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Hyperbolic Function Theory in the Skew-Field of Quaternions

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Abstract. We are studying hyperbolic function theory in the total skew-field of quaternions. Earlier the theory has been studied for quaternion valued functions depending only on three reduced variables. Our functions are depending on all four coordinates of quaternions. We consider functions, called α-hyperbolic harmonic, that are harmonic with respect to the Riemannian metric

$$ds_\alpha^2 = \frac{dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2}{x_3^\alpha}$$

in the upper half space $\mathbb{R}_+^4 = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_3 > 0\}$. If $\alpha = 2$, the metric is the hyperbolic metric of the Poincaré upper half-space. Hempfling and Leutwiler started to study this case and noticed that the quaternionic power function x^m $(m \in \mathbb{Z})$, is a conjugate gradient of a 2-hyperbolic harmonic function. They researched polynomial solutions. Using fundamental α-hyperbolic harmonic functions, depending only on the hyperbolic distance and x_3, we verify a Cauchy type integral formula for conjugate gradient of α-hyperbolic harmonic functions. We also compare these results with the properties of paravector valued α-hypermonogenic in the Clifford algebra $\mathcal{C}\ell_{0,3}$.

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1. Introduction

We study quaternion valued twice continuous differentiable functions $f(x)$ defined in an open subset of the full space $\mathbb{R}^4$ satisfying the following modified

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Cauchy–Riemann system

$$\begin{aligned} x_3\left(\frac{\partial f_0}{\partial x_0}-\frac{\partial f_1}{\partial x_1}-\frac{\partial f_2}{\partial x_2}-\frac{\partial f_3}{\partial x_3}\right)+\alpha f_3=0,\\ \frac{\partial f_0}{\partial x_m}=-\frac{\partial f_m}{\partial x_0} \quad \text{for all} \quad m=1,2,3,\\ \frac{\partial f_m}{\partial x_n}=\frac{\partial f_n}{\partial x_m} \quad \text{for all} \quad m,n=1,2,3. \end{aligned}$$

Earlier the theory has been studied for quaternion valued functions depending only on three reduced variables [5]. In case $\alpha = 2$, this system was studied by Hempfling and Leutwiler in [11]. Recently, we verified Cauchy type formulas for these function in [6]. In this paper, we study integral formulas and operators produced by these formulas. The results are interesting, since we are building hyperbolic function theory in the full skew field of quaternions. We also develop the theory of paravector valued α-hypermonogenic functions in the Clifford algebra $\mathcal{C}\ell_{0,3}$ and find similar integral theorems as in the quaternionic hyperbolic function theory.

2. Preliminaries

The skew-field of quaternions $\mathbb{H}$ is four dimensional associative division algebra over reals with an identity $\mathbf{1}$. We denote by $\mathbf{1}$, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ the generating elements of $\mathbb{H}$ satisfying the relations

$$\boldsymbol{i}^2=\boldsymbol{j}^2=\boldsymbol{k}^2=\boldsymbol{ijk}=-\mathbf{1}.$$

The elements $\beta\mathbf{1}$ and β are identified for any $\beta \in \mathbb{R}$.

Any quaternion x may be represented with respect to the base $\{\mathbf{1},\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ by

$$x=x_0+x_1\boldsymbol{i}+x_2\boldsymbol{j}+x_3\boldsymbol{k}$$

where x_0, x_1, x_2 and x_3 are real numbers. The real vector spaces $\mathbb{R}^4$ and $\mathbb{H}$ may be identified.

We denote the upper half space by

$$\mathbb{R}^4_+=\{(x_0,x_1,x_2,x_3)\ \mid x_m\in\mathbb{R},\ m=0,1,2,3 \text{ and } x_3>0\}$$

and the lower half space by

$$\mathbb{R}^4_-=\{(x_0,x_1,x_2,x_3)\ \mid x_m\in\mathbb{R},\ m=0,1,2,3 \text{ and } x_3<0\}.$$

We recall that the hyperbolic distance $d_h(x,a)$ between the points x and a in $\mathbb{R}^4_+$ is $d_h(x,a)=\operatorname{arcosh}(\lambda(x,a))$ where

$$\begin{aligned}\lambda(x,a)&=\frac{(x_0-a_0)^2+(x_1-a_1)^2+(x_2-a_2)^2+x_3^2+a_3^2}{2x_3a_3}\\ &=\frac{\|x-a\|^2+\|x-a^*\|^2}{4x_3a_3}\\ &=\frac{\|x-a\|^2}{2x_3a_3}+1=\frac{\|x-a^*\|^2}{2x_3a_3}-1,\end{aligned}$$

and

$$a^* = (a_0, a_1, a_2, -a_3),$$
$$\|x-a\| = \sqrt{(x_0-a_0)^2 + (x_1-a_1)^2 + (x_2-a_2)^2 + (x_3-a_3)^2},$$

(see a proof for example in [12]). Similarly, we may compute the hyperbolic distance between the points x and a in $\mathbb{R}^4_-$.

The following simple calculation rules

$$\|x-a\|^2 = 2x_3a_3(\lambda(x,a)-1), \tag{2.1}$$
$$\|x-a^*\|^2 = 2x_3a_3(\lambda(x,a)+1), \tag{2.2}$$
$$\frac{\|x-a\|^2}{\|x-a^*\|^2} = \frac{\lambda(x,a)-1}{\lambda(x,a)+1} = \tanh^2\left(\frac{d_h(x,a)}{2}\right), \tag{2.3}$$

are useful.

We recall that the hyperbolic ball $B_h(a, r_h)$ with the hyperbolic center a in $\mathbb{R}^4_+$ and the radius r_h is the same as the Euclidean ball with the Euclidean center

$$c_a(r_h) = (a_0, a_1, a_2, a_3\cosh r_h)$$

and the Euclidean radius $r_e = a_3 \sinh r_h$.

The inner product $\langle x, y\rangle$ in $\mathbb{R}^4$ is defined as usual by

$$\langle x, y\rangle = \sum_{m=0}^{3} x_m y_m.$$

If $x = x_0 + x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}$ and $y = y_0 + y_1\boldsymbol{i} + y_2\boldsymbol{j} + y_3\boldsymbol{k}$ are quaternions their inner product is defined similarly as in $\mathbb{R}^4$ by

$$\langle x, y\rangle = \sum_{m=0}^{3} x_m y_m.$$

The elements

$$x = x_0 + x_1\boldsymbol{i} + x_2\boldsymbol{j}$$

are called *reduced quaternions*. The set of reduced quaternions is identified with $\mathbb{R}^3$.

The *involution* $(\,)'$ in $\mathbb{H}$ is the mapping $x \to x'$ defined by

$$x' = x_0 - x_1\boldsymbol{i} - x_2\boldsymbol{j} + x_3\boldsymbol{k}$$

and it satisfies

$$(xy)' = x'y'$$

for all quaternions x and y. The *reversion* $(\,)^*$ in $\mathbb{H}$ is the mapping $x \to x^*$ defined by

$$x^* = x_0 + x_1\boldsymbol{i} + x_2\boldsymbol{j} - x_3\boldsymbol{k}$$

and the *conjugation* $\overline{(\,)}$ in $\mathbb{H}$ is the mapping $x \to \overline{x}$ defined by $\overline{x} = (x')^* = (x^*)'$, that is

$$\overline{x} = x_0 - x_1\boldsymbol{i} - x_2\boldsymbol{j} - x_3\boldsymbol{k}.$$

These involutions satisfy the following product rules

$$(xy)^* = y^* x^*$$

and

$$\overline{xy} = \overline{y}\,\overline{x}$$

for all $x, y \in \mathbb{H}$.

The prime involution may be computed as

$$x' = -\boldsymbol{k} x \boldsymbol{k}$$

for all quaternions x. This formula shows, in fact, that the involution $(\ )'$ is the rotation around the x_3 axes. Similarly, the formulas

$$\overline{x} = -\boldsymbol{k} x^* \boldsymbol{k},$$
$$x^* = -\boldsymbol{k} \overline{x} \boldsymbol{k},$$

hold for all quaternions x. Hence we have the identities

$$x\boldsymbol{k} = \boldsymbol{k} x'$$

and

$$x^* \boldsymbol{k} = \boldsymbol{k} \overline{x}$$

valid for all quaternions x.

The real part of a quaternion $x = x_0 + x_1 \boldsymbol{i} + x_2 \boldsymbol{j} + x_3 \boldsymbol{k}$ is defined by

$$\mathrm{Re}\ x = x_0$$

and the vector part by

$$\mathrm{Vec}\ x = x_1 \boldsymbol{i} + x_2 \boldsymbol{j} + x_3 \boldsymbol{k}.$$

if $\mathrm{Re}\ x = \mathrm{Re}\ y = 0$, the product rule

$$xy = -\langle x, y \rangle + x \times y$$

holds, where $\times$ is the usual cross product.

The mappings $S : \mathbb{H} \to \mathbb{R}^3$ and $T : \mathbb{H} \to \mathbb{R}$ are defined by

$$Sa = a_0 + a_1 \boldsymbol{i} + a_2 \boldsymbol{j}$$

and

$$Ta = a_3$$

for $a = a_0 + a_1 \boldsymbol{i} + a_2 \boldsymbol{j} + a_3 \boldsymbol{k} \in \mathbb{H}$. Using the reversion, we compute the formulas

$$Sa = \frac{1}{2}(a + a^*) = \frac{1}{2}(a - \boldsymbol{k} \overline{a} \boldsymbol{k}), \tag{2.4}$$

$$Ta = -\frac{1}{2}(a - a^*) \boldsymbol{k} = \frac{1}{2}(\boldsymbol{k} \overline{a} - a \boldsymbol{k}). \tag{2.5}$$

We use the identities

$$ab + ba = 2a\mathrm{Re}\ b + 2b\mathrm{Re}\ a - 2\langle a, b \rangle, \tag{2.6}$$

$$\langle a, b\rangle = \frac{a\bar{b} + b\bar{a}}{2} = \mathrm{Re}\left(a\bar{b}\right) \tag{2.7}$$

and

$$\frac{1}{2}\left(a\bar{b}c + c\bar{b}a\right) = \langle b, c\rangle\, a - [a, b, c] \tag{2.8}$$

valid for all quaternions a, b and c. The term $[a, b, c]$, called a *triple product*, is defined by

$$[a, b, c] = \langle a, c\rangle b - \langle a, b\rangle c.$$

If Re $a =$ Re $b =$ Re $c = 0$, then (see [10])

$$[a, b, c] = a \times (b \times c).$$

Notice that the triple product is linear with respect to a, b and c. Moreover,

$$[a, b, c]^* = \langle a, c\rangle b^* - \langle a, b\rangle c^* \tag{2.9}$$
$$= \langle a^*, c^*\rangle b^* - \langle a^*, b^*\rangle c^*$$
$$= [a^*, b^*, c^*]\,. \tag{2.10}$$

3. Hyperregular Functions

We define the following hyperbolic generalized Cauchy–Riemann operators $H^l_\alpha(x)$ and $H^r_\alpha(x)$ for $x \in \Omega\backslash\{x_3 = 0\}$ as follows

$$H^l_\alpha f(x) = D^q_l f(x) + \alpha\frac{f_3}{x_3}, \quad \overline{H}^l_\alpha f(x) = \overline{D}^q_l f(x) - \alpha\frac{f_3}{x_3},$$
$$H^r_\alpha f(x) = D^q_r f(x) + \alpha\frac{f_3}{x_3}, \quad \overline{H}^r_\alpha f(x) = \overline{D}^q_r f(x) - \alpha\frac{f_3}{x_3},$$

where the parameter $\alpha \in \mathbb{R}$ and

$$D^q_l f = \frac{\partial f}{\partial x_0} + \boldsymbol{i}\frac{\partial f}{\partial x_1} + \boldsymbol{j}\frac{\partial f}{\partial x_2} + \boldsymbol{k}\frac{\partial f}{\partial x_3} \quad \overline{D}^q_l f = \frac{\partial f}{\partial x_0} - \boldsymbol{i}\frac{\partial f}{\partial x_1} - \boldsymbol{j}\frac{\partial f}{\partial x_2} - \boldsymbol{k}\frac{\partial f}{\partial x_3},$$
$$D^q_r f = \frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_1}\boldsymbol{i} + \frac{\partial f}{\partial x_2}\boldsymbol{j} + \frac{\partial f}{\partial x_3}\boldsymbol{k}, \quad \overline{D}^q_r f = \frac{\partial f}{\partial x_0} - \frac{\partial f}{\partial x_1}\boldsymbol{i} - \frac{\partial f}{\partial x_2}\boldsymbol{j} - \frac{\partial f}{\partial x_3}\boldsymbol{k}.$$

When there is no confusion, we abbreviate $D^q_l f$ by $D^q f$ and H^l_α by H_α.

Definition 3.1. Let $\Omega \subset \mathbb{R}^4$ be open. A function $f : \Omega \to \mathbb{H}$ is called *α-hyperregular*, if $f \in \mathcal{C}^1(\Omega)$ and

$$H^l_\alpha f(x) = H^r_\alpha f(x) = 0$$

for any $x \in \Omega\backslash\{x_3 = 0\}$.

We emphasize that a function is α-hyperregular provided that it is continuous differentiable in the total open set $\Omega \subset \mathbb{R}^4$ and satisfies the preceding equation for all x with $x_3 \neq 0$.

Computing the components of $H^l_\alpha f(x)$ and $H^r_\alpha f(x)$, we obtain

Proposition 3.2. [6] *Let $\Omega \subset \mathbb{R}^4$ be open and a function $f : \Omega \to \mathbb{H}$ continuously differentiable. A function f is α-hyperregular in Ω if and only if*

$$\frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} + \alpha \frac{f_3}{x_3} = 0, \quad \text{if } x_3 \neq 0,$$
$$\frac{\partial f_0}{\partial x_m} = -\frac{\partial f_m}{\partial x_0} \quad \text{for all } m = 1, 2, 3,$$
$$\frac{\partial f_m}{\partial x_n} = \frac{\partial f_n}{\partial x_m} \quad \text{for all } m, n = 1, 2, 3.$$

Our operators are connected to the hyperbolic metric via the hyperbolic Laplace operator as follows.

Proposition 3.3. [6] *Let $\Omega \subset \mathbb{R}^4$ be open, $x \in \Omega \backslash \{x_3 = 0\}$ and $f : \Omega \to \mathbb{R}$ a real twice continuously differentiable function. Then*

$$x_3^\alpha H_\alpha^l \overline{H}_\alpha^l f(x) = x_3^\alpha H_\alpha^r \overline{H}_\alpha^r f(x) = \Delta_\alpha f(x)$$

where the operator

$$\Delta_\alpha = x_3^\alpha \left(\Delta - \frac{\alpha}{x_3} \frac{\partial}{\partial x_3} \right)$$

is the Laplace–Beltrami operator (see [13]) with respect to the Riemannian metric

$$ds_\alpha^2 = \frac{dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2}{x_3^\alpha}. \tag{3.1}$$

Definition 3.4. Let $\Omega \subset \mathbb{R}^4$ be open. A twice continuously real differentiable function $h : \Omega \to \mathbb{R}$ is called *α-hyperbolic harmonic*, if

$$\Delta_\alpha h(x) = 0$$

for all $x \in \Omega \backslash \{x_3 = 0\}$.

We list a couple of simple observations.

Lemma 3.5. *Let Ω be an open subset of $\mathbb{R}^4$. If $h : \Omega \to \mathbb{R}$ is α-hyperbolic on Ω and $h \in \mathcal{C}^3(\Omega)$ then the function $\frac{\partial h}{\partial x_3}$ satisfies the equation*

$$x_3^2 \Delta h(x) - \alpha x_3 \frac{\partial h}{\partial x_3}(x) + \alpha h(x) = 0$$

for all $x \in \Omega$. Moreover, a twice continuously differentiable function $h : \Omega \to \mathbb{R}$ satisfies the preceding equation if and only if the function $x_3^{-\alpha} h(x)$ is $-\alpha$-hyperbolic harmonic for any $x \in \Omega \backslash \{x_3 = 0\}$.

Proof. Assume that $x \in \Omega \backslash \{x_3 = 0\}$. We just compute as follows

$$\begin{aligned} \Delta \left(x_3^{-\alpha} h \right) + \frac{\alpha}{x_3} \frac{\partial h}{\partial x_3} &= x_3^{-\alpha} \Delta h - \frac{2\alpha}{x_3^{\alpha+1}} \frac{\partial h}{\partial x_3} + \alpha(\alpha+1) x_3^{-\alpha-2} h \\ &\quad + \frac{\alpha}{x_3^{\alpha+1}} \frac{\partial h}{\partial x_3} - \alpha^2 x_3^{-\alpha-2} h \\ &= x_3^{-\alpha} \left(x_3^2 \Delta h - \alpha x_3 \frac{\partial h}{\partial x_3} + \alpha h \right). \end{aligned}$$

□

Real valued α-hyperbolic functions are especially important, since they produce α-hyperregular functions.

Theorem 3.6. [6] *Let Ω be an open subset of $\mathbb{R}^4$. If h is α-hyperbolic on Ω then the function $f = \overline{D}^q h$ is α-hyperregular on Ω. Conversely, if f is α-hyperregular on Ω, there exists locally a α-hyperbolic function h satisfying $f = \overline{D}^q h$.*

Theorem 3.7. [6] *Let Ω be an open subset of $\mathbb{R}^4$. If a twice continuously differentiable function $f : \Omega \to \mathbb{H}$ is α-hyperregular then the coordinate functions f_n for $n = 0, 1, 2$ are α-hyperbolic harmonic and f_3 satisfies the equation*

$$x_3^2 \Delta f_3(x) - \alpha x_3 \frac{\partial f_3}{\partial x_3}(x) + \alpha(x) f_3 = 0$$

for any $x \in \Omega$.

The following transformation property is proved in [1,3].

Lemma 3.8. *Let Ω be an open set contained in $\mathbb{R}^4_+$ or in $\mathbb{R}^4_-$. A function a twice continuously differentiable function $f : \Omega \to \mathbb{R}$ is α-hyperbolic harmonic if and only if the function $g(x) = x_3^{\frac{2-\alpha}{2}} f(x)$ satisfies the equation*

$$\Delta_2 g + \frac{1}{4}\left(9 - (\alpha+1)^2\right) g = 0. \tag{3.2}$$

4. Cauchy Type Integral Formulas

We recall the Stokes theorem for T and S-parts proved in [6].

Theorem 4.1. *Let Ω be an open subset of $\mathbb{R}^4 \setminus \{x_3 = 0\}$ and K a 3-chain satisfying $\overline{K} \subset \Omega$. Denote $(\nu_0, \nu_1, \nu_2, \nu_3)$ the outer unit normal and the corresponding quaternion by $\nu = \nu_0 + \nu_1 \boldsymbol{i} + \nu_2 \boldsymbol{j} + \nu_3 \boldsymbol{k}$. If $f, g \in \mathcal{C}^1(\Omega, \mathbb{H})$, then*

$$\int_{\partial K} T(g\nu f + f\nu g)\, d\sigma = \int_K T\left(H^r_{-\alpha} g f + g H^l_{\alpha} f + H^r_{\alpha} f g + f H^l_{-\alpha} g\right) dm,$$

where $d\sigma$ is the surface element and dm the usual Lebesgue volume element in $\mathbb{R}^4$.

Theorem 4.2. *Let Ω be an open subset of $\mathbb{R}^4 \setminus \{x_3 = 0\}$ and K a 3-chain satisfying $\overline{K} \subset \Omega$. Denote $(\nu_0, \nu_1, \nu_2, \nu_3)$ the outer unit normal and the corresponding quaternion by $\nu = \nu_0 + \nu_1 \boldsymbol{i} + \nu_2 \boldsymbol{j} + \nu_3 \boldsymbol{k}$. If $f, g \in \mathcal{C}^1(\Omega, \mathbb{H})$, then*

$$\int_{\partial K} S(g\nu f + f\nu g)\, \frac{d\sigma}{x_3^\alpha} = \int_K S\left(H^r_{\alpha} g f + g H^l_{\alpha} f + H^r_{\alpha} f g + f H^l_{\alpha} g\right) \frac{dm}{x_3^\alpha},$$

where $d\sigma$ is the surface element and dm the usual Lebesgue volume element in $\mathbb{R}^4$.

The fundamental α-hyperbolic harmonic function, that is the fundamental solution of Δ_α, is the following function (see [4,6,7]).

Theorem 4.3. *Let x and y be points in the upper half space. The fundamental α-hyperbolic harmonic function is*

$$E_{\boldsymbol{\alpha}}(x,y)=\begin{cases}\dfrac{x_3^{\frac{\alpha-2}{2}}y_3^{\frac{\alpha-2}{2}}Q^1_{\frac{\alpha}{2}}(\lambda(x,y))}{2^{\nu+1}\omega_3\left(\lambda(x,y)^2-1\right)^{\frac{1}{2}}}, & \text{if } \alpha\geq 0,\\ \dfrac{x_3^{\frac{\alpha-2}{2}}y_3^{\frac{\alpha-2}{2}}Q^1_{\frac{-\alpha-2}{2}}(\lambda(x,y))}{2^{\nu+1}\omega_3\left(\lambda(x,y)^2-1\right)^{\frac{1}{2}}}, & \text{if } \alpha< 0,\end{cases}$$

where the associated Legendre function is defined by

$$Q^1_\nu(\lambda)=\frac{\sqrt{\pi}\Gamma(\nu+2)\lambda^{-\nu}\ {}_2F_1\left(\frac{\nu}{2},\frac{\nu+1}{2};\frac{2\nu+3}{2};\frac{1}{\lambda^2}\right)}{2^{\nu+1}(\lambda^2-1)}$$

and the hypergeometric function by

$${}_2F_1(a,b;c;x)=\frac{1}{\Gamma(c)}\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m}\frac{x^m}{m!}$$

for $|x|<1$.

We remark that the fundamental α-hyperbolic harmonic function is unique up to a harmonic function. The reason why we picked the preceding function is that it leads to nice symmetry properties of a kernel, verified after the following theorem.

Theorem 4.4. *Denote $r_h=d_h(x,y)$, $t=\frac{\alpha-2}{2}$ and define*

$$g_\alpha(r_h)=\frac{\sqrt{\pi}\Gamma(\nu+2)\cosh^{-\nu}r_h\ {}_2F_1\left(\frac{\nu}{2},\frac{\nu+1}{2};\frac{2\nu+3}{2};\frac{1}{\cosh^2 r_h}\right)}{2^{\nu+1}},$$

where

$$\nu=\begin{cases}\frac{\alpha}{2}, & \text{if } \alpha\geq 0,\\ \frac{-\alpha-2}{2}, & \text{if } \alpha<0.\end{cases}$$

The α-hyperregular kernel is the function

$$\begin{aligned}h_\alpha(x,y)&=\overline{D}^x(E_\alpha(x,y))\\ &=x_3^{\frac{\alpha-2}{2}}y_3^{\frac{\alpha+4}{2}}w_\alpha(x,y)s(x,y)\\ &=x_3^{\frac{\alpha-2}{2}}y_3^{\frac{\alpha+4}{2}}s(x,y)v_\alpha(x,y)\end{aligned}$$

where

$$\begin{aligned}w_\alpha(x,y)&=-t\alpha g_\alpha(r_h)\boldsymbol{k}\frac{x-Sy}{y_3}\\ &\quad+\sinh r_h g'_\alpha(r_h)-(t+2)g_\alpha(r_h)\cosh r_h,\\ v_\alpha(x,y)&=-t\alpha g_\alpha(r_h)\frac{x-Sy}{y_3}\boldsymbol{k}\\ &\quad+\sinh r_h g'_\alpha(r_h)-(t+2)g_\alpha(r_h)\cosh r_h,\end{aligned}$$

and

$$s(x,y)=\frac{(x-c_y(r_h))^{-1}}{x_3\|x-c_y(r_h)\|^2}$$

is 2*-hyperregular with respect to* x*.*

The function $s(x,y)$ is the kernel computed in [2] and in [3].

Clearly, the function $h_\alpha(x,y)$ is not symmetrical with respect to x and y. However, it has the following symmetry properties.

Proposition 4.5. *The function* h_α *has the properties*

$$S(h_\alpha(y,x)) = -S(h_\alpha(x,y)),$$
$$y_3^\alpha Th_{-\alpha}(x,y) = -x_3^{-\alpha} Th_\alpha(y,x),$$

and

$$y_3^\alpha Th_{-\alpha}(y,x) = -x_3^{-\alpha} Th_\alpha(x,y)$$

for all x *and* y *outside the hyperplane* $\left\{(u_0,u_1,u_3,u_3)\in\mathbb{R}^4 \mid u_3=0\right\}$.

Proof. Denote

$$F_\alpha(x,y) = x_3^{\frac{\alpha-2}{2}} y_3^{\frac{\alpha-2}{2}} G_\alpha(\lambda(x,y)).$$

If $m=0,1,2$, then

$$\begin{aligned}\frac{\partial F_\alpha(x,y)}{\partial x_m} &= x_3^{\frac{\alpha-2}{2}} y_3^{\frac{\alpha-2}{2}} G'_\alpha(\lambda((x,y)))\frac{\partial\lambda(x,y)}{\partial x_m}\\ &= x_3^{\frac{\alpha-2}{2}} y_3^{\frac{\alpha-2}{2}} G'_\alpha(\lambda(x,y))\frac{x_m-y_m}{x_3y_3}\\ &= -x_3^{\frac{\alpha-2}{2}} y_3^{\frac{\alpha-2}{2}} G'_\alpha(\lambda(y,x))\frac{y_m-x_m}{x_3y_3}\\ &= -\frac{\partial F_\alpha(y,x)}{\partial y_m}\end{aligned}$$

The last properties follow from the tedious calculations

$$y_3^\alpha\partial_{x_3}F_{-\alpha}(x,y) + x_3^{-\alpha}\partial_{y_3}F_\alpha(x,y) = 0,$$
$$y_3^\alpha\partial_{y_3}F_{-\alpha}(x,y) + x_3^{-\alpha}\partial_{x_3}F_\alpha(x,y) = 0$$

which are done in [7]. □

We recall the integral formulas for S- and T-parts verified in [6].

Theorem 4.6. *Let* Ω *and be an open subsets of* $\mathbb{R}^4_+$ *(or* $\mathbb{R}^4_-$*). Assume that* K *is an open subset of* Ω *and* $\overline{K}\subset\Omega$ *is a compact set with the smooth boundary. Let* $(\nu_0,\nu_1,\nu_2,\nu_3)$ *be the outer unit normal and denote the corresponding quaternion by* $\nu=\nu_0+\nu_1\boldsymbol{i}+\nu_2\boldsymbol{j}+\nu_3\boldsymbol{k}$*. If* f *is* α*-hyperregular in* Ω *and* $a\in K$*, then*

$$\begin{aligned}Sf(a) &= -\frac{1}{2}\int_{\partial K} S(h_\alpha(x,a)\nu f + f\nu h_\alpha(x,a))\frac{d\sigma}{x_3^\alpha}\\ &= \int_{\partial K} S[h_\alpha(x,a),\overline{\nu},f]\frac{d\sigma}{x_3^\alpha} - \int_{\partial K} Sh_\alpha(x,a)\langle\overline{\nu},f\rangle\, d\sigma\end{aligned}$$

and

$$\begin{aligned}Tf(a) &= -\frac{a_3^\alpha}{2}\int_{\partial K} T\left(h_{-\alpha}(x,a)\nu f + f\nu h_{-\alpha}(x,a)\right) d\sigma \\ &= a_3^\alpha\left(\int_{\partial K} T\left[h_{-\alpha}(x,a),\overline{\nu},f\right] d\sigma - \int_{\partial K} Th_{-\alpha}(x,a)\langle\overline{\nu},f\rangle\, d\sigma\right).\end{aligned}$$

If we combine these formulas we obtain a new formula.

Theorem 4.7. *Let Ω and be an open subsets of $\mathbb{R}^4_+$ (or $\mathbb{R}^4_-$). Assume that K is an open subset of Ω and $\overline{K}\subset\Omega$ is a compact set with the smooth boundary. Let $(\nu_0,\nu_1,\nu_2,\nu_3)$ be the outer unit normal and denote the corresponding quaternion by $\nu=\nu_0+\nu_1\boldsymbol{i}+\nu_2\boldsymbol{j}+\nu_3\boldsymbol{k}$. If f is $k-$hyperregular in Ω and $a\in K$, then*

$$f(a) = \int_{\partial K} R(x,a,\nu,f)\, d\sigma + \int_{\partial K} h_k(a,x)\langle\overline{\nu},f\rangle\frac{d\sigma}{x_3^\alpha}$$

where

$$\begin{aligned}R(x,a,\nu,Sf) = &-\left\langle x_3^{-\alpha}Sh_\alpha(a,x),Sf\right\rangle S\overline{\nu} + \left\langle a_3^\alpha Sh_{-\alpha}(a,x),Sf\right\rangle T\nu\boldsymbol{k} \\ &+\left\langle x_3^{-\alpha}Sh_\alpha(a,x),S\overline{\nu}\right\rangle Sf - a_3^\alpha Th^a_{-\alpha}(a,x)T\nu Sf\end{aligned}$$

and

$$T\left(R\left(x,a,\nu,Tf\boldsymbol{k}\right)\right) = \left\langle a_3^\alpha Sh_{-\alpha}(a,x),Sf\right\rangle T\nu + \left\langle a_3^\alpha Sh_{-\alpha}(a,x),S\overline{\nu}\right\rangle Tf.$$

Proof. We combine the preceding integral formulas using the formula

$$f(a) = Sf(a) + Tf(a)\boldsymbol{k}.$$

We introduce the following notation

$$\begin{aligned}B &= -\int_{\partial K} x_3^{-\alpha}Sh_\alpha(x,a)\langle\overline{\nu},f\rangle\, d\sigma - \int_{\partial K} a_3^\alpha Th_{-\alpha}(x,a)\boldsymbol{k}\langle\overline{\nu},f\rangle\, d\sigma \\ &= -\int_{\partial K}\left(x_3^{-\alpha}Sh_\alpha(x,a) + a_3^\alpha Th_{-\alpha}(x,a)\boldsymbol{k}\right)\langle\overline{\nu},f\rangle\, d\sigma.\end{aligned}$$

Applying the symmetry properties of the kernels we deduce

$$\begin{aligned}B &= \int_{\partial K}\left(x_3^{-\alpha}Sh_\alpha(a,x) + x_3^{-\alpha}Th_\alpha(a,x)\boldsymbol{k}\right)\langle\overline{\nu},f\rangle\, d\sigma \\ &= \int_{\partial K} x_3^{-\alpha}h_\alpha(a,x)\langle\overline{\nu},f\rangle\, d\sigma.\end{aligned}$$

Applying the properties (2.4) and (2.5), we obtain

$$\begin{aligned}R(x,a,\nu,f) &= S\left(\left[x_3^{-\alpha}h_\alpha(x,a),\overline{\nu},f\right]\right) + T\left(\left[a_3^\alpha h_{-\alpha}(x,a),\overline{\nu},f\right]\right)\boldsymbol{k} \\ &= \frac{1}{2}\left(\left[x_3^{-\alpha}h_\alpha(x,a),\overline{\nu},f\right] + \left[x_3^{-\alpha}h_\alpha(x,a),\overline{\nu},f\right]^*\right) \\ &\quad + \frac{1}{2}\left(\left[a_3^\alpha h_{-\alpha}(x,a),\overline{\nu},f\right] - \left[a_3^\alpha h_{-\alpha}(x,a),\overline{\nu},f\right]^*\right) \\ &= \frac{1}{2}\left[x_3^{-\alpha}h_\alpha(x,a) + a_3^\alpha h_{-\alpha}(x,a),\overline{\nu},f\right] \\ &\quad + \frac{1}{2}\left[x_3^{-\alpha}h_\alpha(x,a) - a_3^\alpha h_{-\alpha}(x,a),\overline{\nu},f\right]^*.\end{aligned}$$

Hence

$$R\left(x,a,\nu,f\right)=R\left(x,a,\nu,Sf\right)+R\left(x,a,\nu,Tf\boldsymbol{k}\right).$$

Using the definition of the triple product we infer

$$\begin{aligned}
R\left(x,a,\nu,Sf\right)&=\frac{1}{2}\left\langle x_3^{-\alpha}h_\alpha\left(x,a\right)+a_3^\alpha h_{-\alpha}\left(x,a\right),Sf\right\rangle\overline{\nu}\\
&\quad-\frac{1}{2}\left\langle x_3^{-\alpha}h_\alpha\left(x,a\right)+a_3^\alpha h_{-\alpha}\left(x,a\right),\overline{\nu}\right\rangle Sf\\
&\quad+\frac{1}{2}\left\langle x_3^{-\alpha}h_\alpha\left(x,a\right)-a_3^\alpha h_{-\alpha}\left(x,a\right),Sf\right\rangle\nu'\\
&\quad-\frac{1}{2}\left\langle x_3^{-\alpha}h_\alpha\left(x,a\right)-a_3^\alpha h_{-\alpha}\left(x,a\right),\overline{\nu}\right\rangle Sf\\
&=\left\langle x_3^{-\alpha}h_\alpha\left(x,a\right),Sf\right\rangle S\overline{\nu}-\left\langle a_3^\alpha h_{-\alpha}\left(x,a\right),Sf\right\rangle T\nu\boldsymbol{k}\\
&\quad-\left\langle x_3^{-\alpha}h_\alpha\left(x,a\right),\overline{\nu}\right\rangle Sf\\
&=\left\langle x_3^{-\alpha}Sh_\alpha\left(x,a\right),Sf\right\rangle S\overline{\nu}-\left\langle a_3^\alpha Sh_{-\alpha}\left(x,a\right),Sf\right\rangle T\nu\boldsymbol{k}\\
&\quad-\left\langle x_3^{-\alpha}Sh_\alpha\left(x,a\right),S\overline{\nu}\right\rangle Sf+x_3^\alpha Th_{-\alpha}(x,a)T\nu Sf.
\end{aligned}$$

Using the symmetry properties, we obtain

$$\begin{aligned}
R\left(x,a,\nu,Sf\right)=&-\left\langle x_3^{-\alpha}Sh_\alpha\left(a,x\right),Sf\right\rangle S\overline{\nu}+\left\langle a_3^\alpha Sh_{-\alpha}\left(a,x\right),Sf\right\rangle T\nu\boldsymbol{k}\\
&+\left\langle x_3^{-\alpha}Sh_\alpha\left(a,x\right),S\overline{\nu}\right\rangle Sf-a_3^\alpha Th_{-\alpha}^a(a,x)T\nu Sf.
\end{aligned}$$

In order to shorten the notations, we abbreviate $g=Tf\boldsymbol{k}$. Then we simply compute

$$\begin{aligned}
R\left(x,a,\nu,g\right)&=\frac{1}{2}\left\langle x_3^{-\alpha}h_\alpha\left(x,a\right)+a_3^\alpha h_{-\alpha}\left(x,a\right),g\right\rangle\overline{\nu}\\
&\quad-\frac{1}{2}\left\langle x_3^{-\alpha}h_\alpha\left(x,a\right)+a_3^\alpha h_{-\alpha}\left(x,a\right),\overline{\nu}\right\rangle g\\
&\quad+\frac{1}{2}\left\langle x_3^{-\alpha}h_\alpha\left(x,a\right)-a_3^\alpha h_{-\alpha}\left(x,a\right),g\right\rangle\nu'\\
&\quad+\frac{1}{2}\left\langle x_3^{-\alpha}h_\alpha\left(x,a\right)-a_3^\alpha h_{-\alpha}\left(x,a\right),\overline{\nu}\right\rangle g\\
&=\left\langle x_3^{-\alpha}h_\alpha\left(x,a\right),g\right\rangle S\overline{\nu}-\left\langle a_3^\alpha h_{-\alpha}\left(x,a\right),g\right\rangle T\nu\boldsymbol{k}\\
&\quad-\left\langle a_3^\alpha h_{-\alpha}\left(x,a\right),\overline{\nu}\right\rangle g\\
&=\left\langle x_3^{-\alpha}Th_\alpha\left(x,a\right)\boldsymbol{k},g\right\rangle S\overline{\nu}-\left\langle a_3^\alpha Th_{-\alpha}\left(x,a\right)\boldsymbol{k},g\right\rangle T\nu\boldsymbol{k}\\
&\quad-\left\langle a_3^\alpha h_{-\alpha}\left(x,a\right),\overline{\nu}\right\rangle g.
\end{aligned}$$

Symmetry properties imply that

$$R\left(x,a,\nu,g\right)=\left\langle a_3^\alpha Sh_{-\alpha}\left(a,x\right),S\overline{\nu}\right\rangle g-a_3^\alpha Th_{-\alpha}\left(a,x\right)TfS\overline{\nu}.$$

□

Corollary 4.8. *Let Ω be an open subsets of $\mathbb{R}^4_+$ (or $\mathbb{R}^4_-$). Assume that K is an open subset of Ω and $\overline{K}\subset\Omega$ is a compact set with the smooth boundary. Let $(\nu_0,\nu_1,\nu_2,\nu_3)$ be the outer unit normal and denote the corresponding* *quaternion by* $\nu = \nu_0 + \nu_1 \boldsymbol{i} + \nu_2 \boldsymbol{j} + \nu_3 \boldsymbol{k}$*. If* f *is k-hyperregular in* Ω *and* $a \in K$*, then the functions*

$$r_1(a) = \int_{\partial K} R(x, a, \nu, f)\, d\sigma$$

and

$$r_2(a) = \int_{\partial K} h_k(a, x) \langle \overline{\nu}, f \rangle \frac{d\sigma}{x_3^\alpha}$$

are α*-hyperregular and* $f = r_1 + r_2$*.*

Theorem 4.9. *Let* Ω *be an open subsets of* $\mathbb{R}^4_+$ *(or* $\mathbb{R}^4_-$*). Assume that* K *is an open subset of* Ω *and* $\overline{K} \subset \Omega$ *is a compact set with the smooth boundary. Let* $(\nu_0, \nu_1, \nu_2, \nu_3)$ *be the outer unit normal and denote the corresponding quaternion by* $\nu = \nu_0 + \nu_1 \boldsymbol{i} + \nu_2 \boldsymbol{j} + \nu_3 \boldsymbol{k}$*. If* $f : \partial K \to \mathbb{H}$ *is a continuous function then the function*

$$r_2(a) = \int_{\partial K} h_\alpha(a, x) \langle \overline{\nu}, f \rangle \frac{d\sigma}{x_3^\alpha}$$

is α*-hyperregular for all* $a \in K$*.*

Theorem 4.10. *Let* Ω *be an open subsets of* $\mathbb{R}^4_+$ *(or* $\mathbb{R}^4_-$*). Assume that* K *is an open subset of* Ω *and* $\overline{K} \subset \Omega$ *is a compact set with the smooth boundary. Let* $(\nu_0, \nu_1, \nu_2, \nu_3)$ *be the outer unit normal and denote the corresponding quaternion by* $\nu = \nu_0 + \nu_1 \boldsymbol{i} + \nu_2 \boldsymbol{j} + \nu_3 \boldsymbol{k}$*. If* $f : \partial K \to \mathbb{H}$ *is a continuous function, then the function*

$$Sr_1(a) = \int_{\partial K} S(R(x, a, \nu, f))\, d\sigma$$

is k-hyperbolic harmonic for all $a \in K$ *and*

$$Tr_1(a) = \int_{\partial K} T(R(x, a, \nu, f))\, d\sigma$$

satisfies the equation

$$x_3^2 \Delta h - \alpha x_3 \frac{\partial h}{\partial x_3} + \alpha h = 0$$

and $a_3^{-\alpha} Tr_1$ *is* $-\alpha$*-hyperbolic harmonic.*

We consider the Teodorescu and Cauchy type operators in subsequent papers. Also the case $a \in \mathbb{R}^4_+ \backslash K$ involves some technical assumptions and left for later work.

5. Comparison of α-Hyperregular and α-Hypermonogenic Functions

The universal real Clifford algebra $\mathcal{C}\ell_{0,3}$ is a real associated algebra with a unit $\mathbf{1}$ and is generated by e_1, e_2 and e_3 satisfying the relation

$$e_s e_t + e_t e_s = -2\delta_{st} \mathbf{1},$$

where δ_{st} is the usual Kronecker delta and $s, t = 1, 2, 3$. We denote $r\mathbf{1}$ briefly by $r \in \mathbb{R}$.

The elements

$$x = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3$$

for $x_0, x_1, x_2, x_3 \in \mathbb{R}$ are called *paravectors*. The real number x_0 is the *real part* of the paravector x.

The *main involution* in $\mathcal{C}\ell_{0,3}$ is the mapping $a \to a'$ defined by $e_s' = -e_s$ for $s = 1, \ldots, 3$ and extended to the total algebra by linearity and the product rule $(ab)' = a'b'$. Similarly the *reversion* is the mapping $a \to a^*$ defined by $e_s^* = -e_s$ for $s = 1, \ldots, 3$ and extended to the total algebra by linearity and the product rule $(ab)^* = b^* a^*$. The *conjugation* is the mapping $a \to \overline{a}$ defined by $\overline{a} = (a')^* = (a^*)'$.

Any element w in $\mathcal{C}\ell_{0,3}$ may be written as

$$w = w_0 + w_1 e_1 + w_2 e_2 + w_3 e_3 + w_{12} e_{12} + w_{13} e_{13} + w_{23} e_{23} + w_{123} e_{123},$$

where $e_{mn} = e_m e_n$ for $1 \leq m < n \leq 3$ and $e_{123} = e_1 e_2 e_3$. The element e_{123},denoted by I, is commuting with all elements and $(e_1 e_2 e_3)^2 = 1$.

We recall that $C\ell_{0,1}$ may be identified with the field of complex numbers. The universal Clifford algebra $\mathcal{C}\ell_{0,2}$ may be identified with the quaternions, by setting $\boldsymbol{i} = e_1$, $\boldsymbol{j} = e_2$ and $\boldsymbol{k} = e_1 e_2$. This identification we used in the first section when we defined involutions.

We generalize the imaginary part of a complex number to $\mathcal{C}\ell_{0,3}$ by decomposing any element $a \in \mathcal{C}\ell_{0,3}$ as

$$a = b + ce_3$$

for $b, c \in \mathcal{C}\ell_{0,2}$. The mappings $P : \mathcal{C}\ell_{0,3} \to \mathcal{C}\ell_{0,2}$ and $Q : \mathcal{C}\ell_{0,3} \to \mathcal{C}\ell_{0,2}$ are defined in [9] by

$$Pa = b, \qquad Qa = c.$$

In order to compute the P- and Q- parts we use the involution $a \to \widehat{a}$ defined by $\widehat{e_i} = (-1)^{\delta_{s3}} e_i$ for $s = 1, 2, 3$ and extended to the total algebra by linearity and the product rule $\widehat{ab} = \widehat{a}\widehat{b}$. Then we obtain the formulas

$$Pa = \frac{1}{2}(a + \widehat{a}) \tag{5.1}$$

and

$$Qa = -\frac{1}{2}(a - \widehat{a})\, e_3. \tag{5.2}$$

The following calculation rules [9] hold

$$P(ab) = (Pa)\, Pb + (Qa)\, Q(b'), \tag{5.3}$$

$$\begin{aligned} Q(ab) &= (Pa)\, Qb + (Qa)\, P'(b) \\ &= aQb + (Qa)\, b'. \end{aligned} \tag{5.4}$$

Note that if $a \in \mathcal{C}\ell_{0,3}$, then

$$a' e_3 = e_3 \widehat{a}$$

Moreover if $a \in \mathcal{C}\ell_{0,2}$ then

$$ae_3 = e_3 a'. \tag{5.5}$$

We consider functions $f : \Omega \to \mathcal{C}\ell_{0,3}$, defined on an open subset Ω of $\mathbb{R}^4$, and assume that its components are continuously differentiable. The left Dirac operator (also called the Cauchy–Riemann operator) in $\mathcal{C}\ell_{0,3}$ is defined by

$$D_l f = \sum_{s=0}^{3} e_s \frac{\partial f}{\partial x_s}$$

and the right Dirac operator by

$$D_r f = \sum_{s=0}^{3} \frac{\partial f}{\partial x_s} e_s.$$

Their conjugate operators $\overline{D_l}$ and $\overline{D_r}$ are defined by

$$\overline{D_l} f = \sum_{s=0}^{3} \overline{e_s} \frac{\partial f}{\partial x_s}, \quad \overline{D_r} f = \sum_{s=0}^{3} \frac{\partial f}{\partial x_s} \overline{e_s}.$$

The modified Dirac operators M_α^l, $\overline{M}_\alpha^l$, M_α^r and $\overline{M}_\alpha^r$, introduced in [8,9], are defined in $\{(x_0, x_1, x_2, x_3) \in \Omega \mid x_3 \neq 0\}$ by

$$M_\alpha^l f(x) = D_l f(x) + \alpha \tfrac{Q' f}{x_3}, \ \overline{M}_\alpha^l f(x) = \overline{D_l} f(x) - \alpha \tfrac{Q' f}{x_3},$$
$$M_\alpha^r f(x) = D_r f(x) + \alpha \tfrac{Q f}{x_3}, \ \overline{M_\alpha}^r f(x) = \overline{D_r} f(x) + \alpha \tfrac{Q f}{x_3},$$

where $(Qf)' = Q' f$. The operator M_2^l is also abbreviated by M.

Definition 5.1. Let $\Omega \subset \mathbb{R}^4$ be open. A function $f : \Omega \to C\ell_{0,3}$ is called left α -hypermonogenic if $f \in \mathcal{C}^1(\Omega)$ and

$$M_\alpha^l f(x) = 0$$

for any $x \in \{x \in \Omega \mid x_3 \neq 0\}$. The right α-hypermonogenic functions are defined similarly. The 2-left hypermonogenic functions are called hypermonogenic functions. A twice continuously differentiable function $f : \Omega \to \mathcal{C}\ell_{0,3}$ is called α-hyperbolic harmonic if $\overline{M}_\alpha^l M_\alpha^l f = 0$.

Computing the components of $M_\alpha^l f(x)$ and $M_\alpha^r f(x)$, we obtain

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^4$ be open and a function $f : \Omega \to \mathcal{C}\ell_{0,3}$ continuously differentiable. If f is paravector valued then f is α-hypermonogenic in Ω if and only if*

$$\begin{aligned}
&\tfrac{\partial f_0}{\partial x_0} - \tfrac{\partial f_1}{\partial x_1} - \tfrac{\partial f_2}{\partial x_2} - \tfrac{\partial f_3}{\partial x_3} + \alpha \tfrac{f_3}{x_3} = 0, \quad \textit{if } x_3 \neq 0,\\
&\tfrac{\partial f_0}{\partial x_m} = -\tfrac{\partial f_m}{\partial x_0} \quad \textit{for all } m = 1,2,3,\\
&\tfrac{\partial f_m}{\partial x_n} = \tfrac{\partial f_n}{\partial x_m} \quad \textit{for all } m,n = 1,2,3.
\end{aligned}$$

Applying Proposition 3.1 we obtain the result.

Theorem 5.3. *Let* $\Omega \subset \mathbb{R}^4$ *be open and a function* $f = (f_0, f_1, f_2, f_3) : \Omega \to \mathbb{R}^4$ *continuously differentiable. Then the function* $f_0 + f_1\boldsymbol{i} + f_2\boldsymbol{j} + f_3\boldsymbol{k}$ *is* α*-hyperregular in* Ω *if and only if the* $f_0+f_1e_1+f_2e_2+f_3e_3$ *is* α*-hypermonogenic in* Ω*.*

We recall the Cauchy type formula for α-hypermonogenic functions.

Theorem 5.4. [7] *Let* Ω *be an open subset of* $\mathbb{R}^4_+$ *and* $K \subset \Omega$ *be a smoothly bounded compact set. Denote* $(\nu_0, \nu_1, \nu_2, \nu_3)$ *the outer unit normal and the corresponding paravector by* $\nu = \nu_0+\nu_1e_1+\nu_2e_2+\nu_3e_3$*. If* f *is* α*-hypermonogenic in* Ω *and* $a \in K$*, then*

$$f(a) = \int_{\partial K} \left(x_3^{-\alpha} h_\alpha^a(a,x)P(\nu f) + a_3^\alpha h_{-\alpha}^a(x,a)e_3 Q'(\nu f)\right) d\sigma$$

where

$$h_\alpha(a,x) = \overline{D}^a E_\alpha(a,x)$$

and $h_\alpha(a,x)$ *and* $a_3^\alpha h_{-\alpha}(a,x)e_3$ *are the* α*-hypermonogenic kernels with respect to* a*.*

Using this formula we may verify the formula also for paravector valued functions. Before this, we present three preliminary results.

Lemma 5.5. *Let* $a \in \Omega \to \mathcal{C}\ell_{0,3}$*. Then*

$$P(a^*) = (P(a))^*,$$
$$Q(a^*) = \overline{Q(a)}$$

and

$$(Q'(a^*))^* = Q(a).$$

Proof. Assume that $a \in \mathcal{C}\ell_{0,3}$. Since

$$a = Pa + Qae_3$$

and $e_3Qa = Q'ae_3$ then

$$a^* = (Pa)^* + e_3(Qa)^*$$
$$= (Pa)^* + \left((Qa)^*\right)' e_3.$$

Noticing that $\left((Qa)^*\right)' = \overline{Qa}$ we conclude

$$P(a^*) = (Pa)^*$$

and therefore

$$Q(a^*) = \overline{Qa}.$$

The last formula follows from if we take $()^*$ and $()'$ from the both side of the equation. □

Lemma 5.6. *Let* a*,* b *be paravectors in* $\mathcal{C}\ell_{0,3}$*. Then* $Q(ab)$ *is a paravector.*

Proof. We just compute

$$Q\,(ab) = Qab' + aQb.$$

Since a, b are paravectors, the elements Qa and Qb are scalars, completing the proof. □

Lemma 5.7. *Let $\Omega \subset \mathbb{R}^4$ be open. A function $f : \Omega \to \mathcal{C}\ell_{0,3}$ is left α-hypermonogenic if and only if $f^*$is right α-hypermonogenic.*

Proof. Assume that f is left α-hypermonogenic then

$$M^l_\alpha f\,(x) = x_3 D^l f\,(x) + \alpha Q' f\,(x) = 0.$$

Since $(a^*)' = \overline{a}$, we infer

$$\begin{aligned} 0 &= \left(M^l_\alpha f\,(x)\right)^* = x_3 D^r f^* + \alpha\left(Q' f\,(x)\right)^* \\ &= x_3 D^r f^* + \alpha\left(\overline{Q} f\,(x)\right). \end{aligned}$$

Using the previous lemma we obtain

$$M^r_\alpha f^*\,(x) = x_3 D^r f^* + \alpha Q f^*\,(x) = 0.$$

Hence f^* is right α-hypermonogenic. Similarly, we verify that if f is right α-hypermonogenic then f is left α-hypermonogenic. □

Theorem 5.8. *Let Ω be an open subset of $\mathbb{R}^4_+$ and $K \subset \Omega$ be a smoothly bounded compact set. Denote $(\nu_0, \nu_1, \nu_2, \nu_3)$ the outer unit normal and the corresponding paravector by $\nu = \nu_0 + \nu_1 e_1 + \nu_2 e_2 + \nu_3 e_3$. If f is right α-hypermonogenic in Ω and $a \in K$, then*

$$f(a) = \int_{\partial K} \left(P(f\nu) x_3^{-\alpha} h_\alpha(a,x) + Q(f\nu) e_3 a_3^\alpha h_{-\alpha}(a,x)\right) d\sigma,$$

where $h_\alpha(a,x)$ and $e_3 a_3^\alpha h_{-\alpha}(a,x)$ are right α-hypermonogenic with respect to the variable a.

Proof. If f is right α-hypermonogenic then f^* is left α-hypermonogenic and therefore

$$f^*(a) = \int_{\partial K} \left(x_3^{-\alpha} h_\alpha(a,x) P(\nu f^*) + a_3^\alpha h_{-\alpha}(a,x) e_3 Q'(\nu f^*)\right) d\sigma.$$

Taking $(\,)^*$ from the both sides we obtain

$$f(a) = \int_{\partial K} \left(x_3^{-\alpha} P(\nu f^*)\right)^* h^*_\alpha(a,x) + \left(Q'(\nu f^*)\right)^* e_3 a_3^\alpha h^*_{-\alpha}(a,x) d\sigma.$$

where $h^*_\alpha(a,x) = (h_\alpha(a,x))^*$. Applying the previous lemma, we infer

$$\left(Q'(\nu f^*)\right)^* = \overline{Q(\nu f^*)} = Q(f\nu)$$

and

$$\left(P(\nu f^*)\right)^* = P(f\nu),$$

since f and ν are paravectors. Hence we have

$$f(a) = \int_{\partial K} \left(x_3^{-\alpha} P(f\nu) h^*_\alpha(a,x) + \left(Q(f\nu)\right) e_3 a_3^\alpha h^*_{-\alpha}(a,x)\right) d\sigma.$$

Since h_α^a is a paravector we infer

$$f(a)=\int_{\partial K}\left(x_3^{-\alpha}P(f\nu)\left(h_\alpha\right)(a,x)+\left(Q(f\nu)\right)e_3a_3^\alpha\left(h_{-\alpha}\right)(a,x)\right)d\sigma$$

completing the proof. □

Theorem 5.9. *Let* Ω *be an open subset of* $\mathbb{R}_+^4$ *and* $K\subset\Omega$ *be a smoothly bounded compact set. Denote* $(\nu_0,\nu_1,\nu_2,\nu_3)$ *the outer unit normal and the corresponding paravector by* $\nu=\nu_0+\nu_1e_1+\nu_2e_2+\nu_3e_3$*. Then, if* f *is a paravector valued* α*-hypermonogenic in* Ω *and* $a\in K$*,*

$$\begin{aligned}f(a)=&\int_{\partial K}\left(h_\alpha(a,x)\left\langle\overline{\nu},f\right\rangle+\left[Ph_\alpha(a,x),\overline{P\nu},Pf\right]\right)\frac{d\sigma}{x_3^\alpha}\\&-\int_{\partial K}a_3^a\left(\left[h_{-\alpha}(a,x),\overline{P\nu},Qfe_3\right]+\left[h_{-\alpha}(a,x),Q\nu e_3,Pf\right]\right)d\sigma.\end{aligned}$$

Proof. If f is a paravector valued α-hypermonogenic in Ω and $a\in K$, then

$$\begin{aligned}f(a)=&\frac{1}{2}\int_{\partial K}\left(x_3^{-\alpha}P(f\nu)h_\alpha(a,x)+Q(f\nu)e_3a_3^\alpha h_{-\alpha}(a,x)\right)d\sigma\\&+\frac{1}{2}\int_{\partial K}\left(x_3^{-\alpha}h_\alpha(a,x)P(\nu f)+h_{-\alpha}(a,x)e_3a_3^\alpha Q'(\nu f)\right)d\sigma,\end{aligned}$$

Since $P(\nu f)=P\nu Pf+Q\nu Q'f$ and f is a paravector we obtain

$$\begin{aligned}&\frac{1}{2}\int_{\partial K}x_3^{-\alpha}h_\alpha(a,x)P(\nu f)+x_3^{-\alpha}P(f\nu)h_\alpha(a,x)d\sigma\\&\quad=\frac{1}{2}\left(\int_{\partial K}x_3^{-\alpha}h_\alpha(a,x)P\nu Pf+x_3^{-\alpha}PfP\nu h_\alpha(a,x)\right)d\sigma\\&\qquad-\int_{\partial K}x_3^{-\alpha}h_\alpha(a,x)Q\nu Qfd\sigma\\&\quad=\int_{\partial K}x_3^{-\alpha}h_\alpha(a,x)\left\langle\overline{\nu},f\right\rangle d\sigma-\int_{\partial K}x_3^{-\alpha}\left[h_\alpha(a,x),\overline{P\nu},Pf\right]d\sigma\\&\quad=\int_{\partial K}x_3^{-\alpha}h_\alpha(a,x)\left\langle\overline{\nu},f\right\rangle d\sigma+\int_{\partial K}x_3^{-\alpha}\left[h_\alpha(a,x),\overline{P\nu},Pf\right]d\sigma.\end{aligned}$$

Similarly we compute

$$\begin{aligned}&\frac{1}{2}\int_{\partial K}h_{-\alpha}(a,x)e_3a_3^\alpha Q'(\nu f)+Q(f\nu)e_3a_3^\alpha\left(h_{-\alpha}\right)(a,x)d\sigma\\&\quad=\frac{1}{2}\int_{\partial K}h_{-\alpha}(a,x)a_3^\alpha P\nu Qfe_3+Qfa_3^\alpha e_3P\nu a_3^\alpha\left(h_{-\alpha}\right)(a,x)d\sigma\\&\qquad+\frac{1}{2}\int_{\partial K}h_{-\alpha}(a,x)a_3^\alpha Q\nu e_3Pf+Pfa_3^\alpha Q\nu e_3\left(h_{-\alpha}\right)(a,x)d\sigma\\&\quad=-\int_{\partial K}a_3^a\left[h_{-\alpha}(a,x),\overline{P\nu},Qfe_3\right]-\int_{\partial K}a_3^\alpha\left[h_{-\alpha}^a(a,x),Q\nu e_3,Pf\right]d\sigma,\end{aligned}$$

completing the proof. □

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