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# Generalized Derivations and Rota-Baxter Operators of $n$-ary Hom-Nambu Superalgebras

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**Abstract.** The aim of this paper is to generalise the construction of $n$-ary Hom-Lie bracket by means of an $(n-2)$-cochain of given Hom-Lie algebra to super case inducing $n$-Hom-Lie superalgebras. We study the notion of generalized derivations and Rota-Baxter operators of $n$-ary Hom-Nambu and $n$-Hom-Lie superalgebras and their relation with generalized derivations and Rota-Baxter operators of Hom-Lie superalgebras. We also introduce the notion of 3-Hom-pre-Lie superalgebras which is the generalization of 3-Hom-pre-Lie algebras.

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**Keywords.** Hom-Lie superalgebras, $n$-ary Nambu superalgebras, $n$-ary Hom-Nambu superalgebras, $n$-Hom-Lie superalgebras, Derivations, Quasiderivations, Rota-Baxter operators, 3-Hom-pre-Lie algebras.

## Introduction

Hom-Lie algebras and more general quasi-Hom-Lie algebras were introduced first by Hartwig, Larsson and Silvestrov in [49], where the general quasi-deformations and discretizations of Lie algebras of vector fields using more general $\sigma$-derivations (twisted derivations) and a general method for construction of deformations of Witt and Virasoro type algebras based on twisted derivations have been developed. The general quasi-Lie algebras and the subclasses of quasi-Hom-Lie algebras and Hom-Lie algebras and their more general color Hom-algebra counterparts as well as corresponding general quasi-Leibniz algebras, and thus also Hom-Leibniz algebras in context of Hom-Lie algebras, have been introduced in [49, 61–63, 83]. In [71], the Hom-associative

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algebras have been introduced and shown to be Hom-Lie admissible, in the usual sense of leading to Hom-Lie algebras using commutator map as new product, thus constituting a natural generalization of associative algebras known to be Lie admissible algebras in the same sense of yielding Lie algebras using the commutator product. Moreover, in [71], more general $G$-Hom-associative algebras including Hom-associative algebras, Hom-Vinberg algebras (Hom-left symmetric algebras), Hom-pre-Lie algebras (Hom-right symmetric algebras) and some other Hom-algebra structures, were introduced and shown to be Hom-Lie admissible. Also, flexible Hom-algebras have been introduced and some connections to some Hom-algebra generalizations of derivations and of adjoint maps have been noticed, and the variety of $n$-dimensional Hom-Lie algebras have been considered and some classes of low-dimensional Hom-Lie algebras have been described. Since the pioneering works [49,60–64,71,78,82], Hom-algebra structures have developed in a popular broad area with increasing number of publications in various directions. In Hom-algebra structures, defining algebra identities are twisted by linear maps. Hom-algebra structures of a given type include their classical counterparts and open more possibilities for deformations, Hom-algebra extensions of cohomological structures and representations, formal deformations of Hom-associative and Hom-Lie algebras, Hom-Lie admissible Hom-coalgebras, Hom-coalgebras, Hom-Hopf algebras [9,35,61,72–74,81,88,89].

The $n$-Lie algebras found their applications in many fields of mathematics and physics. Ternary Lie algebras appeared first in Nambu generalization of Hamiltonian mechanics [76] using ternary bracket generalization of Poisson algebras. The algebraic foundations of Nambu mechanics and foundations of the theory of Nambu-Poisson manifolds have been developed in the works of Takhtajan and Daletskii in [40,84,85]. Filippov, in [42] introduced $n$-Lie algebras. Further properties, classification, and connections to other structures such as bialgebras, Yang-Baxter equation and Manin triples for 3-Lie algebras of $n$-ary algebras were studied in [19–28,50]. Hom-algebra generalization of $n$-ary algebras, such as $n$-Hom-Lie algebras and other $n$-ary Hom algebras of Lie type and associative type, were introduced in [17], by twisting the defining identities using a set of linear maps. A way to generate examples of such $n$-ary Hom-algebras from $n$-ary algebras of the same type has been described. Representations and cohomology of $n$-ary multiplicative Hom-Nambu-Lie algebras have been considered in [10]. Further properties, construction methods, examples, cohomology and central extensions of $n$-ary Hom-algebras have been considered in [14–16,56,57,89–92]. These generalizations include $n$-ary Hom-algebra structures generalizing the $n$-ary algebras of Lie type including $n$-ary Nambu algebras, $n$-ary Nambu-Lie algebras and $n$-ary Lie algebras, and $n$-ary algebras of associative type including $n$-ary totally associative and $n$-ary partially associative algebras.

The construction of $(n+1)$-Lie algebras induced by $n$-Lie algebras using combination of bracket multiplication with a trace, motivated by the work of Awata et al. [18] on the quantization of the Nambu brackets, was generalized using the brackets of general Hom-Lie algebra or $n$-Hom-Lie algebra and trace-like linear forms satisfying some conditions depending on the linear maps defining the Hom-Lie or $n$-Hom-Lie algebras in [15,16]. The structure of 3-Lie algebras induced by Lie algebras, classification of 3-Lie algebras and application to constructions of B.R.S. algebras have been considered in [4,5,7]. Interesting constructions of ternary Lie superalgebras in connection to superspace extension of Nambu-Hamilton equation is considered in [8]. In [33], a method was demonstrated of how to construct $n$-ary multiplications from the binary multiplication of a Hom-Lie algebra and a $(n-2)$-linear function satisfying certain compatibility conditions. Solvability and Nilpotency for $n$-Hom-Lie Algebras and $(n+1)$-Hom-Lie Algebras Induced by $n$-Hom-Lie Algebras have been considered in [59]. In [37], Leibniz $n$-algebras have been studied. The general cohomology theory for $n$-Lie algebras and Leibniz $n$-algebras was established in [80]. The structure and classification of finite-dimensional $n$-Lie algebras were considered in [67] and many other authors. For more details of the theory and applications of $n$-Lie algebras, see [41] and references therein.

Derivations and generalized derivations of different algebraic structures are an important subject of study in algebra and diverse areas. They appear in many fields of mathematics and physics. In particular, they appear in representation theory and cohomology theory among other areas. They have various applications relating algebra to geometry and allow the construction of new algebraic structures. There are many generalizations of derivations. For example, Leibniz derivations [51] and $\delta$-derivations of prime Lie and Malcev algebras [43–45]. The properties and structure of generalized derivations algebras of a Lie algebra and their subalgebras and quasi-derivation algebras were systematically studied in [66], where it was proved for example that the quasi-derivation algebra of a Lie algebra can be embedded into the derivation algebra of a larger Lie algebra. Derivations and generalized derivations of $n$-ary algebras were considered in [77,87] and it was demonstrated substantial differences in structures and properties of derivations on Lie algebras and on $n$-ary Lie algebras for $n > 2$. Generalized derivations of Lie superalgebras have been considered in [93]. Generalized derivations of Lie color algebras and $n$-ary (color) algebras have been studied in [38,52–55]. Generalized derivations of Lie triple systems have been considered in [39]. Generalized derivations of various kinds can be viewed as a generalization of $\delta$-derivation. Quasi-Hom-Lie and Hom-Lie structures for $\sigma$-derivations and $(\sigma,\tau)$-derivations have been considered in [46,49,64,78,79]. Graded $q$-differential algebra and applications to semi-commutative Galois Extensions and Reduced Quantum Plane and $q$-connection was studied in [2,3,6]. Generalized $N$-complexes coming from twisted derivations where considered in [65].

Generalizations of derivations in connection with extensions and enveloping algebras of Hom-Lie color algebras and Hom-Lie superalgebras have been considered in [12,13,31,48]. Generalized derivations of multiplicative $n$-ary Hom-$\Omega$ color algebras have been studied in [36]. Derivations, $L$-modules, $L$-comodules and Hom-Lie quasi-bialgebras have been considered in [29,30]. In [58], constructions of $n$-ary generalizations of BiHom-Lie algebras and BiHom-associative algebras have been considered. Generalized Derivations of $n$-BiHom-Lie algebras have been studied in [34]. Color Hom-algebra structures associated to Rota-Baxter operators have been considered in context of Hom-dendriform color algebras in [32]. Rota-Baxter bisystems and covariant bialgebras, Rota-Baxter cosystems, coquasitriangular mixed bialgebras, coassociative Yang-Baxter pairs, coassociative Yang-Baxter equation and generalizations of Rota-Baxter systems and algebras, curved $\mathcal{O}$-operator systems and their connections with (tri)dendriform systems and pre-Lie algebras have been considered in [68–70]. Generalisations of derivations are important for Hom-Gerstenhaber algebras, Hom-Lie algebroids and Hom-Lie-Rinehart algebras and Hom-Poisson homology [75].

This paper is organized as follows. In Sect. 1 we review basic concepts of Hom-Lie, $n$-ary Hom-Nambu superalgebras and $n$-Hom-Lie algebras. We also recall some examples and classification of Hom-Lie superalgebras of dimension two. We recall the definition of generalized derivations of $n$-Hom-Lie superalgebras and $n$-ary Hom-Nambu superalgebras. In Sect. 2 we provide a construction procedure of $n$-Hom-Lie superalgebras starting from a binary bracket of a Hom-Lie superalgebra and multilinear form satisfying certain conditions. To this end, we give the relation between generalized derivations of Hom-Lie superalgebra and generalized derivations of $n$-Hom-Lie algebras. In Sect. 3, we provide a construction for $n$-ary Hom-Nambu algebra using Hom-Lie algebra. In Sect. 4 the notion of Rota-Baxter operators of $n$-ary Hom-Nambu superalgebras are introduced and some results obtained. Finally, we give the definition of 3-Hom-pre-Lie superalgebras generalizing 3-Hom-pre-Lie algebras in graded case.

## 1. Preliminaries on $n$-ary Hom-Lie Algebras and Hom-Lie Superalgebras

Throughout this paper, we will for simplicity of exposition assume that $\mathbb{K}$ is an algebraically closed field of characteristic zero, even though for most of the general definitions and results in the paper this assumption is not essential.

Let $V = V_0 \oplus V_1$ be a finite-dimensional $\mathbb{Z}_2$-graded linear space. Let $\mathcal{H}(V) = V_0 \cup V_1$ denote the set of homogeneous elements of $V$. If $v \in V$ is a homogenous element, then its degree will be denoted by $|v|$, where $|v| \in \mathbb{Z}_2$ and $\mathbb{Z}_2 = \{0, 1\}$. Let $\mathrm{End}(V)$ be the $\mathbb{Z}_2$-graded linear space of endomorphisms of a $\mathbb{Z}_2$-graded linear space $V = V_0 \oplus V_1$.

The composition of two endomorphisms $a \circ b$ determines the structure of superalgebra in $\mathrm{End}(V)$, and the graded binary commutator $[a, b] = a \circ b - (-1)^{|a||b|} b \circ a$ induces the structure of Lie superalgebras in $\mathrm{End}(V)$.

### 1.1. Definitions and Notations

**Definition 1.1.** [11,62,63] A Hom-Lie superalgebra is a $\mathbb{Z}_2$-graded linear space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ over a field $\mathbb{K}$ equipped with an even bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, (meaning that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, $\forall i, j \in \mathbb{Z}_2$) and an even linear map $\alpha : \mathfrak{g} \to \mathfrak{g}$ (meaning that $\alpha(\mathfrak{g}_i) \subseteq \mathfrak{g}_i, \forall i \in \mathbb{Z}_2$).

$$[x, y] = -(-1)^{|x||y|}[y, x] \quad \text{(super-skew-symmetry)}$$

$$\circlearrowleft_{x,y,z} (-1)^{|x||z|}[\alpha(x),[y,z]] = 0 \quad \text{(super-Hom-Jacobi identity)}$$

for all $x, y, z \in \mathcal{H}(\mathfrak{g})$, where $\circlearrowleft_{x,y,z}$ denotes summation over the cyclic permutations of $x, y, z$.

**Definition 1.2.** A Hom-Lie superalgebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ is called multiplicative if $\alpha([x,y]) = [\alpha(x), \alpha(y)]$ for all $x, y \in \mathfrak{g}$.

For any $x \in \mathfrak{g}$, define $\mathrm{ad}_x \in \mathrm{End}_{\mathbb{K}}(\mathfrak{g})$ by $\mathrm{ad}_x(y) = [x, y]$, for any $y \in \mathfrak{g}$. Then the super-Hom-Jacobi identity can be written as

$$\mathrm{ad}_{[x,y]}(\alpha(z)) = \mathrm{ad}_{\alpha(x)} \circ \mathrm{ad}_y(z) - (-1)^{|x||y|}\mathrm{ad}_{\alpha(y)} \circ \mathrm{ad}_x(z) \tag{1}$$

for all $x, y, z \in \mathcal{H}(\mathfrak{g})$.

*Remark 1.3.* An ordinary Lie superalgebra is a Hom-Lie superalgebra when $\alpha = id$.

*Example 1.4.* [1] Let $\mathcal{A}$ be the complex superalgebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ where $\mathcal{A}_0 = \mathbb{C}[t, t^{-1}]$ is the Laurent polynomials in one variable and $\mathcal{A}_1 = \theta\mathbb{C}[t, t^{-1}]$, where $\theta$ is the Grassman variable ($\theta^2 = 0$). We assume that $t$ and $\theta$ commute. The generators of $\mathcal{A}$ are of the form $t^n$ and $\theta t^n$ for $n \in \mathbb{Z}$. For $q \in \mathbb{C}\backslash\{0, 1\}$ and $n \in \mathbb{N}$, we set $\{n\} = \frac{1-q^n}{1-q}$, a $q$-number. The $q$-numbers have the following properties

$$\{n+1\} = 1 + q\{n\} = \{n\} + q^n \text{ and } \{n+m\} = \{n\} + q^n\{m\}.$$

Let $\mathfrak{A}_q$ be a superspace with basis $\{L_m,\ I_m \,|\, m \in \mathbb{Z}\}$ of parity 0 and $\{G_m,\ T_m \,|\, m \in \mathbb{Z}\}$ of parity 1, where $L_m = -t^m D,\ I_m = -t^m,\ G_m = -\theta t^m D,\ T_m = -\theta t^m$ and $D$ is a $q$-derivation on $\mathcal{A}$ such that

$$D(t^m) = \{m\}t^m,\ D(\theta t^m) = \{m+1\}\theta t^m.$$

We define the bracket $[\cdot,\cdot]_q : \mathfrak{A}_q \times \mathfrak{A}_q \to \mathfrak{A}_q$, with respect the super-skew-symmetry for $n, m \in \mathbb{Z}$ by

$$[L_m, L_n]_q = (\{m\} - \{n\})L_{m+n}, \tag{2}$$
$$[L_m, I_n]_q = -\{n\}I_{m+n}, \tag{3}$$
$$[L_m, G_n]_q = (\{m\} - \{n+1\})G_{m+n}, \tag{4}$$
$$[I_m, G_n]_q = \{m\}T_{m+n}, \tag{5}$$
$$[L_m, T_n]_q = -\{n+1\}T_{m+n}, \tag{6}$$
$$[I_m, I_n]_q = [I_m, T_n]_q = [T_m, G_n]_q = [T_m, T_n]_q = [G_m, G_n]_q = 0. \tag{7}$$

Let $\alpha_q$ be an even linear map on $\mathfrak{A}_q$ defined on the generators by

$$\alpha_q(L_n) = (1+q^n)L_n, \quad \alpha_q(I_n) = (1+q^n)I_n,$$
$$\alpha_q(T_n) = (1+q^{n+1})G_n,\ \alpha_q(T_n) = (1+q^{n+1})T_n.$$

The triple $(\mathfrak{A}_q, [\cdot,\cdot]_q, \alpha_q)$ is a Hom-Lie superalgebra, called *$q$-deformed Heisenberg-Virasoro superalgebra of Hom-type.*

*Example 1.5.* In [11], the authors construct an example of Hom-Lie superalgebra, which is not a Lie superalgebra starting from the orthosymplectic Lie superalgebra. We consider in the sequel the matrix realization of this Lie superalgebra.

Let $\mathfrak{osp}(1,2) = V_0 \oplus V_1$ be the Lie superalgebra where $V_0$ is generated by:

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and $V_1$ is generated by

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Those of the defining relations that have nonzero elements in the right-hand side are

$$[H,X] = 2X,\ [H,Y] = -2Y,\ [X,Y] = H,$$
$$[Y,G] = F,\ [X,F] = G,\ [H,F] = -F,\ [H,G] = G,$$
$$[G,F] = H,\ [G,G] = -2X,\ [F,F] = 2Y.$$

Let $\lambda \in \mathbb{R}^*$, we consider the linear map $\alpha_\lambda : \mathfrak{osp}(1,2) \to \mathfrak{osp}(1,2)$ defined by:

$$\alpha_\lambda(X) = \lambda^2 X, \quad \alpha_\lambda(Y) = \frac{1}{\lambda^2}Y, \quad \alpha_\lambda(H) = H,$$
$$\alpha_\lambda(F) = \frac{1}{\lambda}F, \quad \alpha_\lambda(G) = \lambda G.$$

We provide a family of Hom-Lie superalgebras $\mathfrak{osp}(1,2)_\lambda = (\mathfrak{osp}(1,2), [\cdot,\cdot]_{\alpha_\lambda}, \alpha_\lambda)$, where the Hom-Lie superalgebra bracket $[\cdot,\cdot]_{\alpha_\lambda}$ on the basis elements is given, for $\lambda \neq 0$, by:

$$[H,X]_{\alpha_\lambda} = 2\lambda^2 X, \quad [H,Y]_{\alpha_\lambda} = -\frac{2}{\lambda^2}Y, \quad [X,Y]_{\alpha_\lambda} = H,$$
$$[Y,G]_{\alpha_\lambda} = \frac{1}{\lambda}F, \quad [X,F]_{\alpha_\lambda} = \lambda G, \quad [H,F]_{\alpha_\lambda} = -\frac{1}{\lambda}F, \quad [H,G]_{\alpha_\lambda} = \lambda G,$$
$$[G,F]_{\alpha_\lambda} = H, \quad [G,G]_{\alpha_\lambda} = -2\lambda^2 X, \quad [F,F]_{\alpha_\lambda} = \frac{2}{\lambda^2}Y.$$

These Hom-Lie superalgebras are not Lie superalgebras for $\lambda \neq 1$.

**Theorem 1.6.** [86] *Every 2-dimensional multiplicative Hom-Lie superalgebra $(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, [\cdot,\cdot], \alpha)$ generated by $\{e_1, e_2\}$ is isomorphic to one of the following nonisomorphic Hom-Lie superalgebras. Each algebra is denoted by $\mathfrak{g}^k_{i,j}$, where $i$ is the dimension of $\mathfrak{g}_0$, $j$ is the dimension of $\mathfrak{g}_1$, $k$ is the number.*

1. $\mathfrak{g}^1_{0,2}$*: is an abelian Hom-Lie superalgebra.*
2. $\mathfrak{g}^2_{1,1}$*: is an abelian Hom-Lie superalgebra.*
3. $\mathfrak{g}^3_{1,1}$*:* $[e_0, e_1] = e_1$, $[e_1, e_1] = 0$ *and* $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$, $a \in \mathbb{K}$.
4. $\mathfrak{g}^4_{1,1}$*:*$[e_0, e_1] = e_1$, $[e_1, e_1] = 0$ *and* $\alpha = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $a \neq 0, 1$.
5. $\mathfrak{g}^5_{1,1}$*:* $[e_0, e_1] = 0$, $[e_1, e_1] = e_0$ *and* $\alpha = \begin{pmatrix} a^2 & 0 \\ 0 & a \end{pmatrix}$, $a \neq 0$.

Now, we recall the definitions of $n$-ary Hom-Nambu superalgebras and $n$-Hom-Lie superalgebras, generalizing $n$-ary Nambu superalgebras and $n$-Lie superalgebras (see [1]).

**Definition 1.7.** An *$n$-ary Hom-Nambu* superalgebra $(\mathcal{N}, [\cdot, \dots, \cdot], \widetilde{\alpha})$ is a triple consisting of a linear space $\mathcal{N} = \mathcal{N}_0 \oplus \mathcal{N}_1$, an even $n$-linear map $[\cdot, \dots, \cdot] : \mathcal{N}^n \to \mathcal{N}$ such that $[\mathcal{N}_{k_1}, \dots, \mathcal{N}_{k_n}] \subset \mathcal{N}_{k_1+\cdots+k_n}$ and a family $\widetilde{\alpha} = (\alpha_i)_{1\leq i\leq n-1}$ of even linear maps $\alpha_i : \ \mathcal{N} \to \mathcal{N}$, satisfying

$$\begin{aligned} &\forall (x_1, \dots, x_{n-1}) \in \mathcal{H}(\mathcal{N})^{n-1},\ (y_1, \dots, y_n) \in \mathcal{H}(\mathcal{N})^n :\\ &\quad \big[\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1}), [y_1, \dots, y_n]\big]\\ &= \sum_{i=1}^{n} (-1)^{|X||Y|^{i-1}} \big[\alpha_1(y_1), \dots, \alpha_{i-1}(y_{i-1}),\\ &\quad [x_1, \dots, x_{n-1}, y_i], \alpha_i(y_{i+1}), \dots, \alpha_{n-1}(y_n)\big], \end{aligned} \tag{8}$$

where $|X| = \sum_{k=1}^{n-1} |x_k|$ and $|Y|^{i-1} = \sum_{k=1}^{i-1} |y_k|$.

The identity (8) is called *super-Hom-Nambu identity.*

Let $\widetilde{\alpha} : \mathcal{N}^{n-1} \to \mathcal{N}^{n-1}$ be an even linear map defined for all $X = (x_1, \dots, x_{n-1}) \in \mathcal{N}^{n-1}$ by $\widetilde{\alpha}(X) = (\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1})) \in \mathcal{N}^{n-1}$. For all $X = (x_1, \dots, x_{n-1}) \in \mathcal{N}^{n-1}$, the map $\mathrm{ad}_X : \mathcal{N} \to \mathcal{N}$ defined by

$$\mathrm{ad}_X(y) = [x_1, \dots, x_{n-1}, y], \quad \forall y \in \mathcal{N}, \tag{9}$$

is called adjoint map. Then the super-Hom-Nambu identity (8) may be written in terms of adjoint map as

$$\begin{aligned} \mathrm{ad}_{\widetilde{\alpha}(X)}([y_1, \dots, y_n]) = &\sum_{i=1}^{n-1} (-1)^{|X||Y|^{i-1}} \left[\alpha_1(y_1), \dots, \alpha_{i-1}(y_{i-1}), \mathrm{ad}_X(y_i),\right.\\ &\left.\alpha_i(y_{i+1}) \dots, \alpha_{n-1}(y_n)\right]. \end{aligned}$$

**Definition 1.8.** An $n$-ary Hom-Nambu superalgebra $(\mathcal{N}, [\cdot, \dots, \cdot], \widetilde{\alpha})$ is called $n$-Hom-Lie superalgebra if the bracket $[\cdot, \dots, \cdot]$ is super-skewsymmetric that is

$$\begin{aligned} &\forall\, 1 \leq i \leq n-1 :\\ &\quad [x_1, \dots, x_i, x_{i+1}, \dots, x_n] = -(-1)^{|x_i||x_{i+1}|}[x_1, \dots, x_{i+1}, x_i, \dots, x_n]. \end{aligned} \tag{10}$$

It is equivalent to

$$\begin{aligned} &\forall\, 1 \leq i < j \leq n :\\ &\quad [x_1, \dots, x_i, \dots, x_j, \dots, x_n] = -(-1)^{|X|_{i+1}^{j-1}(|x_i|+|x_j|)+|x_i||x_j|}\\ &\qquad\qquad [x_1, \dots, x_j, \dots, x_i, \dots, x_n] \end{aligned} \tag{11}$$

where $x_1, \dots, x_n \in \mathcal{H}(\mathcal{N})$ and $|X|_i^j = \sum_{k=i}^{j} |x_k|$.

*Remark 1.9.* When the maps $(\alpha_i)_{1\leq i\leq n-1}$ are all identity maps, one recovers the classical $n$-ary Nambu superalgebras.

Let $(\mathcal{N},[\cdot,\dots,\cdot],\widetilde{\alpha})$ and $(\mathcal{N}',[\cdot,\dots,\cdot]',\widetilde{\alpha}')$ be two $n$-ary Hom-Nambu superalgebras where $\widetilde{\alpha}=(\alpha_i)_{1\leq i\leq n-1}$ and $\widetilde{\alpha}'=(\alpha'_i)_{1\leq i\leq n-1}$. A linear map $f:\mathcal{N}\rightarrow\mathcal{N}'$ is an $n$-ary Hom-Nambu superalgebras *morphism* if it satisfies

$$f([x_1,\dots,x_n])=[f(x_1),\dots,f(x_n)]',$$
$$f\circ\alpha_i=\alpha'_i\circ f,\quad \forall i=1,\dots,n-1.$$

In the sequel we deal sometimes with a particular class of $n$-ary Hom-Nambu superalgebras which we call $n$-ary multiplicative Hom-Nambu superalgebras.

**Definition 1.10.** A *multiplicative n-ary Hom-Nambu superalgebra* (resp. *multiplicative n-Hom-Lie superalgebra*) is an $n$-ary Hom-Nambu superalgebra (resp. $n$-Hom-Lie superalgebra) $(\mathcal{N},[\cdot,\dots,\cdot],\widetilde{\alpha})$ with $\widetilde{\alpha}=(\alpha_i)_{1\leq i\leq n-1}$ where $\alpha_1=\cdots=\alpha_{n-1}=\alpha$ and satisfying

$$\alpha([x_1,\dots,x_n])=[\alpha(x_1),\dots,\alpha(x_n)],\ \ \forall x_1,\dots,x_n\in\mathcal{N}. \tag{12}$$

For simplicity, denote the $n$-ary multiplicative Hom-Nambu superalgebra as $(\mathcal{N},[\cdot,\dots,\cdot],\alpha)$ where $\alpha:\mathcal{N}\rightarrow\mathcal{N}$ is an even linear map. Also by misuse of language an element $X\in\mathcal{N}^n$ refers to $X=(x_1,\dots,x_n)$ and $\alpha(X)$ denotes $(\alpha(x_1),\dots,\alpha(x_n))$.

**Definition 1.11.** A multiplicative $n$-ary Hom-Nambu superalgebra $(\mathcal{N},[\cdot,\dots,\cdot],\alpha)$ is called regular if $\alpha$ is bijective.

### 1.2. Derivations, Quasiderivations and Generalized Derivations of Multiplicative $n$-ary Hom-Nambu Superalgebras

In this section we recall the definition of derivation, quasiderivation and generalized derivation of multiplicative $n$-ary Hom-Nambu superalgebras.

Let $(\mathcal{N},[\cdot,\dots,\cdot],\alpha)$ be a multiplicative $n$-ary Hom-Nambu superalgebra. We denote by $\alpha^k$ the $k$-times composition of $\alpha$ (that is $\alpha^k=\underbrace{\alpha\circ\cdots\circ\alpha}_{k-times}$, $\alpha^0=Id$ and $\alpha^1=\alpha$).

**Definition 1.12.** For any $k\geq 1$, we call $\mathfrak{D}\in\mathrm{End}(\mathcal{N})$ an $\alpha^k$-*derivation* of the multiplicative $n$-ary Hom-Nambu superalgebra $(\mathcal{N},[\cdot,\dots,\cdot],\alpha)$ if

$$[\mathfrak{D},\alpha]=0\ \text{ i.e., }\ \mathfrak{D}\circ\alpha=\alpha\circ\mathfrak{D}; \tag{13}$$

$$\mathfrak{D}[x_1,\dots,x_n]=\sum_{i=1}^{n}(-1)^{|\mathfrak{D}||X|^{i-1}}[\alpha^k(x_1),\dots,\alpha^k(x_{i-1}),\mathfrak{D}(x_i),\alpha^k(x_{i+1}),\dots,\alpha^k(x_n)]. \tag{14}$$

We denote by $\mathrm{Der}_{\alpha^k}(\mathcal{N})$ the set of $\alpha^k$-derivations of the multiplicative $n$-Hom-Lie superalgebra $\mathcal{N}$.

For $X=(x_1,\dots,x_{n-1})\in\mathcal{N}^{n-1}$ satisfying $\alpha(X)=X$ and $k\geq 1$, we define the map $\mathrm{ad}^k_X\in\mathrm{End}(\mathcal{N})$ by

$$\mathrm{ad}^k_X(y)=[x_1,\dots,x_{n-1},\alpha^k(y)]\ \ \forall y\in\mathcal{N}. \tag{15}$$

Then, we find the following result.

**Lemma 1.13.** *The map* $ad_X^k$ *is an* $\alpha^{k+1}$*-derivation (called inner* $\alpha^{k+1}$*-derivation), and* $|ad_X^k| = |X|$.

We denote by $Inn_{\alpha^k}(\mathcal{N})$ the space generate by all the inner $\alpha^{k+1}$-derivations. For any $\mathfrak{D} \in \mathrm{Der}_{\alpha^k}(\mathcal{N})$ and $\mathfrak{D}' \in \mathrm{Der}_{\alpha^{k'}}(\mathcal{N})$ we define their supercommutator $[\mathfrak{D}, \mathfrak{D}']$ as usual:

$$[\mathfrak{D}, \mathfrak{D}'] = \mathfrak{D} \circ \mathfrak{D}' - (-1)^{|\mathfrak{D}||\mathfrak{D}'|}\mathfrak{D}' \circ \mathfrak{D}, \tag{16}$$

then $[\mathfrak{D}, \mathfrak{D}'] \in \mathrm{Der}_{\alpha^{k+k'}}(\mathcal{N})$. Set $\mathrm{Der}(\mathcal{N}) = \bigoplus_{k\geq 0} \mathrm{Der}_{\alpha^k}(\mathcal{N})$ and $Inn(\mathcal{N}) = \bigoplus_{k\geq 0} Inn_{\alpha^k}(\mathcal{N})$, the pair $(\mathrm{Der}(\mathcal{N}), [\cdot,\cdot])$ is a Lie superalgebra.

**Definition 1.14.** Let $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ be a multiplicative $n$-ary Hom-Nambu superalgebra. An endomorphism $\mathfrak{D} \in \mathrm{End}(\mathcal{N})$ is said to be an $\alpha^k-$quasiderivation, if there exists an endomorphism $\mathfrak{D}' \in \mathrm{End}(\mathcal{N})$ such that

$$\sum_{i=1}^{n}(-1)^{|\mathfrak{D}||X|^{i-1}}[\alpha^k(x_1), \dots, \mathfrak{D}(x_i), \dots, \alpha^k(x_n)] = \mathfrak{D}'([x_1, \dots, x_n]),$$

for all $x_1, \dots, x_n \in \mathcal{N}$. We call $\mathfrak{D}'$ the endomorphism associated to $\alpha^k-$quasiderivation $\mathfrak{D}$.

We denote the set of $\alpha^k$-quasiderivations by $\mathrm{QDer}_{\alpha^k}(\mathcal{N})$ and

$$\mathrm{QDer}(\mathcal{N}) = \bigoplus_{k\geq 0} \mathrm{QDer}_{\alpha^k}(\mathcal{N}).$$

**Definition 1.15.** An endomorphism $\mathfrak{D}$ of a multiplicative $n$-ary Hom-Nambu superalgebra $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ is called a generalized $\alpha^k$-derivation if there exist linear mappings

$$\mathfrak{D}', \mathfrak{D}'', \dots, \mathfrak{D}^{(n-1)}, \mathfrak{D}^{(n)} \in \mathrm{End}(\mathcal{N})$$

such that

$$\mathfrak{D}^{(n)}([x_1, \dots, x_n]) = \sum_{i=1}^{n}(-1)^{|\mathfrak{D}^{(i-1)}||X|^{i-1}}[\alpha^k(x_1), \dots, \mathfrak{D}^{(i-1)}(x_i), \dots, \alpha^k(x_n)], \tag{17}$$

for all $x_1, \dots, x_n \in \mathcal{N}$. An $(n+1)$-tuple $(\mathfrak{D}, \mathfrak{D}', \mathfrak{D}'', \dots, \mathfrak{D}^{(n-1)}, \mathfrak{D}^{(n)})$ is called an $(n+1)$-ary $\alpha^k$-derivation.

We denote the set of generalized $\alpha^k$-derivations by $\mathrm{GDer}_{\alpha^k}(\mathcal{N})$ and

$$\mathrm{GDer}(\mathcal{N}) = \bigoplus_{k\geq 0} \mathrm{GDer}_{\alpha^k}(\mathcal{N}).$$

## 2. $n$-Hom-Lie Superalgebras Induced by Hom-Lie Superalgebras

In [47], the authors introduced a construction of a 3-Hom-Lie superalgebra from a Hom-Lie superalgebra. It is called 3-Hom-Lie superalgebra induced by Hom-Lie superalgebra. In this section we generalize this construction to the $n$-ary Hom-algebras by the approach in [4].

Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a multiplicative Hom-Lie superalgebra and $\mathfrak{g}^*$ be its dual superspace. Fix an even element of the dual space $\varphi \in \mathfrak{g}^*$. Define the triple product as follows

$$\forall\ x,\ y,\ z \in \mathcal{H}(\mathfrak{g}) : [x,y,z] = \varphi(x)[y,z] + (-1)^{|x|(|y|+|z|)}\varphi(y)[z,x] \\ +(-1)^{|z|(|x|+|y|)}\varphi(z)[x,y]. \tag{18}$$

Obviously this triple product is super-skew-symmetric. It is straightforward to compute the left-hand side and the right-hand side of the super-Hom-Nambu identity (8) if $\varphi \circ \alpha = \varphi$ and

$$\varphi(x)\varphi([y,z]) + (-1)^{|x|(|y|+|z|)}\varphi(y)\varphi([z,x]) + \\ (-1)^{|z|(|x|+|y|)}\varphi(z)\varphi([x,y]) = 0. \tag{19}$$

Now we consider $\varphi$ as an even $\mathbb{K}$-valued cochain of degree one of the Chevalley-Eilenberg complex of a Hom-Lie superalgebra $\mathfrak{g}$. Let coboundary operator $\delta : \wedge^k \mathfrak{g}^* \to \wedge^k \mathfrak{g}^*$ be defined by

$$\delta f(x_1,\ldots,x_{k+1}) \\ = \sum_{i<j} (-1)^{i+j+1}(-1)^{\gamma_{ij}^X} f([x_i,x_j]_{\mathfrak{g}}, \alpha(x_1)\ldots,\widehat{x_i},\ldots,\widehat{x_j},\ldots,\alpha(x_{k+1})), \tag{20}$$

where $\gamma_{ij}^X = |X|_{j+1}^n(|x_i|+|x_j|) + |x_i||X|_{j-1}^{i+1}$, for $f \in \wedge^k \mathfrak{g}^*$ and for all $x_1,\ldots,x_{k+1} \in \mathcal{H}(\mathfrak{g})$. Then, $\delta\varphi(x,y) = \varphi([x,y])$. Finally, we can define the wedge product of two cochains $\varphi$ and $\delta\varphi$, which is the cochain of degree three by

$$\varphi\Lambda\delta\varphi(x,y,z) = \varphi(x)\varphi([y,z]) + (-1)^{|x|(|y|+|z|)}\varphi(y)\varphi([z,x]) \\ +(-1)^{|z|(|x|+|y|)}\varphi(z)\varphi([x,y]).$$

Hence (19) is equivalent to $\varphi\Lambda\delta\varphi = 0$. Thus, if an 1-cochain $\varphi$ satisfies the equation (19), then the triple product (18) is the 3-Hom-Lie bracket, and we will call this multiplicative 3-Hom-Lie bracket the quantum Hom-Nambu bracket induced by an even 1-cochain.

**Definition 2.1.** Let $\phi \in \wedge^{n-2}\mathfrak{g}^*$ be an even $(n-1)$-cochain, we define the $n$-ary product as follows

$$[x_1,\ldots,x_n]_\phi = \\ \sum_{i<j}^{n} (-1)^{i+j+1}(-1)^{\gamma_{ij}^X}\phi(x_1,\ldots,\hat{x_i},\ldots,\hat{x_j},\ldots,x_n)[x_i,x_j], \tag{21}$$

for all $x_1,\ldots,x_n \in \mathcal{H}(\mathfrak{g})$.

It is clair that $[\cdot,\ldots,\cdot]_\phi$ is an even $n$-linear map.

**Proposition 2.2.** *The $n$-ary product $[\cdot,\ldots,\cdot]_\phi$ is super-skew-symmetric.*

*Proof.* Let $x_1,\ldots,x_n \in \mathcal{H}(\mathfrak{g})$ and fix an integer $1 \le i \le n-1$. Then,

$$\begin{aligned}
&[x_1,\ldots,x_i,x_{i+1},\ldots,x_n]_\phi\\
&= \sum_{k<l<i} (-1)^{k+l+1}(-1)^{\gamma^X_{kl}}\phi(x_1,\ldots,\widehat{x}_k,\ldots,\widehat{x}_l\ldots,x_i,\ldots,x_n)[x_k,x_l]\\
&\quad + \sum_{i+1<k<l} (-1)^{k+l+1}(-1)^{\gamma^X_{kl}}\phi(x_1,\ldots,x_i,x_{i+1},\ldots,\widehat{x}_k\ldots,\widehat{x}_l,\ldots,x_n)[x_k,x_l]\\
&\quad + \sum_{k=i<l\neq i+1} (-1)^{i+l+1}(-1)^{\gamma^X_{il}}\phi(x_1,\ldots,\widehat{x}_i,x_{i+1},\ldots,\widehat{x}_l,\ldots,x_n)[x_i,x_l]\\
&\quad + \sum_{k<l=i} (-1)^{k+i+1}(-1)^{\gamma^X_{ki}}\phi(x_1,\ldots,\widehat{x}_k,\ldots,\widehat{x}_i,x_{i+1},\ldots,x_n)[x_k,x_i]\\
&\quad + \sum_{i\neq k<l=i+1} (-1)^{k+i}(-1)^{\gamma^X_{k,i+1}}\phi(x_1,\ldots,\widehat{x}_k,\ldots,x_i,\widehat{x}_{i+1},\ldots,x_n)[x_k,x_{i+1}]\\
&\quad + \sum_{k=i+1<l} (-1)^{l+i}(-1)^{\gamma^X_{i+1,l}}\phi(x_1,\ldots,x_i,\widehat{x}_{i+1},\ldots,\widehat{x}_l,\ldots,x_n)[x_{i+1},x_l]\\
&\quad + (-1)^{\gamma^X_{i,i+1}}\phi(x_1,\ldots,\widehat{x}_i,\widehat{x}_{i+1},\ldots,x_n)[x_i,x_{i+1}]\\
&=: S_1+\cdots+S_7
\end{aligned}$$

and

$$\begin{aligned}
&[x_1,\ldots,x_{i+1},x_i,\ldots,x_n]_\phi\\
&= \sum_{k<l<i} (-1)^{k+l+1}(-1)^{\gamma^X_{kl}}\phi(x_1,\ldots,\widehat{x}_k,\ldots,\widehat{x}_l\ldots,x_{i+1},x_i,\ldots,x_n)[x_k,x_l]\\
&\quad + \sum_{i+1<k<l} (-1)^{k+l+1}(-1)^{\gamma^X_{kl}}\phi(x_1,\ldots,x_{i+1},x_i,\ldots,\widehat{x}_k\ldots,\widehat{x}_l,\ldots,x_n)[x_k,x_l]\\
&\quad + \sum_{k=i<l\neq i+1} (-1)^{i+l+1}(-1)^{\zeta^X_{il}}\phi(x_1,\ldots,\widehat{x}_{i+1},x_i,\ldots,\widehat{x}_l,\ldots,x_n)[x_{i+1},x_l]\\
&\quad + \sum_{k<l=i} (-1)^{k+i+1}(-1)^{\zeta^X_{ki}}\phi(x_1,\ldots,\widehat{x}_k,\ldots,\widehat{x}_{i+1},x_i,\ldots,x_n)[x_k,x_{i+1}]\\
&\quad + \sum_{i\neq k<l=i+1} (-1)^{k+i}(-1)^{\zeta^X_{k,i+1}}\phi(x_1,\ldots,\widehat{x}_k,\ldots,x_{i+1},\widehat{x}_i,\ldots,x_n)[x_k,x_i]\\
&\quad + \sum_{k=i+1<l} (-1)^{l+i}(-1)^{\zeta^X_{i+1,l}}\phi(x_1,\ldots,x_{i+1},\widehat{x}_i,\ldots,\widehat{x}_l,\ldots,x_n)[x_i,x_l]\\
&\quad + (-1)^{\gamma^X_{i,i+1}}\phi(x_1,\ldots,\widehat{x}_i,\widehat{x}_{i+1},\ldots,x_n)[x_{i+1},x_i]\\
&=: S'_1+\cdots+S'_7
\end{aligned}$$

where

$$\begin{aligned}
\zeta^X_{il} &= |X|^n_{l+1}(|x_{i+1}|+|x_l|)+|x_{i+1}|(|x_i|+|x_{i+2}|+\cdots+|x_{l-1}|)\\
&= |X|^n_{l+1}(|x_{i+1}|+|x_l|)+|x_{i+1}|(|x_{i+2}|+\cdots+|x_{l-1}|)+|x_{i+1}||x_i|\\
&= \gamma^X_{i+1,i}+|x_{i+1}||x_i|.
\end{aligned}$$

So we conclude that $S'_3 = -(-1)^{|x_i||x_{i+1}|}S_6$. In the same way, it is easy to see also that $S'_4 = -(-1)^{|x_i||x_{i+1}|}S_5$, $S'_5 = -(-1)^{|x_i||x_{i+1}|}S_4$, and $S'_6 = -(-1)^{|x_i||x_{i+1}|}S_3$.

The super-skew-symmetry of $\phi$ gives that $S_1' = -(-1)^{|x_i||x_{i+1}|}S_1$, $S_2' = -(-1)^{|x_i||x_{i+1}|}S_2$ and $S_7' = -(-1)^{|x_i||x_{i+1}|}S_7$. Finally we get

$$[x_1, \dots, x_i, x_{i+1}, \dots, x_n]_\phi = -(-1)^{|x_i||x_{i+1}|}[x_1, \dots, x_{i+1}, x_i, \dots, x_n]_\phi. \qquad \square$$

Given $X = (x_1, \dots, x_{n-3}) \in \wedge^{n-3}\mathcal{H}(\mathfrak{g})$, $Y = (y_1, \dots, y_n) \in \wedge^n\mathcal{H}(\mathfrak{g})$ and $z \in \mathcal{H}(\mathfrak{g})$, we define the linear map $\phi_X$ by $\phi_X(z) = \phi(X, z)$, and

$$\begin{aligned}\phi\Lambda\delta\phi_X(Y) &= \sum_{i<j}^{n}(-1)^{i+j}(-1)^{\gamma_{ij}^Y}\phi(y_1, \dots \hat{y_i} \dots \hat{y_j} \dots, y_n)\delta\phi_X(y_i, y_j)\\ &= \sum_{i<j}^{n}(-1)^{i+j}(-1)^{\gamma_{ij}^Y}\phi(y_1, \dots \hat{y_i} \dots \hat{y_j} \dots, y_n)\phi_X([y_i, y_j]).\end{aligned}$$

**Theorem 2.3.** *Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a multiplicative Hom-Lie superalgebra, $\mathfrak{g}^*$ its dual and $\phi$ be an even cochain of degree $n-2$, i.e., $\phi \in \wedge^{n-2}\mathfrak{g}^*$. The linear space $\mathfrak{g}$ equipped with the n-ary product (21) and the even linear map $\alpha$ is a multiplicative n-Hom-Lie superalgebra if and only if*

$$\phi\Lambda\delta\phi_X = 0, \ \forall X \in \wedge^{n-3}\mathcal{H}(\mathfrak{g}), \tag{22}$$

$$\phi \circ (\alpha \otimes Id \otimes \cdots \otimes Id) = \phi. \tag{23}$$

*Proof.* Firstly, if $(x_1, \dots, x_n) \in \wedge^n\mathcal{H}(\mathfrak{g})$, then

$$\begin{aligned}&[\alpha(x_1), \dots, \alpha(x_n)]_\phi\\ &\quad= \sum_{i<j}^{n}(-1)^{i+j+1}(-1)^{\gamma_{ij}^Y}\phi(\alpha(x_1), \dots, \widehat{\alpha(x_i)}, \dots, \widehat{\alpha(x_j)}, \dots, \alpha(x_n))[\alpha(x_i), \alpha(x_j)]\\ &\quad= \sum_{i<j}^{n}(-1)^{i+j+1}(-1)^{\gamma_{ij}^X}\phi(x_1, \dots, \hat{x_i}, \dots, \hat{x_j}, \dots, x_n)\alpha([x_i, x_j])\\ &\quad= \alpha([x_1, \dots, x_n]_\phi).\end{aligned}$$

Secondly, for $(x_1, \dots, x_{n-1}) \in \wedge^{n-1}\mathcal{H}(\mathfrak{g})$ and $(y_1, \dots, y_n) \in \wedge^n\mathcal{H}(\mathfrak{g})$, we have

$$\begin{aligned}&[\alpha(x_1), \dots, \alpha(x_{n-1}), [y_1, \dots, y_n]_\phi]_\phi\\ &\quad= \sum_{i<j}(-1)^{i+j+1}(-1)^{\gamma_{ij}^Y}\phi(y_1, \dots, \widehat{y}_i, \dots, \widehat{y}_j, \dots, y_n)\\ &\qquad [\alpha(x_1), \dots, \alpha(x_{n-1}), [y_i, y_j]]_\phi\\ &\quad= \sum_{i<j}\sum_{k<l\leq n-1}(-1)^{i+j+k+l}(-1)^{\gamma_{ij}^Y+\gamma_{kl}^X}(-1)^{(|x_k|+|x_l|)(|x_i|+|x_j|)}\\ &\qquad \phi(\alpha(x_1), \dots, \widehat{\alpha(x_k)}, \dots, \widehat{\alpha(x_l)}, \dots, [y_i, y_j])\phi(y_1, \dots, \widehat{y}_i, \dots, \widehat{y}_j, \dots, y_n)\\ &\qquad\quad [\alpha(x_k), \alpha(x_l)]\\ &\qquad + \sum_{i<j}\sum_{k<n}(-1)^{i+j+k}(-1)^{\gamma_{ij}^Y}(-1)^{|x_k||X^{k+1}|}\\ &\qquad \phi(\alpha(x_1), \dots, \widehat{\alpha(x_k)}, \dots, \alpha(x_{(n-1)}), \dots, \widehat{[y_i, y_j]})\phi(y_1, \dots, \widehat{y}_i, \dots, \widehat{y}_j, \dots, y_n)\\ &\qquad\quad [\alpha(x_k), [y_i, y_j]].\end{aligned}$$

The terms $[\alpha(x_k),[y_i,y_j]]$ are simplified by identity of Jacobi in the second half of the super-Hom-Nambu identity. Now, we group together the other terms according to their coefficient $[\alpha(x_i),\alpha(x_j)]$. For example, if we fixed $(k,l)$ and, if we collect all the terms containing the commutator $[\alpha(x_k),\alpha(x_l)]$, then we get the expression

$$\Big(\sum_{i<j}(-1)^{i+j+k+l}(-1)^{\gamma^Y_{ij}+\gamma^X_{kl}}(-1)^{(|x_k|+|x_l|)(|x_i|+|x_j|)}$$
$$\phi(\alpha(x_1),\ldots,\widehat{\alpha(x_k)},\ldots,\widehat{\alpha(x_l)},\ldots,[y_i,y_j])$$
$$\phi(y_1,\ldots,\widehat{y}_i,\ldots,\widehat{y}_j,\ldots,y_n)\Big)[\alpha(x_k),\alpha(x_l)].$$

Hence the $n$-ary product (21) will satisfy the super-Hom-Nambu identity if for any elements $X=(x_1,\ldots,x_{n-3})\in\wedge^{n-3}\mathcal{H}(\mathfrak{g})$ and $Y=(y_1,\ldots,y_n)\in\wedge^n\mathcal{H}(\mathfrak{g})$ we require

$$\Big(\sum_{i<j}^{n}(-1)^{i+j}(-1)^{\gamma^Y_{ij}}\phi(\alpha(x_1),\ldots,\alpha(x_{n-3}),[y_i,y_j])$$
$$\phi(y_1,\ldots,\widehat{y}_i,\ldots,\widehat{y}_j,\ldots,y_n)\Big)=0.$$ □

*Example 2.4.* [86] Consider a 3-dimensionally graded linear space $L=L_0\oplus L_1$, where $L_0$ is generated by $e_1,e_2$ and $L_1$ is generated by $e_3$. Define an even linear map $\alpha: L\to L$ by

$$\alpha(e_1)=a^2e_1,\quad \alpha(e_2)=e_2\quad \alpha(e_3)=ae_2$$

and an even super-skewsymmetric bilinear map $[\cdot,\cdot]: L\times L\to L$ given by

$$[e_1,e_2]=[e_1,e_3]=0,\quad [e_2,e_3]=e_3,\quad [e_3,e_3]=e_1.$$

Then $(L,[\cdot,\cdot],\alpha)$ is a multiplicative Hom-Lie superalgebra. Define an even linear form $\phi: L\to\mathbb{K}$ given by $\phi(e_1)=0$ and $\phi(e_2)=b$. Then, we have

$$\phi\circ\alpha=\phi\quad\text{and}\quad \phi\Lambda\delta\phi=0.$$

Therefore, using Theorem 2.3, we can construct a multiplicative 3-Hom-Lie superalgebra $(L,[\cdot,\cdot,\cdot]_\phi,\alpha)$, where the ternary bracket $[\cdot,\cdot,\cdot]_\phi$ is given by $[e_2,e_3,e_3]=\phi(e_1)[e_3,e_3]=be_1$ and $a,b$ are parameters.

In the following, we generalise the notion of supertrace introduced in [47] for the even multilinear form.

**Definition 2.5.** Let $\phi:\mathfrak{g}\wedge\cdots\wedge\mathfrak{g}\to\mathbb{K}$ be an even super-skewsymmetric linear form of the multiplicative Hom-Lie superalgebra $(\mathfrak{g},[\cdot,\cdot],\alpha)$, then $\phi$ is called supertrace if:

$$\phi\circ([\cdot,\cdot]\otimes Id\otimes\cdots\otimes Id)=0\text{ and }\phi\circ(\alpha\otimes Id\otimes\cdots\otimes Id)=\phi.$$

**Corollary 2.6.** *Let $\phi:\wedge^{n-2}\mathfrak{g}\to\mathbb{K}$ be a supertrace of Hom-Lie superalgebra $(\mathfrak{g},[\cdot,\cdot],\alpha)$, then $\mathfrak{g}_\phi=(\mathfrak{g},[.,\ldots,.]_\phi,\alpha)$ is an $n$-Hom-Lie superalgebra.*

**Proposition 2.7.** *Let* $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ *be a Hom-Lie superalgebra, and let* $\mathfrak{D} \in \mathrm{Der}(\mathfrak{g})$ *be an* $\alpha^k$*-derivation such that*

$$\sum_{i=1}^{n-2}(-1)^{|\mathfrak{D}||X|^{i-1}}\phi(x_1,\dots\mathfrak{D}(x_i),\dots,x_{n-2}) = 0.$$

*Then* $\mathfrak{D}$ *is an* $\alpha^k$*-derivation of the n-Hom-Lie superalgebra* $(\mathfrak{g}, [\cdot,\dots,\cdot]_\phi, \alpha)$.

*Proof.* Let $X = (x_1,\dots,x_n) \in \wedge^n\mathcal{H}(\mathfrak{g})$, on the one hand we get

$$\begin{aligned}
&\mathfrak{D}([x_1,\dots,x_n]_\phi)\\
&\quad= \mathfrak{D}\Big(\sum_{i<j}(-1)^{i+j+1}\phi(\alpha(x_1),\dots,\alpha(\widehat{x}_i),\dots,\alpha(\widehat{x}_j),\dots,\alpha(x_n))[\alpha(x_i),\alpha(x_j)]\Big)\\
&\quad= \sum_{i<j}(-1)^{i+j+1}\phi(\alpha(x_1),\dots,\alpha(\widehat{x}_i),\dots,\alpha(\widehat{x}_j),\dots,\alpha(x_n))\mathfrak{D}([\alpha(x_i),\alpha(x_j)])\\
&\quad= \sum_{i<j}(-1)^{i+j+1}\phi(x_1,\dots,\widehat{x}_i,\dots,\widehat{x}_j,\dots,x_n)[\alpha(\mathfrak{D}(x_i)),\alpha^{k+1}(x_j)]\\
&\qquad+ \sum_{i<j}(-1)^{i+j+1}\phi(x_1,\dots,\widehat{x}_i,\dots,\widehat{x}_j,\dots,x_n)[\alpha^{k+1}(x_i),\alpha(\mathfrak{D}(x_j))],
\end{aligned}$$

on the other hand, we have

$$\begin{aligned}
&\sum_{l=1}^{n}[\alpha^k(x_1),\dots,\alpha^k(x_{l-1}),\mathfrak{D}(x_l),\dots,\alpha^k(x_{l+1}),\dots,\alpha^k(x_n)]_\phi\\
&\quad= \sum_{l=1}^{n}\sum_{i<j\ ;\ i,j\neq l}(-1)^{i+j+1}\\
&\qquad\phi(\alpha^k(x_1),\dots,\widehat{\alpha^k(x_i)},\dots,\mathfrak{D}(x_l),\dots,\widehat{\alpha^k(x_j)},\dots,\alpha^k(x_n))[\alpha^k(x_i),\alpha^k(x_j)]\\
&\qquad+ \sum_{l=1}^{n}\sum_{i<l}(-1)^{i+l+1}\\
&\qquad\phi(\alpha^k(x_1),\dots,\widehat{\alpha^k(x_i)},\dots,\widehat{\mathfrak{D}(x_l)},\dots,\alpha^k(x_n))[\alpha^k(x_i),\mathfrak{D}(x_l)]\\
&\qquad+ \sum_{l=1}^{n}\sum_{l=i<j}(-1)^{j+l+1}\\
&\qquad\phi(\alpha^k(x_1),\dots,\widehat{\mathfrak{D}(x_l)},\dots,\widehat{\alpha^k(x_j)},\dots,\alpha^k(x_n))[\mathfrak{D}(x_l),\alpha^k(x_j)].
\end{aligned}$$

If $\mathfrak{D}$ is an $\alpha^k$-derivation then

$$\begin{aligned}
\mathfrak{D}([x_1,\dots,x_n]_\phi) &= \sum_{l=1}^{n}(-1)^{|\mathfrak{D}||X|^{i-1}}\\
&\quad[\alpha^k(x_1),\dots,\alpha^k(x_{l-1}),\mathfrak{D}(x_l),\dots,\alpha^k(x_{l+1}),\dots,\alpha^k(x_n)]_\phi,
\end{aligned}$$

which gives

$$\sum_{\substack{i<j\\ i,j\neq l}}(-1)^{i+j+1}\Big(\sum_{l=1}^{n}(-1)^{|\mathfrak{D}||X|^{l-1}}$$
$$\phi(\alpha^k(x_1),\ldots,\widehat{\alpha^k(x_i)},\ldots,\mathfrak{D}(x_l),\ldots,\widehat{\alpha^k(x_j)},\ldots,\alpha^k(x_n))\Big)$$
$$[\alpha^k(x_i),\alpha^k(x_j)]=0.$$

Finally, if we fixed $(i,j)$ we have

$$\sum_{l=1}^{n-2}(-1)^{|\mathfrak{D}||X|^{l-1}}\phi(\alpha^k(x_1),\ldots,\mathfrak{D}(x_l),\ldots,\alpha^k(x_{n-2}))=0. \qquad \square$$

**Proposition 2.8.** *Let* $(\mathfrak{g},[\cdot,\cdot],\alpha)$ *be a Hom-Lie superalgebra and let* $\mathfrak{D}\in \mathrm{QDer}(\mathfrak{g})$ *be an* $\alpha^k$*-quasi-derivation and* $\mathfrak{D}':\mathfrak{g}\to\mathfrak{g}$ *the endomorphism associated to* $\mathfrak{D}$ *such that*

$$\sum_{i=1}^{n-2}(-1)^{|\mathfrak{D}||X|^{i-1}}\phi(x_1,\ldots\mathfrak{D}(x_i),\ldots,x_{n-2})=0.$$

*Then* $\mathfrak{D}$ *is an* $\alpha^k$*-quasi-derivation of the* $n$*-Hom-Lie superalgebra* $(\mathfrak{g},[\cdot,\ldots,\cdot]_\phi,\alpha)$ *with the same endomorphism associated to* $\mathfrak{D}'$.

## 3. $n$-ary Hom-Nambu Superalgebras Induced by Hom-Lie Superalgebras

In this section we construct an $n$-ary Hom-Nambu superalgebras with a help of a given Hom-Lie superalgebra by analogue of Hom-Lie super-triple system given in [91] in graded case. Let $(\mathfrak{g},[\cdot,\cdot],\alpha)$ be a multiplicative Hom-Lie superalgebra. Define the following $n$-linear map $[\cdot,\ldots,\cdot]_n:\mathfrak{g}^{\otimes n}\to\mathfrak{g}$:

$$\Big[x_1,\ldots,x_n\Big]_n=\Big[\big[[\ldots[x_1,x_2],\alpha(x_3)],\alpha^2(x_4)\big]\ldots\alpha^{n-3}(x_{n-1})],\alpha^{n-2}(x_n)\Big]. \quad (24)$$

For n=2, $[x_1,x_2]_2=[x_1,x_2]$ and for $n\geq 3$,

$$[x_1,\ldots,x_n]_n=[[x_1,\ldots,x_{n-1}]_{n-1},\ \alpha^{n-2}(x_n)].$$

**Theorem 3.1.** *Let* $(\mathfrak{g},[\cdot,\cdot],\alpha)$ *be a multiplicative Hom-Lie superalgebra. Then*

$$\mathfrak{g}_n=(\mathfrak{g},[\cdot,\ldots,\cdot]_n,\alpha^{n-1})$$

*is a multiplicative* $n$*-ary Hom-Nambu superalgebra.*

To prove this theorem we need the following lemma.

**Lemma 3.2.** *Let* $(\mathfrak{g},[\cdot,\cdot],\alpha)$ *be a multiplicative Hom-Lie superalgebra, and* ad *the adjoint map defined by* $\mathrm{ad}_x(y)=[x,y]$. *Then, we have*

$$\mathrm{ad}_{\alpha^{n-1}(x)}[y_1,\ldots,y_n]_n$$
$$=\sum_{k=1}^{n}(-1)^{|x||Y|^{k-1}}[\alpha(y_1),\ldots,\alpha(y_{k-1}),\mathrm{ad}_x(y_k),\alpha(y_{k+1}),\ldots,\alpha(y_n)]_n,$$

*where* $x \in \mathcal{H}(\mathfrak{g}), y \in \mathcal{H}(\mathfrak{g})$ *and* $(y_1, \ldots, y_n) \in \mathcal{H}(\mathfrak{g})^n$.

*Proof.* For $n = 2$, using the super-Hom-Jacobi identity we have

$$\begin{aligned}\mathrm{ad}_{\alpha(x)}[y, z] &= [\alpha(x), [y, z]] = [[x, y], \alpha(z)] + (-1)^{|x||y|}[\alpha(y), [x, z]] \\ &= [\mathrm{ad}_x(y), \alpha(z)] + (-1)^{|x||y|}[\alpha(y), \mathrm{ad}_x(z)].\end{aligned}$$

Assume that the property is true up to order $n$, that is

$$\begin{aligned}&\mathrm{ad}_{\alpha^{n-1}(X)}[y_1, \ldots, y_n]_n \\ &\quad= \sum_{k=1}^{n} (-1)^{|X||Y|^{k-1}} [\alpha(y_1), \ldots, \alpha(y_{k-1}), \mathrm{ad}_X(y_k), \alpha(y_{k+1}), \ldots, \alpha(y_n)]_n.\end{aligned}$$

Let $x \in \mathcal{H}(\mathfrak{g})$ and $(y_1, \ldots, y_{n+1}) \in \mathcal{H}(\mathfrak{g})^{n+1}$, we have

$$\begin{aligned}\mathrm{ad}_{\alpha^n(x)}&[y_1, \ldots, y_{n+1}] = \mathrm{ad}_{\alpha^n(x)}[[y_1, \ldots, y_n]_n, \alpha^{n-1}(y_{n+1})]_2 \\ &= \Big[\mathrm{ad}_{\alpha^{n-1}(x)}[y_1, \ldots, y_n]_n, \alpha^n(y_{n+1})\Big]_2 \\ &\quad + (-1)^{|x||Y|}\Big[[\alpha(y_1), \ldots, \alpha(y_n)]_n, \mathrm{ad}_{\alpha^{n-1}(x)}(\alpha^{n-1}(y_{n+1}))\Big]_2 \\ &= \sum_{k=1}^{n} \Big[[\alpha(y_1), \ldots, \alpha(y_{k-1}), \mathrm{ad}_x(y_k), \alpha(y_{k+1}), \ldots, \alpha(y_n)]_n, \alpha^n(y_{n+1})\Big] \\ &\quad + (-1)^{|x||Y|^{k-1}}\Big[[\alpha(y_1), \ldots, \alpha(y_n)]_n, \alpha^{n-1}(\mathrm{ad}_x(y_{n+1}))\Big]_2 \\ &= \sum_{k=1}^{n} (-1)^{|x||Y|^{k-1}}\Big[\alpha(y_1), \ldots, \alpha(y_{k-1}), \mathrm{ad}_x(y_k), \alpha(y_{k+1}), \ldots, \alpha(y_n), \alpha(y_{n+1})\Big]_{n+1} \\ &\quad + (-1)^{|x||Y|^{n}}\Big[\alpha(y_1), \ldots, \alpha(y_n), \mathrm{ad}_x(y_{n+1})\Big]_{n+1} \\ &= \sum_{k=1}^{n+1} (-1)^{|x||Y|^{k-1}}\Big[\alpha(y_1), \ldots, \alpha(y_{k-1}), \mathrm{ad}_x(y_k), \alpha(y_{k+1}), \ldots, \alpha(y_{n+1})\Big]_{n+1}.\end{aligned}$$

The lemma is proved. □

*Proof of Theorem 3.1.* Let $X = (x_1, \ldots, x_{n-1}) \in \mathcal{H}(\mathfrak{g})^{n-1}$ and $Y = (y_1, \ldots, y_n) \in \mathcal{H}(\mathfrak{g})^n$. Using Lemma 3.2, we have

$$\begin{aligned}&\Big[\alpha^{n-1}(x_1), \ldots, \alpha^{n-1}(x_{n-1}), [y_1, \ldots, y_n]_n\Big]_n \\ &\quad= \Big[[\alpha^{n-1}(x_1), \ldots, \alpha^{n-1}(x_{n-1})]_{n-1}, [\alpha^{n-2}(y_1), \ldots, \alpha^{n-2}(y_n)]_n\Big]_2 \\ &\quad= \mathrm{ad}_{\alpha^{n-1}[x_1, \ldots, x_{n-1}]_{n-1}}([\alpha^{n-2}(y_1), \ldots, \alpha^{n-2}(y_n)]_n) \\ &\quad= \sum_{k=1}^{n} (-1)^{|X||Y|^{k-1}} \\ &\qquad \Big[\alpha^{n-1}(y_1), \ldots, \mathrm{ad}_{[x_1, \ldots, x_{n-1}]_{n-1}}(\alpha^{n-2}(y_k)), \ldots, \alpha^{n-1}(y_n)\Big]_n \\ &\quad= \sum_{k=1}^{n} (-1)^{|X||Y|^{k-1}} \\ &\qquad \Big[\alpha^{n-1}(y_1), \ldots, [[x_1, \ldots, x_{n-1}]_{n-1}, \alpha^{n-2}(y_k)]_2, \ldots, \alpha^{n-1}(y_n)\Big]_n\end{aligned}$$

$$= \sum_{k=1}^{n}(-1)^{|X||Y|^{k-1}}\Big[\alpha^{n-1}(y_1),\dots,[x_1,\dots,x_{n-1},y_k]_n,\dots,\alpha^{n-1}(y_n)\Big]_n. \square$$

*Example 3.3.* Consider the 2-dimensional multiplicative Hom-Lie superalgebras $\mathfrak{g}^3_{1,1}$ and $\mathfrak{g}^4_{1,1}$ given in Theorem 1.6. We can construct a multiplicative $n$-ary Hom-Nambu superalgebras structures on $\mathfrak{g}^3_{1,1}$ and $\mathfrak{g}^4_{1,1}$ given respectively by:

$$[e_1,e_0,\dots,e_0]^3_n = (-1)^{n-1}e_1 \quad\text{and}\quad [e_1,e_0,\dots,e_0]^4_n = -(-a)^{n-2}e_1.$$

The other brackets are zero.

*Example 3.4.* [86] Consider a 3-dimensional graded linear space $L = L_0 \oplus L_1$, where $L_0$ is generated by $e_1$ and $L_1$ is generated by $e_2, e_3$. Define an even linear map $\alpha : L \to L$ by

$$\alpha(e_1) = ae_1, \quad \alpha(e_2) = ae_2 \quad \alpha(e_3) = e_3$$

and an even super-skewsymmetric bilinear map $[\cdot,\cdot] : L \times L \to L$ given by

$$[e_1,e_2] = [e_2,e_2] = [e_3,e_3] = 0, \quad [e_1,e_3] = be_2, \quad [e_3,e_3] = ce_1,$$

where $a,b,c$ are parameters and $a \neq 0$. Then $(L,[\cdot,\cdot],\alpha)$ is a multiplicative Hom-Lie superalgebra. Therefore, using Theorem 3.1, we can construct a multiplicative 3-ary Hom-Nambu superalgebra $(L,[\cdot,\cdot,\cdot]_3,\alpha^2)$, where the ternary bracket $[\cdot,\cdot,\cdot]_3$ is given by $[e_1,e_3,e_3] = bce_1$ and $[e_2,e_3,e_3] = bce_2$. We can also construct a multiplicative $n$-ary Hom-Nambu superalgebra $(L,[\cdot,\dots,\cdot]_n, \alpha^{n-1})$, where the $n$-ary bracket $[\cdot,\dots,\cdot]_n$ is given by:

- If $n = 4p$, then $[e_1,e_3,e_3,\dots,e_3] = b^pc^{p+1}e_1$ and $[e_2,e_3,e_3,\dots,e_3] = b^{p+1}c^pe_2$.
- If $n = 4p+1$, $[e_1,e_3,e_3,\dots,e_3] = b^{p+1}c^{p+1}e_2$ and $[e_2,e_3,e_3,\dots,e_3] = b^{p+1}c^{p+1}e_1$.
- If $n = 4p+2$, then $[e_1,e_3,e_3,\dots,e_3] = b^{p+1}c^{p+2}e_1$ and $[e_2,e_3,e_3,\dots,e_3] = b^{p+2}c^{p+1}e_2$.
- If $n = 4p+3$, then $[e_1,e_3,e_3,\dots,e_3] = b^{p+2}c^{p+2}e_2$ and $[e_2,e_3,e_3,\dots,e_3] = b^{p+2}c^{p+2}e_1$.

**Proposition 3.5.** *Let $(\mathfrak{g},[\cdot,\cdot],\alpha)$ be a multiplicative Hom-Lie superalgebra and $\mathfrak{D} : \mathfrak{g} \to \mathfrak{g}$ an $\alpha^k$-derivation of $\mathfrak{g}$ for an integer $k$. Then $\mathfrak{D}$ is an $\alpha^k$-derivation of $\mathfrak{g}_n$.*

*Proof.* We use the mathematical induction. For $n = 3$, given $x,y,z \in \mathcal{H}(\mathfrak{g})$, we have

$$\begin{aligned}
\mathfrak{D}([x,y,z]) &= \mathfrak{D}([[x,y],\alpha(z)])\\
&= [\mathfrak{D}([x,y]),\alpha^{k+1}(z)] + (-1)^{|\mathfrak{D}||[x,y]|}[[\alpha^k(x),\alpha^k(y)],\mathfrak{D}(\alpha(z))]\\
&= [[\mathfrak{D}(x),\alpha^k(y)],\alpha^{k+1}(z)] + (-1)^{|\mathfrak{D}||x|}[[\alpha^k(x),\mathfrak{D}(y)],\alpha^{k+1}(z)]\\
&\quad + (-1)^{|\mathfrak{D}|(|x|+|y|)}[[\alpha^k(x),\alpha^k(y)],\alpha(\mathfrak{D}(z))]\\
&= [\mathfrak{D}(x),\alpha^k(y),\alpha^k(z)] + (-1)^{|\mathfrak{D}||x|}[\alpha^k(x),\mathfrak{D}(y),\alpha^k(z)]
\end{aligned}$$

$$+ (-1)^{|\mathfrak{D}|(|x|+|y|)}[\alpha^k(x), \alpha^k(y), \mathfrak{D}(z)].$$

Now, suppose that the property is true to order $n-1$:

$$\mathfrak{D}([x_1, \ldots, x_{n-1}]_{n-1}) = \sum_{i=1}^{n}(-1)^{|\mathfrak{D}||X|^{i-1}}[\alpha^k(x_1), \ldots, D(x_k), \ldots, \alpha^k(x_{n-1})]_{n-1}.$$

If $(x_1, \ldots, x_n) \in \mathfrak{g}^n$, then

$$\begin{aligned}
&\mathfrak{D}([x_1, \ldots, x_n]_n)\\
&\quad= \mathfrak{D}(\Big[[x_1, \ldots, x_{n-1}]_{n-1}, \alpha^{n-2}(x_n)\Big])\\
&\quad= \Big[\mathfrak{D}([x_1, \ldots, x_{n-1}]_{n-1}), \alpha^{n+k-2}(x_n)\Big]\\
&\qquad+ (-1)^{|\mathfrak{D}||[x_1,\ldots,x_{n-1}]_{n-1}|}\Big[[\alpha^k(x_1), \ldots, \alpha^k(x_{n-1})]_{n-1}, \mathfrak{D}(\alpha^{n-2}(x_n))\Big]\\
&\quad= \Big[\mathfrak{D}([x_1, \ldots, x_{n-1}]_{n-1}), \alpha^{n-2}(\alpha^k(x_n))\Big]\\
&\qquad+ (-1)^{|\mathfrak{D}||X|^{n-1}}\Big[[\alpha^k(x_1), \ldots, \alpha^k(x_{n-1})]^{n-1}, \alpha^{n-2}(\mathfrak{D}(x_n))\Big]\\
&\quad= \sum_{i=1}^{n-1}(-1)^{|\mathfrak{D}||X|^{i-1}}\Big[[\alpha^k(x_1), \ldots, \mathfrak{D}(x_i), \ldots, \alpha^k(x_{n-1})]_{n-1}, \alpha^{n-2}(\alpha^k(x_n))\Big]\\
&\qquad+ (-1)^{|\mathfrak{D}||X|^{n-1}}[\alpha^k(x_1), \ldots, \alpha^k(x_{n-1}), \mathfrak{D}(x_n)]_n\\
&\quad= \sum_{i=1}^{n-1}(-1)^{|\mathfrak{D}||X|^{i-1}}[\alpha^k(x_1), \ldots, \mathfrak{D}(x_i), \ldots, \alpha^k(x_{n-1}), \alpha^k(x_n)]_n\\
&\qquad+ (-1)^{|\mathfrak{D}||X|^{n-1}}[\alpha^k(x_1), \ldots, \alpha^k(x_{n-1}), \mathfrak{D}(x_n)]_n\\
&\quad= \sum_{i=1}^{n}(-1)^{|\mathfrak{D}||X|^{i-1}}[\alpha^k(x_1), \ldots, \mathfrak{D}(x_i), \ldots, \alpha^k(x_{n-1}), \alpha^k(x_n)]_n,
\end{aligned}$$

which completes the proof. □

**Proposition 3.6.** *Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a multiplicative Hom-Lie superalgebra. For endomorphisms $\mathfrak{D}$,$\mathfrak{D}'$,...,$\mathfrak{D}^{(n-1)}$ of $\mathfrak{g}$ such that $\mathfrak{D}^{(i)}$ is an $\alpha^k$-quasiderivation with associated endomorphism $\mathfrak{D}^{(i+1)}$ for $0 \le i \le n-2$, the $(n+1)$-tuple $(\mathfrak{D}, \mathfrak{D}, \mathfrak{D}', \mathfrak{D}'', \ldots, \mathfrak{D}^{(n-1)})$ is an $(n+1)$-ary $\alpha^k$-derivation of $\mathfrak{g}_n$.*

*Proof.* Let $x_1, \ldots, x_n \in \mathfrak{g}$, then

$$\begin{aligned}
&\mathfrak{D}^{(n-1)}([x_1, \ldots, x_n]_n)\\
&\quad= \mathfrak{D}^{(n-1)}([[x_1, \ldots, x_{n-1}]_{n-1}, x_n])\\
&\quad= [\mathfrak{D}^{(n-2)}([x_1, \ldots, x_{n-1}]_{n-1}), \alpha^k(x_n)]\\
&\qquad+ (-1)^{|\mathfrak{D}^{(n-2)}||X|^{n-1}}[[\alpha^k(x_1), \ldots, \alpha^k(x_{n-1})]_{n-1}, \mathfrak{D}^{(n-2)}(x_n)]\\
&\quad= [[\mathfrak{D}^{(n-3)}([x_1, \ldots, x_{n-2}]), \alpha^k(x_n)], \alpha^k(x_n)]\\
&\qquad+ (-1)^{|\mathfrak{D}^{(n-3)}||X|^{n-2}}\Big[\Big[[\alpha^k(x_1), \ldots, \alpha^k(x_{n-2})], \mathfrak{D}^{(n-3)}(x_{n-1})\Big], \alpha^k(x_n)\Big]\\
&\qquad+ (-1)^{|\mathfrak{D}^{(n-2)}||X|^{n-1}}\Big[[\alpha^k(x_1), \ldots, \alpha^k(x_{n-1})]_{n-1}, \mathfrak{D}^{(n-2)}(x_n)\Big]
\end{aligned}$$

$$\vdots$$
$$= [\mathfrak{D}(x_1), \alpha^k(x_2), \dots, \alpha^k(x_n)]_n + (-1)^{|\mathfrak{D}||x_1|}[\alpha^k(x_1), \mathfrak{D}(x_2), \dots, \alpha^k(x_n)]_n$$
$$+ (-1)^{|\mathfrak{D}|(|x_1|+|x_2|)}[\alpha^k(x_1), \alpha^k(x_2), \mathfrak{D}'(x_3), \dots, \alpha^k(x_n)]_n$$
$$+ \dots + (-1)^{|\mathfrak{D}^{(n-2)}||X|^{n-1}}[\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), \mathfrak{D}^{(n-2)}(x_n)]_n).$$

Therefore $(n+1)$-tuple $(\mathfrak{D}, \mathfrak{D}, \mathfrak{D}', \mathfrak{D}'', \dots, \mathfrak{D}^{(n-1)})$ is a generalized $\alpha^k$-derivation of $\mathfrak{g}_n$. □

## 4. Rota-Baxter $n$-ary Hom-Nambu Superalgebras

In this section, we introduce the notion of Rota-Baxter operators of Hom-Nambu superalgebras and 3-Hom-pre-Lie algebras. Then we introduce the notion of a 3-Hom-pre-Lie superalgebra which is closely related to Rota-Baxter operators of weight 0.

### 4.1. Rota-Baxter Operator on $n$-ary Hom-Nambu Superalgebras

Let $(A, \cdot, \alpha)$ be a $\mathbb{K}$-super-linear space with an even binary operation $\cdot$ and an even linear map $\alpha : A \to A$ and let $\lambda \in \mathbb{K}$. If an even linear map $R : A \to A$ satisfies, for all $x, y \in A$,

$$R\alpha = \alpha R, \quad R(x) \cdot R(y) = R(R(x) \cdot y + x \cdot R(y) + \lambda x \cdot y), \tag{25}$$

then $R$ is called a Rota-Baxter operator of weight $\lambda$ on Hom-superalgebra $(A, \cdot, \alpha)$.

We generalize the concepts of a Rota-Baxter operator to $n$-ary Hom-Nambu superalgebras.

**Definition 4.1.** Let $\lambda \in \mathbb{K}$ and an $n$-ary Hom-Nambu superalgebra $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$. A Rota-Baxter operator of weight $\lambda$ on $(A, [\cdot, \dots, \cdot], \alpha)$ is an even linear map $R : \mathcal{N} \to \mathcal{N}$ such that $R\alpha = \alpha R$ satisfying

$$[R(x_1), \dots, R(x_n)] = R\Big(\sum_{\emptyset \neq I \subseteq [n]} \lambda^{|I|-1}[\hat{R}(x_1), \dots, \hat{R}(x_i), \dots, \hat{R}(x_n)]\Big), \tag{26}$$

where $\hat{R}(x_i) := \hat{R}_I(x_i) := \begin{cases} x_i, & i \in I, \\ R(x_i), & i \notin I \end{cases}$ for all $x_1, \dots, x_n \in \mathcal{N}$. In particular, a Rota-Baxter operator of weight $\lambda$ of ternary Hom-Nambu superalgebra $(\mathcal{N}, [\cdot, \cdot, \cdot], \alpha)$ is an even linear map $R : \mathcal{N} \to \mathcal{N}$ commuting with $\alpha$ such that

$$\begin{aligned}[R(x_1), R(x_2), R(x_3)] = R\Big(&[R(x_1), R(x_2), x_3] + [R(x_1), x_2, R(x_3)] \\ &+ [x_1, R(x_2), R(x_3)] \\ &+ \lambda[R(x_1), x_2, x_3] + \lambda[x_1, R(x_2), x_3] \\ &+ \lambda[x_1, x_2, R(x_3)] \\ &+ \lambda^2[x_1, x_2, x_3]\Big).\end{aligned}$$

**Proposition 4.2.** *Let $(\mathcal{N}, [\cdot, \dots, \cdot], \alpha)$ be a $n$-ary Hom-Nambu superalgebra over a field $\mathbb{K}$. An invertible even linear mapping $R : \mathcal{N} \to \mathcal{N}$ is a Rota-Baxter operator of weight $0$ on $\mathcal{N}$ if and only if $R^{-1}$ is an even derivation on $\mathcal{N}$.*

*Proof.* $R$ is an even invertible Rota-Baxter operator of weight 0 on $\mathcal{N}$ if and only if

$$\forall x_1,\ldots,x_n \in A:\ [R(x_1),\ldots,R(x_n)] = R\Big(\sum_{i=1}^{n}[R(x_1),\ldots,x_i,\ldots,R(x_n)]\Big).$$

For $X_k = R(x_k),\ k \in \{1,\ldots,n\}$.

$$[X_1,\ldots,X_n] = R\Big(\sum_{i=1}^{n}[X_1,\ldots,R^{-1}(X_i),\ldots,X_n]\Big).$$

Hence, $R^{-1}([X_1,\ldots,X_n]) = \sum_{i=1}^{n}[X_1,\ldots,R^{-1}(X_i),\ldots,X_n]$. Thus $R^{-1}$ is an even derivation on $A$. □

**Proposition 4.3.** *Let $R$ be a Rota-Baxter of weight 0 of Hom-Lie superalgebra $(\mathfrak{g},[\cdot,\cdot],\alpha)$ and $\phi \in \wedge^{n-2}\mathfrak{g}^*$ an even $(n-2)$-cochaine satisfying the conditions (22) and (23). Then $R$ is a Rota-Baxter operator of weight 0 on the $n$-ary Hom-Nambu superalgebra $(\mathfrak{g},[\cdot,\ldots,\cdot]_\phi,\alpha)$ defined in (3) if and only if $R$ satisfies*

$$\begin{aligned}&\forall x_1,\ldots,x_n \in \mathcal{H}(\mathfrak{g}):\\ &\sum_{k<l}^{n}\Big(\sum_{i\neq k,l}^{n}\phi\big(R(x_1),\ldots,\widehat{R(x_k)},\ldots,\widehat{R(x_l)},\ldots,R(x_n)\big)\Big)\\ &[R(x_k),R(x_l)] \in \ker(R).\end{aligned} \tag{27}$$

*Proof.* For $X = (x_1,\ldots,x_n) \in \mathcal{H}(\mathfrak{g})^{\otimes n}$,

$$\begin{aligned}&[R(x_1),\ldots,R(x_n)]_\phi\\ &\quad= \sum_{k<l}(-1)^{|k|+|l|+1}(-1)^{\gamma^X_{kl}}\phi(R(x_1),\ldots,\widehat{R(x_k)},\ldots,\widehat{R(x_l)},\ldots,R(x_n))\\ &\qquad[R(x_k),R(x_l)]\\ &\quad= \sum_{k<l}(-1)^{|k|+|l|+1}(-1)^{\gamma^X_{kl}}\phi\Big(R(x_1),\ldots,\widehat{R(x_k)},\ldots,\widehat{R(x_l)},\ldots,R(x_n)\Big)\\ &\qquad([R(x_k),x_l]+[x_k,R(x_l)]).\end{aligned}$$

On the other hand,

$$\begin{aligned}&R\Big(\sum_{i=1}^{n}[R(x_1),\ldots,x_i,\ldots,R(x_n)]_\phi\Big)\\ &\quad= R\Big(\sum_{i=1}^{n}\sum_{k<l;k,l\neq i}(-1)^{k+l+1}(-1)^{\gamma^X_{kl}}\\ &\qquad\phi(R(x_1),\ldots,\widehat{R(x_k)},\ldots,x_i,\ldots,\widehat{R(x_l)},\ldots,R(x_n))\\ &\qquad[R(x_k),R(x_l)]\Big) + R\Big(\sum_{i=1}^{n}\sum_{k<i}(-1)^{k+i+1}(-1)^{\gamma^X_{ki}}\end{aligned}$$

$$\phi(R(x_1),\ldots,\widehat{R(x_k)},\ldots,\widehat{x_i},\ldots,R(x_n))[R(x_k),x_i]\Big)$$
$$+R\Big(\sum_{i=1}^{n}\sum_{i<k}(-1)^{k+i+1}(-1)^{\gamma^X_{ik}}$$
$$\phi(R(x_1),\ldots,\widehat{x_i},\ldots,\widehat{R(x_k)},\ldots,R(x_n))[x_i,R(x_k)]\Big).$$

It is easy to see that

$$\begin{aligned}
&R\Big(\sum_{i=1}^{n}[R(x_1),\ldots,x_i,\ldots,R(x_n)]_\phi\Big)-[R(x_1),\ldots,R(x_n)]_\phi\\
&\quad=R\Big(\sum_{i=1}^{n}\sum_{\substack{k<l\\k,l\neq i}}(-1)^{k+l+1}(-1)^{\gamma^X_{kl}}\\
&\quad\phi(R(x_1),\ldots,\widehat{R(x_k)},\ldots,x_i,\ldots,\widehat{R(x_l)},\ldots,R(x_n))[R(x_k),R(x_l)]\Big).
\end{aligned}$$

Then, $R$ is a Rota-Baxter operator on the $n$-ary Hom-Nambu superalgebra $(\mathfrak{g},[\cdot,\ldots,\cdot]_\phi,\alpha)$ defined in (3) if and only if

$$\begin{aligned}
&R\Big(\sum_{i=1}^{n}\sum_{k<l;k,l\neq i}(-1)^{k+l+1}(-1)^{\gamma^X_{kl}}\\
&\quad\phi(R(x_1),\ldots,\widehat{R(x_k)},\ldots,x_i,\ldots,\widehat{R(x_l)},\ldots,R(x_n))\\
&\quad[R(x_k),R(x_l)]\Big)=0,
\end{aligned}$$

which gives

$$\sum_{k<l}^{n}(-1)^{k+l+1}(-1)^{\gamma^X_{kl}}\Big(\sum_{i\neq k,l}^{n}\phi\big(R(x_1),\ldots,\widehat{R(x_k)},\ldots,x_i,\ldots,\widehat{R(x_l)},\ldots,R(x_n)\big)$$
$$[R(x_k),R(x_l)]\in\ker(R). \qquad \square$$

**Proposition 4.4.** *A Rota-Baxter $R$ of weight 0 of Hom-Lie superalgebra $(\mathfrak{g},[\cdot,\cdot],\alpha)$ is a Rota-Baxter operator on the associated $n$-ary Hom-Nambu superalgebra $(\mathfrak{g},[\cdot,\ldots,\cdot]_n,\alpha^{n-2})$ defined in (24).*

*Proof.* It easy to show that $\alpha^pR=R\alpha^p$ for any integer $p\geq 0$. We use the mathematical induction on the integer $n\geq 3$:

(i) For $n=3$: For $x,y,z\in\mathcal{H}(\mathfrak{g})$, we have:

$$\begin{aligned}
&[R(x),R(y),R(z)]_3=[[R(x),R(y)],\alpha(R(z))]\\
&\quad=[R([R(x),y]),R(\alpha(z))]+[R([x,R(y)]),R(\alpha(z))]\\
&\quad=R([R([R(x),y]),\alpha(z)])+R([[R(x),y],R(\alpha(z))])\\
&\qquad+R([R([x,R(y)]),\alpha(z)])+R([[x,R(y)],R(\alpha(z))])\\
&\quad=R([R([R(x),y]),\alpha(z)])+R([R([x,R(y)]),\alpha(z)])\\
&\qquad+R([R(x),y,R(z)]_3)+R([x,R(y),R(\alpha(z))]_3)
\end{aligned}$$

$$= R([R([R(x), y]) + R([x, R(y)]), \alpha(z)])$$
$$\quad + R([R(x), y, R(z)]_3) + R([x, R(y), R(z)]_3)$$
$$= R([R(x), R(y), z]_3) + R([R(x), y, R(z)]_3) + R([x, R(y), R(z)]_3)$$
$$= [R(x), R(y), R(z)]_3$$

(ii) Assume the property is true to order $n > 3$, that is:

$\forall (x_1, \ldots, x_{n-1}) \in \mathcal{H}(\mathfrak{g})^{\otimes n-1}$ :

$$[R(x_1), \ldots, R(x_{n-1})]_{n-1} = R\Big(\sum_{i=1}^{n-1}[R(x_1), \ldots, x_i, \ldots, R(x_{n-1})]_{n-1}\Big).$$

For $(x_1, \ldots, x_n) \in \mathcal{H}(\mathfrak{g})^{\otimes n}$,

$$[R(x_1), \ldots, R(x_n)]_n = [[R(x_1), \ldots, R(x_{n-1})]_{n-1}, \alpha^{n-2}(R(x_n))]$$
$$= \sum_{i=1}^{n-1}\Big[R([R(x_1), \ldots, x_i, \ldots, R(x_{n-1})]_{n-1}), R(\alpha^{n-2}(x_n))\Big]$$
$$= R\Big(\sum_{i=1}^{n-1}\Big[[R(x_1), \ldots, x_i, \ldots, R(x_{n-1})]_{n-1}, \alpha^{n-2}(R(x_n))\Big]\Big)$$
$$\quad + R\Big(\sum_{i=1}^{n-1}\Big[R([R(x_1), \ldots, x_i, \ldots, R(x_{n-1})]_{n-1}), \alpha^{n-2}(x_n)\Big]\Big)$$
$$= R\Big(\sum_{i=1}^{n-1}[R(x_1), \ldots, x_i, \ldots, R(x_{n-1}), R(x_n)]_n\Big)$$
$$\quad + R\Big(\Big[\sum_{i=1}^{n-1}R([R(x_1), \ldots, x_i, \ldots, R(x_{n-1})]_{n-1}), \alpha^{n-2}(x_n)\Big]\Big)$$
$$= R\Big(\sum_{i=1}^{n-1}[R(x_1), \ldots, x_i, \ldots, R(x_{n-1}), R(x_n)]_n\Big)$$
$$\quad + R\big([[R(x_1), \ldots, R(x_{n-1})]_{n-1}, \alpha^{n-2}(x_n)]\big)$$
$$= R\Big(\sum_{i=1}^{n-1}[R(x_1), \ldots, x_i, \ldots, R(x_{n-1}), R(x_n)]_n\Big)$$
$$\quad + R\big([R(x_1), \ldots, R(x_{n-1}), x_n]_n\big)$$
$$= R\Big(\sum_{i=1}^{n}[R(x_1), \ldots, x_i, R(x_n)]_n\Big).$$

The theorem is proved. □

### 4.2. 3-Hom-pre-Lie Superalgebras

In this subsection, we generalize the notion of a 3-Hom-pre-Lie algebra introduced in [19] to the super case, which is closely related to Rota-Baxter operators. In particular, there is a construction of 3-Hom-pre-Lie superalgebras obtained from 3-Hom-Lie superalgebras.

**Definition 4.5.** A triple $(A, \{\cdot,\cdot,\cdot\}, \alpha)$, consisting of a linear super-space $A$ and two even linear maps $\{\cdot,\cdot,\cdot\} : A\otimes A\otimes A \to A$ and $\alpha : A \to A$, is called a 3-*Hom-pre-Lie superalgebra* if the following identities hold:

$$\{x, y, z\} = -(-1)^{|x||y|}\{y, x, z\}, \tag{28}$$

$$\begin{aligned}\{\alpha(x_1), \alpha(x_2), \{x_3, x_4, x_5\}\} &= \{[x_1, x_2, x_3]_C, \alpha(x_4), \alpha(x_5)\}\\ &+(-1)^{|x_3|(|x_1|+|x_2|)}\{\alpha(x_3), [x_1, x_2, x_4]_C, \alpha(x_5)\}\\ &+(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)}\{\alpha(x_3), \alpha(x_4), \{x_1, x_2, x_5\}\},\end{aligned} \tag{29}$$

$$\begin{aligned}\{[x_1, x_2, x_3]_C, \alpha(x_4), \alpha(x_5)\} &= \{\alpha(x_1), \alpha(x_2), \{x_3, x_4, x_5\}\}\\ &+(-1)^{|x_1|(|x_2|+|x_3|)}\{\alpha(x_2), \alpha(x_3), \{x_1, x_4, x_5\}\}\\ &+(-1)^{|x_3|(|x_1|+|x_2|)}\{\alpha(x_3), \alpha(x_1), \{x_2, x_4, x_5\}\},\end{aligned} \tag{30}$$

where $x, y, z, x_i \in \mathcal{H}(A), 1 \le i \le 5$ and $[\cdot,\cdot,\cdot]_C$ is called 3-supercommutator and defined by

$$\begin{aligned}&\forall x, y, z \in \mathcal{H}(A) :\\ &\quad [x, y, z]_C = \{x, y, z\} + (-1)^{|x|(|y|+|z|)}\{y, z, x\} + (-1)^{|z|(|x|+|y|)}\{z, x, y\}.\end{aligned} \tag{31}$$

**Proposition 4.6.** *Let* $(A, \{\cdot,\cdot,\cdot\}, \alpha)$ *be a* 3*-Hom-pre-Lie superalgebra. Then the induced* 3*-supercommutator in* (31) *and the linear map* $\alpha$ *define a* 3*-Hom-Lie superalgebra on* $A$.

*Proof.* By (28), the induced 3-supercommutator $[\cdot,\cdot,\cdot]_C$ in (31) is super-skew-symmetric. For $x_1, x_2, x_3, x_4, x_5 \in \mathcal{H}(A)$,

$$\begin{aligned}&[\alpha(x_1), \alpha(x_2), [x_3, x_4, x_5]_C]_C - [[x_1, x_2, x_3]_C, \alpha(x_4), \alpha(x_5)]_C\\
&\quad - (-1)^{|x_3|(|x_1|+|x_2|)}[\alpha(x_3), [x_1, x_2, x_4]_C, \alpha(x_5)]_C\\
&\quad - (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)}[\alpha(x_3), \alpha(x_4), [x_1, x_2, x_5]_C]_C\\
&= \{\alpha(x_1), \alpha(x_2), \{x_3, x_4, x_5\}\}\\
&\quad + (-1)^{|x_3|(|x_4|+|x_5|)}\{\alpha(x_1), \alpha(x_2), \{x_4, x_5, x_3\}\}\\
&\quad + (-1)^{|x_5|(|x_3|+|x_4|)}\{\alpha(x_1), \alpha(x_2), \{x_5, x_3, x_4\}\}\\
&\quad + (-1)^{|x_1|(|x_2|+|x_3|+|x_4|+|x_5|)}\{\alpha(x_2), [x_3, x_4, x_5]_C, \alpha(x_1)\}\\
&\quad + (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|+|x_5|)}\{[x_3, x_4, x_5]_C, \alpha(x_1), \alpha(x_2)\}\\
&\quad - \{[x_1, x_2, x_3]_C, \alpha(x_4), \alpha(x_5)\}\\
&\quad - (-1)^{(|x_1|+|x_2|+|x_3|)(|x_4|+|x_5|)}\{\alpha(x_4), \alpha(x_5), \{x_1, x_2, x_3\}\}\\
&\quad - (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|+|x_5|)}(-1)^{|x_1|(|x_2|+|x_3|)}\{\alpha(x_4), \alpha(x_5), \{x_2, x_3, x_1\}\}\\
&\quad - (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|+|x_5|)}(-1)^{|x_3|(|x_1|+|x_2|)}\{\alpha(x_4), \alpha(x_5), \{x_3, x_1, x_2\}\}\\
&\quad - (-1)^{|x_5|(|x_1|+|x_2|+|x_3|+|x_4|+|x_5|)}\{\alpha(x_5), [x_1, x_2, x_3]_C, \alpha(x_4)\}\\
&\quad - (-1)^{|x_3|(|x_1|+|x_2|)}\{\alpha(x_3), [x_1, x_2, x_4]_C, \alpha(x_5)\}\end{aligned}$$

$$- (-1)^{|x_3|(|x_4|+|x_5|)}\{[x_1,x_2,x_4]_C,\alpha(x_5),\alpha(x_3)\}$$
$$- (-1)|x_5|(|x_1|+|x_2|+|x_3|+|x_4|)(-1)^{|x_3|(|x_1|+|x_2|)}$$
$$\{\alpha(x_5),\alpha(x_3),\{x_1,x_2,x_4\}\}$$
$$- (-1)^{|x_5|(|x_1|+|x_2|+|x_3|+|x_4|)}(-1)^{|x_1|(|x_2|+|x_4|)+|x_3|(|x_1|+|x_2|)}$$
$$\{\alpha(x_5),\alpha(x_3),\{x_2,x_4,x_1\}\}$$
$$- (-1)^{|x_5|(|x_1|+|x_2|+|x_3|+|x_4|)}(-1)^{|x_4|(|x_1|+|x_2|)+|x_3|(|x_1|+|x_2|)}$$
$$\{\alpha(x_5),\alpha(x_3),\{x_4,x_1,x_2\}\}$$
$$- (-1)^{|x_3|(|x_4|+|x_5|)+|x_4|(|x_1|+|x_2|)}\{\alpha(x_4),[x_1,x_2,x_5]_C,\alpha(x_3)\}$$
$$- (-1)^{|x_5|(|x_3|+|x_4|)}\{[x_1,x_2,x_5]_C,\alpha(x_3),\alpha(x_4)\}$$
$$- (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)}\{\alpha(x_3),\alpha(x_4),\{x_1,x_2,x_5\}\}$$
$$- (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|+|x_5|)}\{\alpha(x_3),\alpha(x_4),\{x_5,x_1,x_2\}\}$$
$$- (-1)^{|x_1|(|x_2|+|x_5|)}(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)}\{\alpha(x_3),\alpha(x_4),\{x_2,x_5,x_1\}\} = 0,$$

when applying identities (29) and (30), thus the proof is completed. □

**Definition 4.7.** Let $(A,\{\cdot,\cdot,\cdot\},\alpha)$ be a 3-Hom-pre-Lie superalgebra. The 3-Hom-Lie superalgebra $(A,[\cdot,\cdot,\cdot]_C,\alpha)$ is called the *sub-adjacent* 3*-Hom-Lie superalgebra* of $(A,\{\cdot,\cdot,\cdot\},\alpha)$ and $(A,\{\cdot,\cdot,\cdot\},\alpha)$ is called a *compatible* 3*-Hom-pre-Lie superalgebra* of the 3-Hom-Lie superalgebra $(A,[\cdot,\cdot,\cdot]_C,\alpha)$.

New identities of 3-pre-Hom-Lie superalgebras can be derived from Proposition 4.6.

**Corollary 4.8.** *Let $(A,\{\cdot,\cdot,\cdot\},\alpha)$ be a* 3*-Hom-pre-Lie algebra. The following identities hold:*

$$\{[x_1,x_2,x_3]_C,\alpha(x_4),\alpha(x_5)\} - (-1)^{|x_3||x_4|}\{[x_1,x_2,x_4]_C,\alpha(x_3),\alpha(x_5)\}$$
$$+(-1)^{|x_2|(|x_3|+|x_4|)}\{[x_1,x_3,x_4]_C,\alpha(x_2),\alpha(x_5)\}$$
$$-(-1)^{|x_1|(|x_2|+|x_3|+|x_4|)}\{[x_2,x_3,x_4]_C,\alpha(x_1),\alpha(x_5)\} = 0,$$
$$\{\alpha(x_1),\alpha(x_2),\{x_3,x_4,x_5\}\} + (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)}\{\alpha(x_3),\alpha(x_4),\{x_1,x_2,x_5\}\}$$
$$+(-1)^{|x_1|(|x_2|+|x_3|+|x_4|)+|x_3||x_4|}\{\alpha(x_2),\alpha(x_4),\{x_3,x_1,x_5\}\}$$
$$+(-1)^{|x_3|(|x_1|+|x_2|)}\{\alpha(x_3),\alpha(x_1),\{x_2,x_4,x_5\}\}$$
$$+(-1)^{|x_1|(|x_2|+|x_3|)}\{\alpha(x_2),\alpha(x_3),\{x_1,x_4,x_5\}\}$$
$$+(-1)^{|x_4|(|x_2|+|x_3|)}\{\alpha(x_1),\alpha(x_4),\{x_2,x_3,x_5\}\} = 0,$$

*for $x_i \in \mathcal{H}(A), 1 \le i \le 5$.*

**Proposition 4.9.** *Let $(A,[\cdot,\cdot,\cdot],\alpha)$ be a* 3*-Hom-Lie superalgebra and $R : A \to A$ is an operator Rota-Baxter of weight* 0*. Then there exists a* 3*-Hom-pre-Lie superalgebra structure on $A$ given by*

$$\{x,y,z\} = [R(x),R(y),z], \quad \forall\ x,y,z \in \mathcal{H}(A). \tag{32}$$

*Proof.* Let $x,y,z \in \mathcal{H}(A)$. It is obvious that

$$\{x,y,z\} = [R(x),R(y),z] = -(-1)^{|x||y|}[R(y),R(x),z] = -(-1)^{|x||y|}\{y,x,z\}.$$

Furthermore, the following equation holds:

$$[x,y,z]_C = [R(x),R(y),z] + (-1)^{|z|(|x|+|y|)}[R(z),R(x),y] \\ +(-1)^{|x|(|y|+|z|)}[R(y),R(z),x].$$

Since $R$ is a Rota-Baxter operator, we have

$$R([x,y,z]_C) = [R(x),R(y),R(z)].$$

For $x_1,x_2,x_3,x_4,x_5 \in \mathcal{H}(A)$,

$$\begin{aligned}
\{\alpha(x_1),\alpha(x_2),\{x_3,x_4,x_5\}\} &= [R(\alpha(x_1)),R(\alpha(x_2)),[R(x_3),R(x_4),x_5]] \\
&= [\alpha(R(x_1)),\alpha(R(x_2)),[R(x_3),R(x_4),x_5]]; \\
\{[x_1,x_2,x_3]_C,\alpha(x_4),\alpha(x_5)\} &= [R([x_1,x_2,x_3]_C),R(\alpha(x_4)),\alpha(x_5)] \\
&= [[R(x_1),R(x_2),R(x_3)],\alpha(R(x_4)),\alpha(x_5)]; \\
\{\alpha(x_3),[x_1,x_2,x_4]_C,x_5\} &= [R(\alpha(x_3)),R([x_1,x_2,x_4]_C),\alpha(x_5)] \\
&= [\alpha(R(x_3)),[R(x_1),R(x_2),R(x_4)],\alpha(x_5)]; \\
\{\alpha(x_3),\alpha(x_4),\{x_1,x_2,x_5\}\} &= [R(\alpha(x_3)),R(\alpha(x_4)),[R(x_1),R(x_2),x_5]] \\
&= \alpha(R(x_3)),\alpha(R(x_4)),[R(x_1),R(x_2),x_5]].
\end{aligned}$$

By Condition (8), (29) holds. On the other hand, we have

$$\begin{aligned}
\{[x_1,x_2,x_3]_C,\alpha(x_4),\alpha(x_5)\} &= [R([x_1,x_2,x_3]_C),R(\alpha(x_4)),\alpha(x_5)] \\
&= [[R(x_1),R(x_2),R(x_3)],\alpha(R(x_4)),\alpha(x_5)]; \\
\{\alpha(x_1),\alpha(x_2),\{x_3,x_4,x_5\}\} &= [R(\alpha(x_1)),R(\alpha(x_2)),[R(x_3),R(x_4),x_5]] \\
&= [\alpha(R(x_1)),\alpha(R(x_2)),[R(x_3),R(x_4),x_5]; \\
\{\alpha(x_2),\alpha(x_3),\{x_1,x_4,x_5\}\} &= [R(\alpha(x_2)),R(\alpha(x_3)),[R(x_1),R(x_4),x_5]] \\
&= [\alpha(R(x_2)),\alpha(R(x_3)),[R(x_1),R(x_4),x_5]; \\
\{\alpha(x_3),\alpha(x_1),\{x_2,x_4,x_5\}\} &= [R(\alpha(x_3)),R(\alpha(x_1)),[R(x_2),R(x_4),x_5]] \\
&= [\alpha(R(x_3)),\alpha(R(x_1)),[R(x_2),R(x_4),x_5]].
\end{aligned}$$

By super-Hom-Nambu identity, (30) holds. This completes the proof. □

*Example 4.10.* Consider a 3-dimensional 3-Lie superalgebra ($A = A_0 \oplus A_1$, $[\cdot,\cdot,\cdot]$) (see [5]), where $A_0$ is generated by $< e_1 >$ and $A_1$ is generated by $< e_2, e_3 >$ and the only non-trivial bracket is

$$[e_2,e_2,e_2] = e_3.$$

Define an even linear map $\alpha : A \to A$ by

$$\alpha(e_1) = ae_1, \quad \alpha(e_2) = be_2 \quad \alpha(e_3) = b^3 e_3,$$

where $a, b \in \mathbb{K}$ . Then $\alpha$ is a morphisme of 3-Lie superalgebra $(A,[\cdot,\cdot,\cdot])$. Thus $(A,[\cdot,\cdot,\cdot]_\alpha = \alpha \circ [\cdot,\cdot,\cdot],\alpha)$ is a 3-Hom-Lie superalgebra, where the only non-trivial bracket is

$$[e_2,e_2,e_2]_\alpha = \alpha(e_3) = b^3 e_3.$$

Now, define an even linear map $R : A \to A$ by

$$R(e_1) = a'e_1, \quad R(e_2) = b'e_2 + c'e_3 \quad \alpha(e_3) = \frac{1}{3}b'e_3,$$

where $a', b', c'$ are parameters. Then $R$ is Rota-Baxter operator of 3-Hom-Lie superalgebra $(A, [\cdot,\cdot,\cdot]_\alpha, \alpha)$. By Proposition 4.9, $(A, \{\cdot,\cdot,\cdot\}, \alpha)$ is 3-Hom-pre-Lie superalgebra, where the only non-trivial bracket is

$$\{e_2, e_2, e_2\} = [R(e_2), R(e_2), e_2]_\alpha = b'^2 b^3 e_3.$$

**Corollary 4.11.** *With the above conditions, $(A, [\cdot,\cdot,\cdot]_C, \alpha)$ is a 3-Hom-Lie superalgebra as the sub-adjacent 3-Hom-Lie superalgebra of the 3-Hom-pre-Lie superalgebra given in Proposition 4.9, and $R$ is a 3-Hom-Lie superalgebra morphism from $(A, [\cdot,\cdot,\cdot]_C, \alpha)$ to $(A, [\cdot,\cdot,\cdot], \alpha)$. Furthermore, $R(A) = \{R(x)|x \in A\} \subset A$ is a 3-Hom-Lie super-subalgebra of $A$ and there is an induced 3-Hom-pre-Lie superalgebra structure $(\{\cdot,\cdot,\cdot\}_{R(A)}, \alpha)$ on $R(A)$ given by*

$$\{R(x), R(y), R(z)\}_{R(A)} := R(\{x, y, z\}), \quad \forall x, y, z \in \mathcal{H}(A). \tag{33}$$

**Proposition 4.12.** *Let $(A, [\cdot,\cdot,\cdot], \alpha)$ be a 3-Hom-Lie superalgebra. Then there exists a compatible 3-Hom-pre-Lie superalgebra if and only if there exists an invertible Rota-Baxter operator $R$ on $A$.*

*Proof.* Let $R$ be an invertible Rota-Baxter operator of $A$. Then there exists a 3-Hom-pre-Lie superalgebra structure $(\{x, y, z\}, \alpha)$ on $A$ defined by

$$\{x, y, z\} = ad_{(R(x),R(y))}(z), \quad \forall x, y, z \in \mathcal{H}(A).$$

Also, there is an induced 3-Hom-pre-Lie superalgebra structure $(\{\cdot,\cdot,\cdot\}_A, \alpha)$ on $A = R(A)$ given by

$$\{x, y, z\}_A = R\{R^{-1}(x), R^{-1}(y), R^{-1}(z)\} = R(ad_{(x,y)}(R^{-1}(z)))$$

for all $x, y, z \in \mathcal{H}(A)$. Since $R$ is a Rota-Baxter operator on $A$, we have

$$\begin{aligned}
[x, y, z] &= R\Big([x, y, R^{-1}(z)] + [x, R^{-1}(y), z] + [R^{-1}(x), y, z]\Big) \\
&= R\Big(ad_{(x,y)}(R^{-1}(z)) + (-1)^{|z|(|x|+|y|)} ad_{(z,x)}(R^{-1}(y)) \\
&\quad + (-1)^{|x|(|y|+|z|)} ad_{(y,z)}(R^{-1}(x))\Big) \\
&= \{x, y, z\}_A + (-1)^{|z|(|x|+|y|)}\{z, x, y\}_A + (-1)^{|x|(|y|+|z|)}\{y, z, x\}_A.
\end{aligned}$$

Therefore $(A, \{\cdot,\cdot,\cdot\}_A, \alpha)$ is a compatible 3-Hom-pre-Lie superalgebra of $(A, [\cdot,\cdot,\cdot])$. □

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## References

[1] Abdaoui, K., Mabrouk, S., Makhlouf, A.: Cohomology of Hom-Leibniz and $n$-ary Hom-Nambu-Lie superalgebras, pp. 24. arXiv:1406.3776 [math.RT] (2019)

[2] Abramov, V.: On a graded $q$-differential algebra. J. Nonlinear Math. Phys. **13**(1), 1–8 (2006)

[3] Abramov, V.: Graded $q$-Differential Algebra Approach to $q$-Connection. In: Silvestrov, S., Paal, E., Abramov, V., Stolin, A. (Eds.), Generalized Lie Theory in Mathematics, Physics and Beyond, Springer-Verlag, Berlin, Heidelberg, Ch. 6, 71–79 (2009)

[4] Abramov, V.: Super 3-Lie algebras induced by super Lie algebras. Adv. Appl. Clifford Algebr. **27**(1), 9–16 (2017)

[5] Abramov, V., Lätt, P.: Classification of Low Dimensional 3-Lie Superalgebras. In: Silvestrov, S., Rancic, M. (eds.) Engineering Mathematics II, Springer Proceedings in Mathematics and Statistics, vol. 179, pp. 1–12. Springer, Cham (2016)

[6] Abramov, V.: Raknuzzaman, Md: Semi-commutative Galois Extensions and Reduced Quantum Plane. In: Silvestrov, S., Rancic, M. (eds.) Engineering Mathematics II, Springer Proceedings in Mathematics and Statistics, vol. 179, pp. 13–31. Springer, Cham (2016)

[7] Abramov, V.: Weil Algebra, 3-Lie algebra and B.R.S. algebra, In: Silvestrov, S., Malyarenko, A., Rancic, M. (Eds.), Algebraic Structures and Applications, Springer Proceedings in Mathematics and Statistics, Vol 317, Ch 1 (2020). arXiv:1802.05576 [math.RA]

[8] Abramov, V., Lätt, P.: Ternary Lie superalgebras and Nambu-Hamilton equation in superspace, In: Silvestrov, S., Malyarenko, A., Rancic, M. (Eds.), Algebraic Structures and Applications, Springer Proceedings in Mathematics and Statistics, Vol. 317, Ch. 3 (2020)

[9] Ammar, F., Ejbehi, Z., Makhlouf, A.: Cohomology and deformations of Hom-algebras. J. Lie Theory **21**(4), 813–836 (2011)

[10] Ammar, F., Mabrouk, S., Makhlouf, A.: Representation and cohomology of $n$-ary multiplicative Hom-Nambu-Lie algebras. J. Geom. Phys. **61**(10), 1898–1913 (2011)

[11] Ammar, F., Makhlouf, A.: Hom-Lie superalgebras and Hom-Lie admissible superalgebras. J. Algebra **324**(7), 1513–1528 (2010)

[12] Armakan, A., Silvestrov, S., Farhangdoost, M.: Enveloping algebras of color hom-Lie algebras. Turk. J. Math. **43**, 316–339 (2019). https://doi.org/10.3906/mat-1808-96. arXiv:1709.06164 [math.QA]

[13] Armakan, A., Silvestrov, S., Farhangdoost, M.: Extensions of hom-Lie color algebras Georgian Math. J. **28**(1), 15–27 (2021). https://doi.org/10.1515/gmj-2019-2033. arXiv:1709.08620 [math.QA]

[14] Arnlind, J., Kitouni, A., Makhlouf, A., Silvestrov, S.: Structure and Cohomology of 3-Lie algebras induced by Lie algebras, In: Makhlouf, A., Paal, E., Silvestrov, S., Stolin, A. (Eds.), Algebra, Geometry and Mathematical Physics, Springer Proceedings in Mathematics and Statistics, Vol 85, 123-144 (2014). arXiv:1312.7599 [math.RA]

[15] Arnlind, J., Makhlouf, A., Silvestrov, S.: Ternary Hom-Nambu-Lie algebras induced by Hom-Lie algebras. J. Math. Phys. **51**(043515), 11 (2010)

[16] Arnlind, J., Makhlouf, A., Silvestrov, S.: Construction of $n$-Lie algebras and $n$-ary Hom-Nambu-Lie algebras. J. Math. Phys. **52**(123502), 13 (2011)

[17] Ataguema, H., Makhlouf, A., Silvestrov, S.: Generalization of $n$-ary Nambu algebras and beyond. J. Math. Phys. **50**, 083501 (2009)

[18] Awata, H., Li, M., Minic, D., Yoneya, T.: On the quantization of Nambu brackets. J. High Energy Phys. **2**(13), 17 (2001)

[19] Bai, C., Guo, L., Sheng, Y.: Bialgebras, the classical Yang-Baxter equation and Manin triples for 3-Lie algebras. (2016). arXiv:1604.05996

[20] Bai, R., Bai, C., Wang, J.: Realizations of 3-Lie algebras. J. Math. Phys. **51**, 063505 (2010)

[21] Bai, R., Wu, Y., Li, J., Zhou, H.: Constructing $(n+1)$-Lie algebras from $n$-Lie algebras. J. Phys. A **45**, 47 (2012)

[22] Bai, R., Song, G., Zhang, Y.: On classification of $n$-Lie algebras. Front. Math. China **6**, 581–606 (2011)

[23] Bai, R., Wang, X., Xiao, W., An, H.: Structure of low dimensional $n$-Lie algebras over a field of characteristic 2. Linear Algebra Appl. **428**(8–9), 1912–1920 (2008)

[24] Bai, R., Chen, L., Meng, D.: The Frattini subalgebra of $n$-Lie algebras. Acta. Math. Sin. Engl. Ser. **23**(5), 847–856 (2007)

[25] Bai, R., Meng, D.: The central extension of $n$-Lie algebras. Chin. Ann. Math. **27**(4), 491–502 (2006)

[26] Bai, R., Meng, D.: The centroid of $n$-Lie algebras. Algebras Groups Geom. **25**(2), 29–38 (2004)

[27] Bai, R., Zhang, Z., Li, H., Shi, H.: The inner derivation algebras of $(n+1)$-dimensional $n$-Lie algebras. Comm. Algebra **28**(6), 2927–2934 (2000)

[28] Bai, R., An, H., Li, Z.: Centroid structures of $n$-Lie algebras. Linear Algebra Appl. **430**, 229–240 (2009)

[29] Bakayoko, I.: Laplacian of Hom-Lie quasi-bialgebras. Int. J. Algebra **8**(15), 713–727 (2014)

[30] Bakayoko, I.: $L$-modules, $L$-comodules and Hom-Lie quasi-bialgebras. Afr. Diaspora J. Math. **17**, 49–64 (2014)

[31] Bakayoko, I., Silvestrov, S.: Multiplicative $n$-Hom-Lie color algebras, In: Silvestrov, S., Malyarenko, A., Rancic, M. (Eds.), Algebraic Structures and Applications, Springer Proceedings in Mathematics and Statistics, Vol 317, Ch. 7, (2020). arXiv:1912.10216 [math.QA]

[32] Bakayoko, I., Silvestrov, S.: Hom-left-symmetric color dialgebras, Hom-tridendriform color algebras and Yau's twisting generalizations, pp. 24 (2019). arXiv:1912.01441 [math.RA]

[33] Ben Hassine, A., Mabrouk, S., Ncib, O.: Some Constructions of Multiplicative $n$-ary hom-Nambu Algebras. Adv. Appl. Clifford Algebras **29**, 88 (2019)

[34] Ben Abdeljelil, A., Elhamdadi, M., Kaygorodov, I., Makhlouf, A.: Generalized Derivations of $n$-BiHom-Lie algebras, In: Silvestrov, S., Malyarenko, A., Rancic, M. (Eds.), Algebraic Structures and Applications, Springer Proceedings in Mathematics and Statistics, Vol 317, Ch. 4 (2020). arXiv:1901.09750 [math.RA]

[35] Benayadi, S., Makhlouf, A.: Hom-Lie algebras with symmetric invariant nondegenerate bilinear forms. J. Geom. Phys. **76**, 38–60 (2014)

[36] Beites, P.D., Kaygorodov, I., Popov, Y.: Generalized derivations of multiplicative $n$-ary Hom-$\Omega$ color algebras. Bull. Malay. Math. Sci. Soc. **2018**, 41 (2018)

[37] Casas, J.M., Loday, J.-L., Pirashvili, T.: Leibniz $n$-algebras. Forum Math. **14**, 189–207 (2002)

[38] Chen, L., Ma, Y., Ni, L.: Generalized Derivations of Lie color algebras. Results Math. **63**(3–4), 923–936 (2013)

[39] Chen, L., Ma, Y., Zhou, J.: Generalized Derivations of Lie triple systems (2019). arXiv:1412.7804

[40] Daletskii, Y.L., Takhtajan, L.A.: Leibniz and Lie algebra structures for nambu algebra. Lett. Math. Phys. **39**, 127–141 (1997). https://doi.org/10.1023/A:1007316732705

[41] De Azcárraga, J.A., Izquierdo, J.M.: $n$-Ary algebras: a review with applications. J. Phys. A Math. Theor. **43**, 293001 (2010)

[42] Filippov, V.T.: $n$-Lie algebras. Sib. Math. J. **26**, 879–891 (1985). (Transl. from Russian: Sib. Mat. Zh. 26:126-140 (1985))

[43] Filippov, V.T.: On $\delta$-derivations of Lie algebras. Sib. Math. J. **39**, 1218–1230 (1998). (Translated from Sibirskii Matematicheskii Zhurnal, Vol. 39, No. 6, pp. 1409-1422, November-December, 1998.)

[44] Filippov, V.T.: $\delta$-Derivations of prime Lie algebras. Sib. Math. J. **40**, 174–184 (1999)

[45] Filippov, V.T.: $\delta$-derivations of prime alternative and Mal'tsev algebras. Algebra Logic **39**, 354–358 (2000). (Translated from Algebra i Logika, Vol. 39, No. 5, pp. 618-625, September-October, 2000)

[46] Elchinger, O., Lundengård, K., Makhlouf, A., Silvestrov, S.: Brackets with $(\tau,\sigma)$-derivations and $(p,q)$-deformations of Witt and Virasoro algebras. Forum Math. **28**, 657–673 (2016)

[47] Guan, B., Chen, L., Sun, B.: 3-Ary Hom-Lie Superalgebras Induced By Hom-Lie Superalgebras. Adv. Appl. Clifford Algebras **27**, 3063–3082 (2017)

[48] Guan, B., Chen, L., Sun, B.: On Hom-Lie Superalgebras. Adv. Appl. Clifford Algebras **29**, 16 (2019). https://doi.org/10.1007/s00006-018-0932-1

[49] Hartwig, J.T., Larsson, D., Silvestrov, S.D.: Deformations of Lie algebras using $\sigma$-derivations. J. Algebra **295**, 314–361 (2006). (Preprint in Mathematical

Sciences 2003:32, LUTFMA-5036-2003, Centre for Mathematical Sciences, Department of Mathematics, Lund Institute of Technology, pp. 52 (2003))

[50] Kasymov, S.M.: Theory of $n$-Lie algebras. Algebra Logic **26**, 155–166 (1987). (Transl. from Russian: Algebra i Logika, Vol. 26, No. 3, pp. 277-297, (1987))

[51] Kaygorodov, I., Popov, Y.U.: Alternative algebras admitting derivations with invertible values and invertible derivations. Izv. Math. **78**, 922–935 (2014)

[52] Kaygorodov, I.: On $\delta$-Derivations of $n$-ary algebras. Izv.: Math. **76**(5), 1150–1162 (2012)

[53] Kaygorodov, I.: $(n+1)$-Ary derivations of simple $n$-ary algebras. Algebra Logic **50**(5), 470–471 (2011)

[54] Kaygorodov, I.: $(n+1)$-Ary derivations of semisimple Filippov algebras. Math. Notes **96**(2), 208–216 (2014)

[55] Kaygorodov, I., Popov, Yu.: Generalized derivations of (color) $n$-ary algebras. Linear Multilinear Algebra **64**, 6 (2016)

[56] Kitouni, A., Makhlouf, A.: On structure and central extensions of $(n+1)$-Lie algebras induced by $n$-Lie algebras (2019). arXiv:1405.5930 [math.RA] (2014)

[57] Kitouni, A., Makhlouf, A., Silvestrov, S.: On $(n+1)$-Hom-Lie algebras induced by $n$-Hom-Lie algebras. Georgian Math. J. **23**(1), 75–95 (2016)

[58] Kitouni, A., Makhlouf, A., Silvestrov, S.: On $n$-ary generalization of BiHom-Lie algebras and BiHom-associative algebras, In: Silvestrov, S., Malyarenko, A., Rancic, M. (Eds.), Algebraic Structures and Applications, Springer Proceedings in Mathematics and Statistics, Vol 317, Ch 5 (2020)

[59] Kitouni, A., Makhlouf, A., Silvestrov, S.: On Solvability and Nilpotency for $n$-Hom-Lie Algebras and $(n+1)$-Hom-Lie Algebras Induced by $n$-Hom-Lie Algebras, In: Silvestrov, S., Malyarenko, A., Rancic, M. (Eds.), Algebraic Structures and Applications, Springer Proceedings in Mathematics and Statistics, Vol 317, Ch 6 (2020)

[60] Larsson, D., Sigurdsson, G., Silvestrov, S.D.: Quasi-Lie deformations on the algebra $\mathbb{F}[t]/(t^N)$. J. Gen. Lie Theory Appl. **2**(3), 201–205 (2008)

[61] Larsson, D., Silvestrov, S.D.: Quasi-Hom-Lie algebras, central extensions and 2-cocycle-like identities. J. Algebra **288**, 321–344 (2005). (Preprints in Mathematical Sciences 2004:3, LUTFMA-5038-2004, Centre for Mathematical Sciences, Department of Mathematics, Lund Institute of Technology, Lund University (2004))

[62] Larsson, D., Silvestrov, S. D.: Quasi-Lie algebras. In: Noncommutative Geometry and Representation Theory in Mathematical Physics. Contemp. Math., 391, Amer. Math. Soc., Providence, RI, 241-248 (2005) (Preprints in Mathematical Sciences 2004:30, LUTFMA-5049-2004, Centre for Mathematical Sciences, Department of Mathematics, Lund Institute of Technology, Lund University (2004))

[63] Larsson, D., Silvestrov, S.D.: Graded quasi-Lie agebras. Czechoslovak J. Phys. **55**, 1473–1478 (2005)

[64] Larsson, D., Silvestrov, S.D.: Quasi-deformations of $sl_2(\mathbb{F})$ using twisted derivations. Comm. Algebra **35**, 4303–4318 (2007)

[65] Larsson, D., Silvestrov, S. D.: On Generalized $N$-Complexes Comming from Twisted Derivations, In: Silvestrov, S., Paal, E., Abramov, V., Stolin, A. (Eds.), Generalized Lie Theory in Mathematics, Physics and Beyond. Springer, Berlin, Heidelberg, Ch. 7, pp. 81–88 (2009)

[66] Leger, G., Luks, E.: Generalized derivations of Lie algebras. J. Algebra **228**, 165–203 (2000)

[67] Ling, W.X.: On the structure of $n$-Lie algebras. University-GHS-Siegen, Siegen (1993). PhD Thesis

[68] Ma, T., Makhlouf, A., Silvestrov, S.: Curved $\mathcal{O}$-operator systems, pp. 17 (2017). arXiv:1710.05232 [math.RA]

[69] Ma, T., Makhlouf, A., Silvestrov, S.: Rota-Baxter bisystems and covariant bialgebras, p. 30 (2017). arXiv:1710.05161 [math.R]

[70] Ma, T., Makhlouf, A., Silvestrov, S.: Rota-Baxter cosystems and coquasitriangular mixed bialgebras. J. Algebra Appl **2019**, 5 (2019)

[71] Makhlouf, A., Silvestrov, S.D.: Hom-algebra structures. J. Gen. Lie Theory Appl. **2**(2), 51–64 (2008). (Preprints in Mathematical Sciences 2006:10, LUTFMA-5074-2006, Centre for Mathematical Sciences, Department of Mathematics, Lund Institute of Technology, Lund University (2006))

[72] Makhlouf, A., Silvestrov, S.: Hom-Lie admissible Hom-coalgebras and Hom-Hopf algebras, In: Silvestrov, S., Paal, E., Abramov, V., Stolin, A. (Eds.), Generalized Lie Theory in Mathematics, Physics and Beyond, Springer-Verlag, Berlin, Heidelberg, Ch. 17, pp. 189-206 (2009)

[73] Makhlouf, A., Silvestrov, S.D.: Hom-Algebras and Hom-Coalgebras. J. Algebra Appl. **9**(04), 553–589 (2010). arXiv:0811.0400 [math.RA]

[74] Makhlouf, A., Silvestrov, S.D.: Notes on Formal deformations of Hom-Associative and Hom-Lie algebras. Forum Math. **22**(4), 715–739 (2010)

[75] Mishra, S. K., Silvestrov, S.: A Review on Hom-Gerstenhaber algebras and Hom-Lie algebroids, In: Silvestrov, S., Malyarenko, A., Rancic, M. (Eds.), Algebraic Structures and Applications, Springer Proceedings in Mathematics and Statistics, Vol 317, Ch 11 (2020)

[76] Nambu, Y.: Generalized Hamiltonian dynamics. Phys. Rev. D **7**(8), 2405–2412 (1973)

[77] Pojidaev, A., Saraiva, P.: On derivations of the ternary Malcev algebra M8. Comm. Algebra. **34**, 3593–3608 (2006)

[78] Richard, L., Silvestrov, S.D.: Quasi-Lie structure of $\sigma$-derivations of $\mathbb{C}[t^{\pm 1}]$. J. Algebra **319**(3), 1285–1304 (2008)

[79] Richard, L., Silvestrov, S.: A Note on Quasi-Lie and Hom-Lie Structures of $\sigma$-Derivations of $\mathbb{C}[z_1^{\pm 1}, \cdots, z_n^{\pm 1}]$, In: Silvestrov, S., Paal, E., Abramov, V., Stolin, A. (Eds.), Generalized Lie Theory in Mathematics, Physics and Beyond, Springer-Verlag, Berlin, Heidelberg, Ch. 22, pp. 257–262 (2009)

[80] Rotkiewicz, M.: Cohomology ring of $n$-Lie algebras. Extracta Math. **20**, 219–232 (2005)

[81] Sheng, Y.: Representation of Hom-Lie algebras. Algebr. Represent. Theory **15**(6), 1081–1098 (2012). arXiv:1005.0140 [math-ph]

[82] Sigurdsson, G., Silvestrov, S.: Lie color and Hom-Lie algebras of Witt type and their central extensions, In: Silvestrov, S., Paal, E., Abramov, V., Stolin, A. (Eds.), Generalized Lie Theory in Mathematics, Physics and Beyond. Springer, Berlin, Heidelberg, Ch. 21, pp. 247-255 (2009)

[83] Sigurdsson, G., Silvestrov, S.: Graded quasi-Lie algebras of Witt type. Czech. J. Phys. **56**, 1287–1291 (2006)

[84] Takhtajan, L.A.: On foundation of the generalized Nambu mechanics. Comm. Math. Phys. **160**(2), 295–315 (1994)

[85] Takhtajan, L.A.: Higher order analog of Chevalley-Eilenberg complex and deformation theory of $n$-gebras. St. Petersburg Math. J. **6**(2), 429–438 (1995)

[86] Wang, C., Zhang, Q., Wei, Z.: A classification of low dimensional multiplicative Hom-Lie superalgebras. Open Math. **14**(1), 613–628 (2016)

[87] Williams, M.P.: Nilpotent $n$-Lie Algebras. Comm. Algebra **37**(6), 1843–1849 (2009)

[88] Yau, D.: Enveloping algebras of Hom-Lie algebras. J. Gen. Lie Theory Appl. **2**(2), 95–108 (2008)

[89] Yau, D.: Hom-algebras and homology. J. Lie Theory **19**(2), 409–421 (2009)

[90] Yau, D.: A Hom-associative analogue of $n$-ary Hom-Nambu algebras (2019). arXiv:1005.2373

[91] Yau, D.: On $n$-ary Hom-Nambu and Hom-Nambu-Lie algebras. J. Geom. Phys. **62**, 506–522 (2012). arXiv:1004.2080 [math.RA]

[92] Yau, D.: On $n$-ary Hom-Nambu and Hom-Maltsev algebras (2019). arXiv:1004.4795 [math.RA]

[93] Zhang, R., Zhang, Y.: Generalized derivations of Lie superalgebras. Comm. Algebra **38**(10), 3737–3751 (2010)

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