcal{P}_{\underline{\eta},\mathrm{k}}(\boldsymbol{x}') = \frac{1}{k!} \sum_{(\sigma(j_1), \sigma(j_2), \ldots, \sigma(j_k)) \in \mathcal{P}(j_1, j_2, \ldots, j_k)} z_{\sigma(j_1)} z_{\sigma(j_2)} \cdots z_{\sigma(j_k)},$$

where the sum is computed over the $\dfrac{k!}{\mathrm{k}!}$ different permutations σ of k_ℓ elements equal to $\ell = 0, 1, \ldots, p$. When $\mathrm{k} = (0, \ldots, 0) = 0$ we set $\mathcal{P}_{\underline{\eta},0}(\boldsymbol{x}') = 1$. Similarly, define

$$\mathcal{P}^R_{\underline{\eta},\mathrm{k}}(\boldsymbol{x}') = \frac{1}{k!} \sum_{(\sigma(j_1), \sigma(j_2), \ldots, \sigma(j_k)) \in \mathcal{P}(j_1, j_2, \ldots, j_k)} z^R_{\sigma(j_1)} z^R_{\sigma(j_2)} \cdots z^R_{\sigma(j_k)}.$$

Using the Representation Formula we can define the homogeneous polynomials

$$\mathcal{P}_{\mathrm{k}}(\boldsymbol{x}) = \mathcal{P}_{\mathrm{k}}(\boldsymbol{x}_p + r\underline{\omega}) = \frac{1}{2}\big((1 - \underline{\omega}\underline{\eta}) \mathcal{P}_{\underline{\eta},\mathrm{k}}(\boldsymbol{x}') + (1 + \underline{\omega}\underline{\eta}) \mathcal{P}_{\underline{\eta},\mathrm{k}}(\boldsymbol{x}'_\diamond)\big)$$

where $\boldsymbol{x}'_\diamond = (\boldsymbol{x}_p, -r)$. Analogously, one can define $\mathcal{P}^R_{\mathrm{k}}(\boldsymbol{x})$ using the version of the Representation Formula for functions in $\mathcal{GSM}^R(\mathbb{R}^{p+q+1})$:

$$\mathcal{P}^R_{\mathrm{k}}(\boldsymbol{x}) = \mathcal{P}^R_{\mathrm{k}}(\boldsymbol{x}_p + r\underline{\omega}) = \frac{1}{2}\big(\mathcal{P}^R_{\underline{\eta},\mathrm{k}}(\boldsymbol{x}')(1-\underline{\eta}\underline{\omega}) + \mathcal{P}^R_{\underline{\eta},\mathrm{k}}(\boldsymbol{x}'_\diamond)(1+\underline{\eta}\underline{\omega})\big).$$

In the particular case when k is such that $k_i = 0$ for $i \neq \ell$ and $k_\ell = 1$ for some fixed $\ell \in \{0, \dots, p\}$, the polynomial $\mathcal{P}_{\mathrm{k}}$ (here denoted by $\mathcal{Z}_\ell$) is the extension of z_ℓ in (4) and

$$\mathcal{Z}_\ell(\boldsymbol{x}) = \mathcal{Z}_\ell(\boldsymbol{x}_p + r\underline{\omega}) = x_\ell + r\underline{\omega}e_\ell.$$

Unlike what happens in classical Clifford analysis where the counterparts of the polynomials $\mathcal{P}_{\mathrm{k}}$ are both left and right monogenic, the polynomials $\mathcal{P}_{\mathrm{k}}$ (resp. $\mathcal{P}^R_{\mathrm{k}}$) belong only to $\mathcal{GSM}^L(\mathbb{R}^{p+q+1})$ (resp. $\mathcal{GSM}^R(\mathbb{R}^{p+q+1})$). This is a crucial difference with the classical monogenic case. As we know, this difference emerges even more while writing the Laurent expansion for generalized partial-slice monogenic functions [14]. These polynomials are used to express the Taylor series of a left (resp. right) generalized partial-slice monogenic functions.

Denote by $B(0,\rho) = \{\boldsymbol{x} \in \mathbb{R}^{p+q+1} : |\boldsymbol{x}| < \rho\}$ the ball centered in 0 with radius $\rho > 0$.

Theorem 3.2. (Taylor series) *Let $f : B(0,\rho) \to \mathbb{R}_{p+q}$ be a generalized partial-slice monogenic function. For any $\boldsymbol{x} \in B(0,\rho)$, we have*

$$f(\boldsymbol{x}) = \sum_{k=0}^{+\infty}\left(\sum_{|\mathrm{k}|=k}\mathcal{P}_{\mathrm{k}}(\boldsymbol{x})a_{\mathrm{k}}\right), \quad a_{\mathrm{k}} = \partial_{\mathrm{k}} f(0),$$

where the series converges uniformly on compact subsets of $B(0,\rho)$.

Remark 3.3. Reasoning in an analogous way, in case f is a right generalized partial-slice monogenic function on $B(0,\rho)$ then we have

$$f(\boldsymbol{x}) = \sum_{k=0}^{+\infty}\left(\sum_{|\mathrm{k}|=k}a_{\mathrm{k}}\mathcal{P}^R_{\mathrm{k}}(\boldsymbol{x})\right), \qquad a_{\mathrm{k}} = \partial_{\mathrm{k}} f(0).$$

Remark 3.4. When $\rho = +\infty$ the function f is generalized partial-slice monogenic function on the whole $\mathbb{R}^{p+q+1}$ and so it is entire.

As we suggested in [14] it is natural to define the $*$-product between two functions $f, g \in \mathcal{GSM}^L(\mathbb{R}^{p+q+1})$, making a suitable use of the CK-extension. This idea can be used to define a $*$-product in $\mathcal{GSM}^L(\mathbb{R}^{p+q+1})$ using results from [15]. Before to follow a general approach, we consider two functions which are generalized partial-slice monogenic on balls in $\mathbb{R}^{p+q+1}$.

Given two multi-indices $\mathrm{s} = (s_0, \dots, s_p)$, $\mathrm{m} = (m_0, \dots, m_p) \in \mathbb{N}^{p+1}$, let us define $\mathrm{s} + \mathrm{m} := (s_0 + m_0, \dots, s_p + m_p)$.

Definition 3.5. Let f and g be two left generalized partial-slice monogenic functions on $B(0,\rho)$ given by

$$f(\boldsymbol{x}) = \sum_{k=0}^{+\infty}\Big(\sum_{|\mathrm{k}|=k}\mathcal{P}_{\mathrm{k}}(\boldsymbol{x})a_{\mathrm{k}}\Big), \qquad \textit{and} \qquad g(\boldsymbol{x}) = \sum_{k=0}^{+\infty}\Big(\sum_{|\mathrm{k}|=k}\mathcal{P}_{\mathrm{k}}(\boldsymbol{x})b_{\mathrm{k}}\Big).$$

Define

$$(f * g)(\boldsymbol{x}) := \sum_{k=0}^{+\infty} \Big(\sum_{|\mathrm{k}|=k} \frac{(\mathrm{s}+\mathrm{m})!}{\mathrm{s}!\mathrm{m}!} \mathcal{P}_{\mathrm{k}}(\boldsymbol{x}) c_{\mathrm{k}} \Big), \qquad c_{\mathrm{k}} = \sum_{\mathrm{s}+\mathrm{m}=\mathrm{k}} a_{\mathrm{s}} b_{\mathrm{m}}.$$

The operation just defined acts on $\mathcal{GSM}^L(B(0,\rho))$ and so it should have been more precise to denote it by $*_L$. An analogous operation $*_R$ can be defined on $\mathcal{GSM}^R(B(0,\rho))$ by taking the Cauchy product of the coefficients, written on the left. More precisely,

$$(f *_R g)(\boldsymbol{x}) := \sum_{k=0}^{+\infty} \Big(\sum_{|\mathrm{k}|=k} \frac{(\mathrm{s}+\mathrm{m})!}{\mathrm{s}!\mathrm{m}!} c_{\mathrm{k}} \mathcal{P}^R_{\mathrm{k}}(\boldsymbol{x}) \Big), \qquad c_{\mathrm{k}} = \sum_{\mathrm{s}+\mathrm{m}=\mathrm{k}} a_{\mathrm{s}} b_{\mathrm{m}}.$$

Remark 3.6. In particular, we immediately deduce:

1. $1 * f = f * 1$;
2. if $a \in \mathbb{R}_{p+q}$, then $f * a = fa$;
3. $(\mathcal{P}_{\mathrm{s}} * \mathcal{P}_{\mathrm{m}})(\boldsymbol{x}) = \frac{(\mathrm{s}+\mathrm{m})!}{\mathrm{s}!\mathrm{m}!} \mathcal{P}_{\mathrm{s}+\mathrm{m}}(\boldsymbol{x})$.

To define the $*$-product in general, not necessarily on balls centered at the origin, we make use of Theorem 2.11 and proceed as follows.

Definition 3.7. Let $\Omega \subseteq \mathbb{R}^{p+q+1}$ be a p-symmetric slice domain and let $f, g \in \mathcal{GSM}^L(\Omega)$. Set $\Omega_0 = \Omega \cap \mathbb{R}^{p+1}$ and denote by f_0 and g_0 the real analytic functions defined by $f_0 = f_{|\Omega_0}$, $g_0 = g_{|\Omega_0}$. Define, on a suitable p-symmetric slice domain $\tilde{\Omega} \subseteq \Omega$:

$$(f * g)(\boldsymbol{x}) = CK[(f_0 \cdot g_0)(\boldsymbol{x}_p)], \tag{5}$$

where the product $f_0 \cdot g_0$ is meant as product of real analytic functions.

Remark 3.8. 1. The product defined in Definition 3.5 is a particular case of the product in Definition 3.7. Indeed, since $\mathrm{s}!\mathcal{P}_{\mathrm{s}}(\boldsymbol{x}) = CK[\boldsymbol{x}_p^{\mathrm{s}}]$, so that we have $(\mathrm{s}+\mathrm{m})!\mathcal{P}_{\mathrm{s}+\mathrm{m}}(\boldsymbol{x}) = CK[\boldsymbol{x}_p^{\mathrm{s}+\mathrm{m}}]$, we deduce

$$(\mathrm{s}+\mathrm{m})!\mathcal{P}_{\mathrm{s}+\mathrm{m}}(\boldsymbol{x}) = CK[\boldsymbol{x}_p^{\mathrm{s}+\mathrm{m}}] = CK[\boldsymbol{x}_p^{\mathrm{s}}] * CK[\boldsymbol{x}_p^{\mathrm{m}}] = \mathrm{s}!\mathcal{P}_{\mathrm{k}}(\boldsymbol{x}) * \mathrm{m}!\mathcal{P}_{\mathrm{m}}(\boldsymbol{x}),$$

that yields

$$\mathcal{P}_{\mathrm{s}}(\boldsymbol{x}) * \mathcal{P}_{\mathrm{m}}(\boldsymbol{x}) = \frac{(\mathrm{s}+\mathrm{m})!}{\mathrm{s}!\mathrm{m}!} \mathcal{P}_{\mathrm{s}+\mathrm{m}}(\boldsymbol{x}),$$

which is consistent with point 3, in Remark 3.6.

2. If $f_0 \cdot g_0 = g_0 \cdot f_0$ then (5) yields that $f * g = g * f$.
3. From the formula (5) and the fact that a Clifford algebra is associative, we deduce that the $*$-product is associative.
4. The set $\mathcal{GSM}^L(\Omega)$ equipped with the sum and the $*$-product is an algebra over $\mathbb{R}$.

4. Generalized Partial-Slice Functions and Their Smoothness

Let $D \subseteq \mathbb{R}^{p+2}$ be a domain, which is invariant under the reflection of the $(p+2)$-th variable, i.e.

$$\boldsymbol{x}' := (\boldsymbol{x}_p, r) \in D \Longrightarrow \boldsymbol{x}'_\diamond := (\boldsymbol{x}_p, -r) \in D.$$

We define the *p-symmetric completion* $\Omega_D \subseteq \mathbb{R}^{p+q+1}$ of D as

$$\Omega_D = \bigcup_{\underline{\omega} \in \mathbb{S}} \big\{\boldsymbol{x}_p + r\underline{\omega} \ \exists\, \boldsymbol{x}_p \in \mathbb{R}^{p+1},\ \exists\, r \geq 0,\ \text{s.t. } (\boldsymbol{x}_p, r) \in D\big\}.$$

Note that a domain $\Omega \subseteq \mathbb{R}^{p+q+1}$ is p-symmetric if and only if $\Omega = \Omega_D$ for some domain $D \subseteq \mathbb{R}^{p+2}$.

Definition 4.1. Let $D \subseteq \mathbb{R}^{p+2}$ be a domain, invariant under the reflection with respect to the $(p+2)$-th variable. A function $f : \Omega_D \longrightarrow \mathbb{R}_{p+q}$ of the form

$$f(\boldsymbol{x}) = F_1(\boldsymbol{x}') + \underline{\omega} F_2(\boldsymbol{x}'), \qquad \boldsymbol{x} = \boldsymbol{x}_p + r\underline{\omega} \in \Omega_D,\ \underline{\omega} \in \mathbb{S}, \tag{6}$$

where the $\mathbb{R}_{p+q}$-valued components F_1, F_2 of f are an even-odd pair in the $(p+2)$-th variable, i.e. satisfy

$$F_1(\boldsymbol{x}'_\diamond) = F_1(\boldsymbol{x}'), \qquad F_2(\boldsymbol{x}'_\diamond) = -F_2(\boldsymbol{x}'), \qquad \boldsymbol{x}' \in D, \tag{7}$$

is called a (left) generalized partial-slice function.

We denote by $\mathcal{GS}(\Omega_D)$ the set of all generalized partial-slice functions f on Ω_D. When the components F_1, F_2 are of class $C^k(D)$, we denote the set of all such functions by $\mathcal{GS}^k(\Omega_D)$.

Definition 4.2. Let $f(\boldsymbol{x}) = F_1(\boldsymbol{x}') + \underline{\omega} F_2(\boldsymbol{x}') \in \mathcal{GS}^1(\Omega_D)$. The function f is called generalized partial-slice monogenic of type (p,q) if F_1, F_2 satisfy the generalized Cauchy–Riemann equations

$$\begin{cases} D_{\boldsymbol{x}_p} F_1(\boldsymbol{x}') - \partial_r F_2(\boldsymbol{x}') = 0, \\ \overline{D}_{\boldsymbol{x}_p} F_2(\boldsymbol{x}') + \partial_r F_1(\boldsymbol{x}') = 0, \end{cases} \tag{8}$$

for all $\boldsymbol{x}' \in D$.

We denote by $\mathcal{GSR}(\Omega_D)$ the set of all generalized partial-slice monogenic functions on Ω_D, omitting to specify the type (p,q) if no confusion arises.

Now we give a example to explain that the assumption that a function $f \in \mathcal{GS}^1(\Omega_D)$ does not necessarily imply that the function f is of class $C^1(\Omega_D)$.

Example 4.3. Let $\alpha \in (2,3)$. Consider the odd function $g \in C^1(\mathbb{R})$ given by

$$g(x) = \begin{cases} |x|^\alpha \sin\frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Set $f(\boldsymbol{x}) = \underline{\omega} g(r)$, where $\boldsymbol{x} = \boldsymbol{x}_p + r\underline{\omega}$ with $\boldsymbol{x}_p \in \mathbb{R}^{p+1}, r \geq 0$ and $\underline{\omega} \in \mathbb{S}$. Then one can get immediately $f \in \mathcal{GS}^1(\mathbb{R}^{p+q+1})$, while $f \notin C^1(\mathbb{R}^{p+q+1})$.

Inspired by [9, Proposition 7], in the next result we prove conditions that guarantee that a function belongs to a certain class.

Proposition 4.4. *Let* $k \in \mathbb{N}\backslash\{0\}$ *and* $f(\boldsymbol{x}) = F_1(\boldsymbol{x}') + \underline{\omega} F_2(\boldsymbol{x}') \in \mathcal{GS}(\Omega_D)$*.*

(i) *If* $F_1, F_2 \in C(D)$*, then* $f \in C(\Omega_D)$*.*
(ii) *If* $F_1, F_2 \in C^{2k+1}(D)$*, then* $f \in C^k(\Omega_D)$*.*
(iii) *If* $F_1, F_2 \in C^\infty(D)$*, then* $f \in C^\infty(\Omega_D)$*.*

Proof. Let $f(\boldsymbol{x}) = F_1(\boldsymbol{x}') + \underline{\omega} F_2(\boldsymbol{x}') \in \mathcal{GS}(\Omega_D)$. It is immediate that the generalized partial-slice function f on $\Omega_D\backslash\mathbb{R}^{p+1}$ has the same smoothness as the pair (F_1, F_2) on $D\backslash\mathbb{R}^{p+1}$. To obtain the conclusion, it remains to consider the case $\Omega_D \cap \mathbb{R}^{p+1}$, where the fact that (F_1, F_2) is an even-odd pair.

(i) Assume $F_1, F_2 \in C(D)$. The continuity of f at $\Omega_D \cap \mathbb{R}^{p+1}$ follows from the fact that F_2 is odd in the $(p+2)$-th variable.
(ii) Assume $F_1, F_2 \in C^{2k+1}(D)$. Since (F_1, F_2) is an even-odd pair in the $(p+2)$-th variable, from [13, Theorem 1] and [13, Theorem 2], there exist $G_1, G_2 \in C^k(D)$ such that
$$F_1(\boldsymbol{x}_p, r) = G_1(\boldsymbol{x}_p, r^2),\ F_2(\boldsymbol{x}_p, r) = rG_2(\boldsymbol{x}_p, r^2), \quad (\boldsymbol{x}_p, r) \in D.$$
which gives
$$f(\boldsymbol{x}) = f(\boldsymbol{x}_p + \underline{\boldsymbol{x}}_q) = G_1(\boldsymbol{x}_p, r^2) + \underline{\boldsymbol{x}}_q G_2(\boldsymbol{x}_p, r^2), \quad r = |\underline{\boldsymbol{x}}_q|.$$
Hence, $f \in C^k(\Omega_D)$.
(iii) The conclusion can be obtained by the same strategy used to prove in (ii). □

The approach via generalized partial-slice monogenic functions to introduce monogenic functions of plane waves is obtained as follows: given $\underline{\eta} \in \mathbb{S}$ and $f(\boldsymbol{x}) = F_1(\boldsymbol{x}') + \underline{\omega} F_2(\boldsymbol{x}') \in \mathcal{GSR}(\Omega_D)$, we substitute the variable $r = |\underline{\boldsymbol{x}}_q|$ by $\langle \underline{\eta}, \underline{\boldsymbol{x}}_q \rangle$, namely consider
$$g(\boldsymbol{x}) = g(\boldsymbol{x}_p + \underline{\boldsymbol{x}}_q) = F_1(\boldsymbol{x}_p, \langle \underline{\eta}, \underline{\boldsymbol{x}}_q \rangle) + \underline{\eta} F_2(\boldsymbol{x}_p, \langle \underline{\eta}, \underline{\boldsymbol{x}}_q \rangle). \tag{9}$$
We have the following result:

Theorem 4.5. *Let* $\underline{\eta} \in \mathbb{S}$ *and* $f(\boldsymbol{x}) = F_1(\boldsymbol{x}') + \underline{\omega} F_2(\boldsymbol{x}') \in \mathcal{GSR}(\Omega_D)$*. Then the function of plane wave given by* (9) *is monogenic in* $\Omega(\underline{\eta}) := \{\boldsymbol{x}_p + \underline{\boldsymbol{x}}_q : (\boldsymbol{x}_p, \langle \underline{\eta}, \underline{\boldsymbol{x}}_q \rangle) \in D\}$*.*

Proof. Let $f(\boldsymbol{x}) = f(\boldsymbol{x}_p + r\underline{\omega}) = F_1(\boldsymbol{x}') + \underline{\omega} F_2(\boldsymbol{x}') \in \mathcal{GSR}(\Omega_D)$, where $\boldsymbol{x}' = (\boldsymbol{x}_p, r) \in D$. Set $r = \langle \underline{\eta}, \underline{\boldsymbol{x}}_q \rangle$. From the identities
$$D_{\underline{\boldsymbol{x}}_q} F_1(\boldsymbol{x}') = \sum_{i=p+1}^{p+q} e_i \partial_r F_1(\boldsymbol{x}')(\partial_{x_i} r) = \sum_{i=p+1}^{p+q} e_i \eta_i \partial_r F_1(\boldsymbol{x}') = \eta \partial_r F_1(\boldsymbol{x}'),$$
and
$$\begin{aligned} D_{\underline{\boldsymbol{x}}_q}(\underline{\eta} F_2(\boldsymbol{x}')) &= \sum_{i=p+1}^{p+q} e_i \underline{\eta} \partial_r F_2(\boldsymbol{x}')(\partial_{x_i} r) \\ &= \sum_{i=p+1}^{p+q} e_i \eta_i \underline{\eta} \partial_r F_2(\boldsymbol{x}') = \underline{\eta}^2 \partial_r F_2(\boldsymbol{x}') = -\partial_r F_2(\boldsymbol{x}'), \end{aligned}$$

we obtain

$$\begin{aligned} D_{\boldsymbol{x}} g(\boldsymbol{x}) &= (D_{\boldsymbol{x}_p} + D_{\underline{\boldsymbol{x}}_q})(F_1(\boldsymbol{x}') + \underline{\eta} F_2(\boldsymbol{x}')) \\ &= D_{\boldsymbol{x}_p} F_1(\boldsymbol{x}') + D_{\boldsymbol{x}_p}(\underline{\eta} F_2(\boldsymbol{x}')) + \underline{\eta}\partial_r F_2(\boldsymbol{x}') - \partial_r F_2(\boldsymbol{x}') \\ &= D_{\boldsymbol{x}_p} F_1(\boldsymbol{x}') - \partial_r F_2(\boldsymbol{x}') + \underline{\eta}(\overline{D}_{\boldsymbol{x}_p} F_2(\boldsymbol{x}') + \partial_r F_1(\boldsymbol{x}')). \end{aligned}$$

Hence, (8) implies that $D_{\boldsymbol{x}} g(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \Omega(\underline{\eta})$. The proof is complete. □

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