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Some Constructions of Multiplicative n-ary hom–Nambu Algebras

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Abstract. We show that given a hom–Lie algebra one can construct the n-ary hom–Lie bracket by means of an $(n-2)$-cochain of the given hom–Lie algebra and find the conditions under which this n-ary bracket satisfies the Filippov–Jacobi identity, thereby inducing the structure of n-hom–Lie algebra. We introduce the notion of a hom–Lie n-tuple system which is the generalization of a hom–Lie triple system. We construct hom–Lie n-tuple system using a hom–Lie algebra.

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Introduction

The first instance of n-ary algebras in Physics appeared with a generalization of the Hamiltonian mechanics proposed in 1973 by Nambu [23]. More recent motivation comes from string theory and M-branes involving naturally an algebra with ternary operation called Bagger–Lambert algebra which gives impulse to a significant development. It was used in [7] as one of the main ingredients in the construction of a new type of supersymmetric gauge theory that is consistent with all the symmetries expected of a multiple M2-brane theory: 16 supersymmetries, conformal invariance, and SO(8) R-symmetry acting on the eight transverse scalars. On the other hand, in the study of supergravity solutions describing M2-branes ending on M5-branes, the Lie algebra appearing in the original Nahm equations has to be replaced with a generalization involving a ternary bracket in the lifted Nahm equations (see [8]).

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In [6], generalizations of n-ary algebras of Lie type and associative type by twisting the identities using linear maps were introduced. The notions of representations, derivations, cohomology and deformations were studied in [3,12,15,21,24]. These generalizations include n-ary Hom-algebra structures generalizing the n-ary algebras of Lie type including n-ary Nambu algebras, n-Lie algebras (called also n-ary Nambu–Lie algebras) and n-ary Lie algebras, and n-ary algebras of associative type including n-ary totally associative and n-ary partially associative algebras. In [4], a method was demonstrated how to construct ternary multiplications from the binary multiplication of a hom–Lie algebra, a linear twisting map, and a trace function satisfying certain compatibility conditions; and it was shown that this method can be used to construct ternary hom–Nambu–Lie algebras from hom–Lie algebras. This construction was generalized to n-Lie algebras and n-hom–Nambu–Lie algebras in [5].

It is well known that algebras of derivations and generalized derivations are very important in the study of Lie algebras and its generalizations. The notion of δ-derivation appeared in the paper of Filippov [14]. The results for δ-derivations and generalized derivations were studied by many authors. For example, Zhang and Zhang [26] generalized the above results to the case of Lie superalgebras; Chen, Ma, Ni and Zhou considered the generalized derivations of color Lie algebras, hom–Lie superalgebras and Lie triple systems [10,11]. Derivations and generalized derivations of n-ary algebras were considered in [17,18] and other papers. In [9], the authors generalize these results in the color n-ary hom–Nambu case.

This paper is organized as follows. In Sect. 1, we review some basic concepts of hom–Lie algebras, n-ary hom–Nambu algebras and n-hom–Lie algebras. We also recall the definitions of derivations, α^k-derivations, α^k-quasiderivations and α^k-centroid. In Sect. 2, we provide a construction procedure of n-hom–Lie algebras starting from a binary bracket of a hom–Lie algebra and multilinear form satisfying certain conditions. To this end, we give the relation between α^k-derivations, (resp. α^k-quasiderivations and α^k-centroid) of hom–Lie algebras and α^k-derivations (resp. α^k-quasiderivations and α^k-centroid) of n-hom–Lie algebras. In Sect. 3, we introduce the notion of a hom–Lie n-tuple system which is the generalization of a Lie n-tuple system which is introduced in [13]. We construct a hom–Lie n-tuple system using a hom–Lie algebra. Finally, we give a relation between α^k-quasiderivations of a hom–Lie algebra and $(n+1)$-ary α^k-derivations of associated hom–Lie n-tuple system.

1. hom–Lie Algebra and n-ary hom–Nambu Algebras

Throughout this paper, we will, for simplicity of exposition, assume that $\mathbb{K}$ is an algebraically closed field of characteristic zero, even though, for most of the general definitions and results in the paper, this assumption is not essential.

1.1. Definitions

The notion of a hom–Lie algebra was initially motivated by examples of deformed Lie algebras coming from twisted discretizations of vector fields (see [16,19]). We will follow notation conventions in [22].

Definition 1.1. A hom–Lie algebra is a triple $(\mathfrak{g}, [\ ,\ ], \alpha)$, where $[\ ,\ ] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ is a bilinear map and $\alpha : \mathfrak{g} \to \mathfrak{g}$ a linear map satisfying

$$[x, y] = -[y, x], \text{ (skew-symmetry)}$$
$$\circlearrowleft_{x,y,z} [\alpha(x), [y, z]] = 0, \text{ (hom–Jacobi condition)}$$

for all x, y, z from $\mathfrak{g}$, where $\circlearrowleft_{x,y,z}$ denotes summation over the cyclic permutations of x, y, z.

Definition 1.2. A hom–Lie algebra $(\mathfrak{g}, [\ ,\ ], \alpha)$ is called multiplicative if $\alpha([x, y]) = [\alpha(x), \alpha(y)]$ for all $x, y \in \mathfrak{g}$.

We define a linear map $ad : \mathfrak{g} \to End(\mathfrak{g})$ by $\mathrm{ad}_x(y) = [x, y]$. Thus, the hom–Jacobi identity is equivalent to

$$\mathrm{ad}_{[x,y]}(\alpha(z)) = \mathrm{ad}_{\alpha(x)} \circ \mathrm{ad}_y(z) - \mathrm{ad}_{\alpha(y)} \circ \mathrm{ad}_x(z), \quad \text{for all } x, y, z \in \mathfrak{g}. \quad (1.1)$$

Remark 1.3. An ordinary Lie algebra is a hom–Lie algebra when $\alpha = id$.

Example 1.4. Let $\mathcal{A}$ be the complex algebra where $\mathcal{A} = \mathbb{C}[t, t^{-1}]$ is the ring of Laurent polynomials in one variable. The generators of $\mathcal{A}$ are of the form of t^n for $n \in \mathbb{Z}$.

Let $q \in \mathbb{C}\backslash\{0, 1\}$ and $n \in \mathbb{N}$, we set $\{n\} = \frac{1-q^n}{1-q}$, a q-number. The q-numbers have the following properties: $\{n + 1\} = 1 + q\{n\} = \{n\} + q^n$ and $\{n + m\} = \{n\} + q^n\{m\}$.

Let $\mathfrak{A}_q$ be a space with basis $\{L_m,\ I_m|m \in \mathbb{Z}\}$ where $L_m = -t^m D,\ I_m = -t^m$ and D is a q-derivation on $\mathcal{A}$ such that

$$D(t^m) = \{m\}t^m.$$

We define the bracket $[\ ,\ ]_q : \mathfrak{A}_q \times \mathfrak{A}_q \longrightarrow \mathfrak{A}_q$, with respect to the super-skew-symmetry for $n, m \in \mathbb{Z}$ by

$$[L_m, L_n]_q = (\{m\} - \{n\})L_{m+n}, \quad (1.2)$$
$$[L_m, I_n]_q = -\{n\}I_{m+n}, \quad (1.3)$$
$$[I_m, I_n]_q = 0. \quad (1.4)$$

Let α be an even linear map on $\mathfrak{A}_q$ defined on the generators by

$$\alpha_q(L_n) = (1 + q^n)L_n, \quad \alpha_q(I_n) = (1 + q^n)I_n,$$

The triple $(\mathfrak{A}_q, [\ ,\ ]_q, \alpha_q)$ is a hom–Lie algebra, called the q-deformed Heisenberg–Virasoro algebra of hom-type.

Example 1.5. We consider the matrix construction of the algebra $\mathfrak{sl}_2(\mathbb{R})$ generated by the following three vectors:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The defining relations are

$$[H, X] = 2X; \quad [H, Y] = -2Y; \quad [X, Y] = H.$$

Let $\lambda \in \mathbb{R}^*$ and consider the linear maps $\alpha_\lambda : \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{sl}_2(\mathbb{R})$ defined by:

$$\alpha_\lambda(H) = H; \quad \alpha_\lambda(X) = \lambda^2 X; \quad \alpha_\lambda(Y) = \frac{1}{\lambda^2} Y.$$

Note that α_λ is a Lie algebra automorphism.

In [2], the authors have shown that $(\mathfrak{sl}_2(\mathbb{R}))_\lambda = (\mathfrak{sl}_2(\mathbb{R}), [\ ,\ ]_{\alpha_\lambda}, \alpha_\lambda)$ is a family of multiplicative hom–Lie algebras where the hom–Lie bracket $[\ ,\ ]_{\alpha_\lambda}$ on the basis elements is given, for $\lambda \neq 0$, by

$$[H, X]_{\alpha_\lambda} = 2\lambda^2 X; \quad [H, Y]_{\alpha_\lambda} = -\frac{2}{\lambda^2} Y; \quad [X, Y]_{\alpha_\lambda} = H.$$

Now, we recall the definitions of n-ary hom–Nambu algebras and n-ary hom–Nambu–Lie algebras, generalizing n-ary Nambu algebras and n-ary Nambu–Lie algebras (also called Filippov algebras), respectively, which were introduced by Ataguema et al. [6].

Definition 1.6. An *n-ary hom–Nambu* algebra $(\mathcal{N}, [\ ,\dots,\ ], \widetilde{\alpha})$ consists of a vector space $\mathcal{N}$, an n-linear map $[\ ,\dots,\ ] : \mathcal{N}^n \longrightarrow \mathcal{N}$ and a family $\widetilde{\alpha} = (\alpha_i)_{1 \le i \le n-1}$ of linear maps $\alpha_i : \ \mathcal{N} \longrightarrow \mathcal{N}$, satisfying

$$\begin{aligned} &\big[\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1}), [y_1, \dots, y_n]\big] \\ &\quad = \sum_{i=1}^{n} \big[\alpha_1(y_1), \dots, \alpha_{i-1}(y_{i-1}), [x_1, \dots, x_{n-1}, y_i], \alpha_i(y_{i+1}), \dots, \alpha_{n-1}(y_n)\big], \end{aligned} \tag{1.5}$$

for all $(x_1, \dots, x_{n-1}) \in \mathcal{N}^{n-1}$, $(y_1, \dots, y_n) \in \mathcal{N}^n$.

The identity (1.5) is called the *hom–Nambu identity*.

Let $X = (x_1, \dots, x_{n-1}) \in \mathcal{N}^{n-1}$, $\widetilde{\alpha}(X) = (\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1})) \in \mathcal{N}^{n-1}$ and $y \in \mathcal{N}$. We define an adjoint map $\mathrm{ad}(X)$ as a linear map on $\mathcal{N}$, such that

$$\mathrm{ad}_X(y) = [x_1, \dots, x_{n-1}, y]. \tag{1.6}$$

Then, the hom–Nambu identity (1.5) may be written in terms of the adjoint map as

$$\begin{aligned} &\mathrm{ad}_{\widetilde{\alpha}(X)}([x_n, \dots, x_{2n-1}]) \\ &\quad = \sum_{i=n}^{2n-1} [\alpha_1(x_n), \dots, \alpha_{i-n}(x_{i-1}), \mathrm{ad}_X(x_i), \alpha_{i-n+1}(x_{i+1}) \dots, \alpha_{n-1}(x_{2n-1})]. \end{aligned}$$

Definition 1.7. An n-ary hom–Nambu algebra is a triple $(\mathcal{N}, [\ ,\dots,\ ], \widetilde{\alpha})$ that is called n-hom–Lie algebra if the bracket $[\ ,\dots,\ ]$ is skew-symmetric, i.e $[x_{\sigma(1)}, \dots, x_{\sigma(n)}] = (-1)^{sign(\sigma)}[x_1, \dots, x_n]$ for $\sigma \in S_n$.

Remark 1.8. When the maps $(\alpha_i)_{1 \le i \le n-1}$ are all identity maps, one recovers the classical n-ary Nambu algebras. The hom–Nambu identity (1.5), for $n = 2$, corresponds to the hom–Jacobi identity (see [22]), which reduces to the Jacobi identity when $\alpha_1 = id$.

Let $(\mathcal{N}, [\ ,\dots,\ ], \widetilde{\alpha})$ and $(\mathcal{N}', [\cdot,\dots,\cdot]', \widetilde{\alpha}')$ be two n-ary hom–Nambu algebras where $\widetilde{\alpha} = (\alpha_i)_{i=1,\dots,n-1}$ and $\widetilde{\alpha}' = (\alpha'_i)_{i=1,\dots,n-1}$. A linear map $f : \mathcal{N} \to \mathcal{N}'$ is an n-ary hom–Nambu algebra *morphism* if it satisfies

$$f([x_1,\dots,x_{2n-1}]) = [f(x_1),\dots,f(x_{2n-1})]'$$
$$f \circ \alpha_i = \alpha'_i \circ f \quad \forall i = 1,\dots,n-1.$$

In the sequel, we deal sometimes with a particular class of n-ary hom–Nambu algebras which we call n-ary multiplicative hom–Nambu algebras.

Definition 1.9. A *multiplicative n-ary hom–Nambu algebra* (resp. *multiplicative n-hom–Lie algebra*) is an n-ary hom–Nambu algebra (resp. n-hom–Lie algebra) $(\mathcal{N}, [\ ,\dots,\ ], \widetilde{\alpha})$ with $\widetilde{\alpha} = (\alpha_i)_{1\le i\le n-1}$ where $\alpha_1 = \cdots = \alpha_{n-1} = \alpha$ and satisfying

$$\alpha([x_1,\dots,x_n]) = [\alpha(x_1),\dots,\alpha(x_n)], \quad \forall x_1,\dots,x_n \in \mathcal{N}. \tag{1.7}$$

For simplicity, we will denote the n-ary multiplicative hom–Nambu algebra as $(\mathcal{N}, [\ ,\dots,\ ], \alpha)$ where $\alpha : \mathcal{N} \to \mathcal{N}$ is a linear map. Also by misuse of language, an element $x \in \mathcal{N}^n$ refers to $x = (x_1,\dots,x_n)$ and $\alpha(x)$ denotes $(\alpha(x_1),\dots,\alpha(x_n))$.

1.2. Derivations, Quasiderivations and Centroids of Multiplicative n-hom–Lie Algebras

In this section, we recall the definition of derivation, generalized derivation, quasiderivation and centroids of multiplicative n-hom–Lie algebras.

Let $(\mathcal{N}, [\ ,\dots,\ ], \alpha)$ be a multiplicative n-hom–Lie algebra. We denote by α^k the k-times composition of α (i.e. $\alpha^k = \alpha\circ\cdots\circ\alpha$ k-times). In particular, $\alpha^{-1} = 0$, $\alpha^0 = id$.

Definition 1.10. For any $k \ge 1$, we call $D \in End(\mathcal{N})$ an α^k-*derivation* of the multiplicative n-hom–Lie algebra $(\mathcal{N}, [\ ,\dots,\ ], \alpha)$ if

$$[D,\alpha] = 0 \ \text{ i.e. } \ D\circ\alpha = \alpha\circ D, \tag{1.8}$$

and

$$D[x_1,\dots,x_n] = \sum_{i=1}^{n} \Big[\alpha^k(x_1),\dots,\alpha^k(x_{i-1}), D(x_i), \alpha^k(x_{i+1}),\dots,\alpha^k(x_n)\Big]. \tag{1.9}$$

We denote by $Der_{\alpha^k}(\mathcal{N})$ the set of α^k-derivations of the multiplicative n-hom–Lie algebra $\mathcal{N}$.

For $X = (x_1,\dots,x_{n-1}) \in \mathcal{N}^{n-1}$ satisfying $\alpha(X) = X$ and $k \ge 1$, we define the map $\mathrm{ad}^k_X \in End(\mathcal{N})$ by

$$\mathrm{ad}^k_X(y) = \Big[x_1,\dots,x_{n-1},\alpha^k(y)\Big] \quad \forall y \in \mathcal{N}. \tag{1.10}$$

Lemma 1.11. *The map ad^k_X is an α^{k+1}-derivation that we call the inner α^{k+1}-derivation.*

We denote by $Inn_{\alpha^k}(\mathcal{N})$ the space generated by all the inner α^{k+1}-derivations. For any $D \in Der_{\alpha^k}(\mathcal{N})$ and $D' \in Der_{\alpha^k}(\mathcal{N})$, we define their commutator $[D, D']$ as usual:

$$[D, D'] = D \circ D' - D' \circ D. \tag{1.11}$$

Set $Der(\mathcal{N}) = \bigoplus_{k \geq -1} Der_{\alpha^k}(\mathcal{N})$ and $Inn(\mathcal{N}) = \bigoplus_{k \geq -1} Inn_{\alpha^k}(\mathcal{N})$.

Definition 1.12. An endomorphism D of a multiplicative n-ary hom–Nambu algebra $(\mathcal{N}, [\ ,\dots,\ ], \alpha)$ is called a generalized α^k-derivation if there exist linear mappings $D', D'', \dots, D^{(n-1)}, D^{(n)} \in End(\mathcal{N})$ such that

$$D^{(n)}([x_1, \dots, x_n]) = \sum_{i=1}^{n} \Big[\alpha^k(x_1), \dots, D^{(i-1)}(x_i), \dots, \alpha^k(x_n)\Big], \tag{1.12}$$

for all $x_1, \dots, x_n \in \mathcal{N}$. An $(n+1)$-tuple $(D, D', D'', \dots, D^{(n-1)}, D^{(n)})$ is called an $(n+1)$-ary α^k-derivation.

The set of generalized α^k-derivations is denoted by $GDer_{\alpha^k}(\mathcal{N})$. Set $GDer(\mathcal{N}) = \bigoplus_{k \geq -1} GDer_{\alpha^k}(\mathcal{N})$.

Definition 1.13. Let $(\mathcal{N}, [\ ,\dots,\ ], \alpha)$ be a multiplicative n-ary hom–Nambu algebra and $End(\mathcal{N})$ be the endomorphism algebra of $\mathcal{N}$. An endomorphism $D \in End(\mathcal{N})$ is said to be an α^k-quasiderivation, if there exists an endomorphism $D' \in End(\mathcal{N})$ such that

$$\sum_{i=1}^{n} \Big[\alpha^k(x_1), \dots, D(x_i), \dots, \alpha^k(x_n)\Big] = D'([x_1, \dots, x_n]),$$

for all $x_1, \dots, x_n \in \mathcal{N}$. We call D' the endomorphism associated with the α^k-quasiderivation D.

The set of α^k-quasiderivations will be denoted by $QDer_{\alpha^k}(\mathcal{N})$. Set $QDer(\mathcal{N}) = \bigoplus_{k \geq -1} QDer_{\alpha^k}(\mathcal{N})$.

Definition 1.14. Let $(\mathcal{N}, [\ ,\dots,\ ], \alpha)$ be a multiplicative n-ary hom–Nambu algebra and $End(\mathcal{N})$ be the endomorphism algebra of $\mathcal{N}$. Then the following subalgebra of $End(\mathcal{N})$

$$Cent(\mathcal{N}) = \{\theta \in End(N) : \theta([x_1, \dots, x_n]) = [\theta(x_1), \dots, x_n],\ \forall x_i \in \mathcal{N}\}$$

is said to be the centroid of the n-ary hom–Nambu algebra. The definition is the same for the classical case of n-ary Nambu algebra. We may also consider the same definition for any n-ary hom–Nambu algebra.

Now, let $(\mathcal{N}, [\ ,\dots,\ ], \alpha)$ be a multiplicative n-ary hom–Nambu algebra.

Definition 1.15. An α^k-centroid of a multiplicative n-ary hom–Nambu algebra $(\mathcal{N}, [\ ,\dots,\ ], \alpha)$ is a subalgebra of $End(\mathcal{N})$, denoted $Cent_{\alpha^k}(\mathcal{N})$, given by

$$Cent_{\alpha^k}(\mathcal{N}) \\ = \Big\{\theta \in End(\mathcal{N}) : \theta[x_1,\dots,x_n] = \Big[\theta(x_1),\alpha^k(x_2),\dots,\alpha^k(x_n)\Big], \forall x_i \in \mathcal{N}\Big\}.$$

We recover the definition of the centroid when $k = 0$.

If $\mathcal{N}$ is a multiplicative n-hom–Lie algebra, then it is a simple fact that

$$\theta[x_1,\dots,x_n] = \Big[\alpha^k(x_1),\dots,\theta(x_p),\dots,\alpha^k(x_n)\Big], \ \forall p \in \{1,\dots,n\}.$$

2. n-hom–Lie Algebras Induced by hom–Lie Algebras

In [4], the authors introduced a construction of a 3-hom–Lie algebra from a hom–Lie algebra, and more generally of an $(n+1)$-hom–Lie algebra from an n-hom–Lie algebra. It is called the $(n+1)$-hom–Lie algebra induced by n-hom–Lie algebra. In this context, Abramov gave a new approach of this construction (see [1]). Now, we generalize this approach in the Hom case.

Let $(\mathfrak{g},[\ ,\ ],\alpha)$ be a multiplicative hom–Lie algebra and $\mathfrak{g}^*$ be its dual space. Fix an element of the dual space $\varphi \in \mathfrak{g}^*$. Define the triple product as follows:

$$[x,y,z] = \varphi(x)[y,z] + \varphi(y)[z,x] + \varphi(z)[x,y], \quad \forall\ x,\ y,\ z \in \mathfrak{g}. \tag{2.1}$$

Obviously, this triple product is skew-symmetric. Straightforward computation of the left hand side and the right hand side of the Filippov–Jacobi identity (1.5) if $\varphi \circ \alpha = \varphi$ yields

$$\varphi(x)\varphi([y,z]) + \varphi(y)\varphi([z,x]) + \varphi(z)\varphi([x,y]) = 0. \tag{2.2}$$

Now, we consider φ as a $\mathbb{K}$-valued cochain of degree one of the Chevalley–Eilenberg complex of a Lie algebra $\mathfrak{g}$. Making use of the coboundary operator $\delta : \wedge^k\mathfrak{g}^* \to \wedge^{k+1}\mathfrak{g}^*$ defined by

$$\delta f(u_1,\dots,u_{k+1}) \\ = \sum_{i<j}(-1)^{i+j+1} f([u_i,u_j]_{\mathfrak{g}},\alpha(u_1)\dots,\widehat{u_i},\dots,\widehat{u_j},\dots,\alpha(u_{k+1})), \tag{2.3}$$

for $f \in \wedge^k\mathfrak{g}^*$ and for all $u_1,\dots,u_{k+1} \in \mathfrak{g}$, we obtain that $\delta\varphi(x,y) = \varphi([x,y])$.

Finally, we can define the wedge product of two cochains φ and $\delta\varphi$, which is a cochain of degree three, by

$$\varphi \wedge \delta\varphi(x,y,z) = \varphi(x)\varphi([y,z]) + \varphi(y)\varphi([z,x]) + \varphi(z)\varphi([x,y]).$$

Hence, (2.2) is equivalent to $\varphi \wedge \delta\varphi = 0$. Thus, if a 1-cochain φ satisfies the equation (2.2), then the triple product (2.1) is the ternary Lie bracket and we will call this multiplicative 3-hom–Lie bracket the quantum Nambu bracket induced by a 1-cochain.

Definition 2.1. For $\phi \in \wedge^{n-2}\mathfrak{g}^*$, we define the n-ary product as follows:

$$[x_1,\dots,x_n]_\phi = \sum_{i<j}^{n}(-1)^{i+j+1}\phi(x_1,\dots,\hat{x_i},\dots,\hat{x_j},\dots,x_n)[x_i,x_j], \tag{2.4}$$

for all $x_1,\dots,x_n \in \mathfrak{g}$.

Proposition 2.2. *The n-ary product $[\ ,\dots,\ ]_\phi$ is skew-symmetric.*

Proof. Let $x_1,\dots,x_n\in\mathfrak{g}$ and, fixing two integers $i<j$, we have

$$\begin{aligned}
&[x_1,\dots,x_i,\dots,x_j,\dots,x_n]_\phi\\
&=\sum_{k<l:k,l\neq i,j}(-1)^{k+l+1}\phi(x_1,\dots,x_i\dots,\widehat{x}_k,\dots,x_j,\dots,\widehat{x}_l,\dots,x_n)[x_l,x_k]\\
&\quad+\sum_{i<l\neq j}(-1)^{i+l+1}\phi(x_1,\dots,\widehat{x}_i\dots,x_j,\dots,\widehat{x}_l,\dots,x_n)[x_i,x_l]\\
&\quad+\sum_{l<i}(-1)^{i+l+1}\phi(x_1,\dots,\widehat{x}_l,\dots,\widehat{x}_i,\dots,x_j,\dots,x_n)[x_l,x_i]\\
&\quad+\sum_{j<l}(-1)^{j+l+1}\phi(x_1,\dots,x_i\dots,\widehat{x}_j,\dots,\widehat{x}_l,\dots,x_n)[x_j,x_l]\\
&\quad+\sum_{l<j,i\neq l}(-1)^{j+l+1}\phi(x_1,\dots,x_i,\dots,\widehat{x}_l,\dots,\widehat{x}_j,\dots,x_n)[x_l,x_j]\\
&\quad+(-1)^{i+j+1}\phi(x_1,\dots,\widehat{x}_i,\dots,\widehat{x}_j,\dots,x_n)[x_i,x_j]\\
&=-[x_1,\dots,x_j,\dots,x_i,\dots,x_n]_\phi.
\end{aligned}$$

Given $X=(x_1,\dots,x_{n-3})\in\wedge^{n-3}\mathfrak{g}$, $Y=(y_1,\dots,y_n)\in\wedge^n\mathfrak{g}$ and $z\in\mathfrak{g}$, we define the linear map ϕ_X by

$$\phi_X(z)=\phi(X,z),$$

and

$$\begin{aligned}
\phi\wedge\delta\phi_X(Y)&=\sum_{i<j}^{n}(-1)^{i+j}\phi(y_1,\dots\hat{y}_i\dots\hat{y}_j\dots,y_n)\delta\phi_X(y_i,y_j)\\
&=\sum_{i<j}^{n}(-1)^{i+j}\phi(y_1,\dots\hat{y}_i\dots\hat{y}_j\dots,y_n)\phi_X([y_i,y_j]).
\end{aligned}$$

Theorem 2.3. *Let $(\mathfrak{g},[\ ,\ ],\alpha)$ be a multiplicative hom–Lie algebra, $\mathfrak{g}^*$ be its dual and ϕ be a cochain of degree $n-2$, i.e. $\phi\in\wedge^{n-2}\mathfrak{g}^*$. The vector space $\mathfrak{g}$ is equipped with the n-ary product (2.4) and the linear map α is a multiplicative n-hom–Lie algebra if and only if*

$$\phi\wedge\delta\phi_X=0,\quad\forall X\in\wedge^{n-3}\mathfrak{g},\tag{2.5}$$

$$\phi\circ(\alpha\otimes Id\otimes\cdots\otimes Id)=\phi.\tag{2.6}$$

Proof. Firstly, let $(x_1,\dots,x_n)\in\wedge^n\mathfrak{g}$. We have

$$\begin{aligned}
&[\alpha(x_1),\dots,\alpha(x_n)]_\phi\\
&=\sum_{i<j}^{n}(-1)^{i+j+1}\phi(\alpha(x_1),\dots,\alpha(\hat{x}_i),\dots,\alpha(\hat{x}_j),\dots,\alpha(x_n))[\alpha(x_i),\alpha(x_j)]\\
&=\sum_{i<j}^{n}(-1)^{i+j+1}\phi(x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_n)\alpha([x_i,x_j])\\
&=\alpha([x_1,\dots,x_n]_\phi).
\end{aligned}$$

Secondly, for $(x_1,\ldots,x_{n-1})\in\wedge^{n-1}\mathfrak{g}$ and $(y_1,\ldots,y_n)\in\wedge^n\mathfrak{g}$, we have

$$\begin{aligned}
&[\alpha(x_1),\ldots,\alpha(x_{n-1}),[y_1,\ldots,y_n]_\phi]_\phi\\
&\quad=\sum_{i<j}(-1)^{i+j+1}\phi(y_1,\ldots,\widehat{y}_i,\ldots,\widehat{y}_j,\ldots,y_n)\\
&\quad\quad\times[\alpha(x_1),\ldots,\alpha(x_{n-1}),[y_i,y_j]]_\phi\\
&\quad=\sum_{i<j}\sum_{k<l\leq n-1}(-1)^{i+j+k+l}\phi(\alpha(x_1),\ldots,\widehat{\alpha(x_k)},\ldots,\widehat{\alpha(x_l)},\ldots,[y_i,y_j])\\
&\quad\quad\times\phi(y_1,\ldots,\widehat{y}_i,\ldots,\widehat{y}_j,\ldots,y_n)[\alpha(x_k),\alpha(x_l)]\\
&\quad\quad+\sum_{i<j}\sum_{k<n}(-1)^{i+j+k}\phi(\alpha(x_1),\ldots,\widehat{\alpha(x_k)},\ldots,\alpha(x_{(n-1)}),\ldots,\widehat{[y_i,y_j]})\\
&\quad\quad\times\phi(y_1,\ldots,\widehat{y}_i,\ldots,\widehat{y}_j,\ldots,y_n)[\alpha(x_k),[y_i,y_j]].
\end{aligned}$$

The terms $[\alpha(x_k),[y_i,y_j]]$ are simplified by the hom–Jacobi condition in the second half of the Filippov identity. Now, we group together the other terms according to their coefficient $[\alpha(x_i),\alpha(x_j)]$. For example, if we fix (k,l), and if we collect all the terms containing the commutator $[\alpha(x_k),\alpha(x_l)]$, then we get the expression

$$\left(\sum_{i<j}(-1)^{i+j+k+l}\phi(\alpha(x_1),\ldots,\widehat{\alpha(x_k)},\ldots,\widehat{\alpha(x_l)},\ldots,[y_i,y_j])\right.$$
$$\left.\times\,\phi(y_1,\ldots,\widehat{y}_i,\ldots,\widehat{y}_j,\ldots,y_n)\right)[\alpha(x_k),\alpha(x_l)].$$

Hence, the n-ary product (2.4) will satisfy the n-ary Filippov–Jacobi identity; if for any elements $X=(x_1,\ldots,x_{n-3})\in\wedge^{n-3}\mathfrak{g}$ and $Y=(y_1,\ldots,y_n)\in\wedge^n\mathfrak{g}$ we require

$$\left(\sum_{i<j}^{n}(-1)^{i+j}\phi(\alpha(x_1),\ldots,\alpha(x_{n-3}),[y_i,y_j])\phi(y_1,\ldots,\widehat{y}_i,\ldots,\widehat{y}_j,\ldots,y_n)\right)=0.$$

Definition 2.4. Let $\phi:\mathfrak{g}\otimes\cdots\otimes\mathfrak{g}\to\mathbb{K}$ be a skew-symmetric multilinear form of the multiplicative hom–Lie algebras $(\mathfrak{g},[\ ,\ ],\alpha)$, then ϕ is called a trace if

$$\phi\circ(Id\otimes\ldots\otimes Id\otimes[\ ,\ ])=0\ \text{and}\ \phi\circ(\alpha\otimes Id\otimes\ldots\otimes Id)=\phi.$$

Corollary 2.5. *If $\phi:\mathfrak{g}^{\otimes n-2}\to\mathbb{K}$ is a trace of the hom–Lie algebra $(\mathfrak{g},[\ ,\ ],\alpha)$, then $\mathfrak{g}_\phi=(\mathfrak{g},[.,\ldots,.]_\phi,\alpha)$ is a n-hom–Lie algebra.*

Proposition 2.6. *Let $(\mathfrak{g},[\ ,\ ],\alpha)$ be a hom–Lie algebra and $D\in Der(\mathfrak{g})$ be an α^k-derivation such that*

$$\sum_{i=1}^{n-2}\phi(x_1,\ldots D(x_i),\ldots,x_{n-2})=0.$$

Then, D is an α^k-derivation of the n-hom–Lie algebra $(\mathfrak{g},[\ ,\ldots,\ ]_\phi,\alpha)$.

Proof. Let $X = (x_1, \ldots, x_n) \in \wedge^n \mathfrak{g}$. On the one hand, we get

$$
\begin{aligned}
&D([x_1, \ldots, x_n]_\phi) \\
&\quad = D\left(\sum_{i<j} (-1)^{i+j+1} \phi(\alpha(x_1), \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, \alpha(x_n))[\alpha(x_i), \alpha(x_j)]\right) \\
&\quad = \sum_{i<j} (-1)^{i+j+1} \phi(\alpha(x_1), \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, \alpha(x_n)) D([\alpha(x_i), \alpha(x_j)]) \\
&\quad = \sum_{i<j} (-1)^{i+j+1} \phi(x_1, \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_n) \left[\alpha(D(x_i)), \alpha^{k+1}(x_j)\right] \\
&\qquad + \sum_{i<j} (-1)^{i+j+1} \phi(x_1, \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_n) \left[\alpha^{k+1}(x_i), \alpha(D(x_j))\right],
\end{aligned}
$$

and, on the other hand, we have

$$
\begin{aligned}
&\sum_{l=1}^{n} \left[\alpha^k(x_1), \ldots, \alpha^k(x_{l-1}), D(x_l), \ldots, \alpha^k(x_{l+1}), \ldots, \alpha^k(x_n)\right]_\phi \\
&\quad = \sum_{l=1}^{n} \sum_{i<j\ ;\ i,j \neq l} (-1)^{i+j+1} \phi(\alpha^k(x_1), \ldots, \widehat{\alpha^k(x_i)}, \ldots, \\
&\qquad D(x_l), \ldots, \widehat{\alpha^k(x_j)}, \ldots, \alpha^k(x_n)) \left[\alpha^k(x_i), \alpha^k(x_j)\right] \\
&\qquad + \sum_{l=1}^{n} \sum_{i<l} (-1)^{i+l+1} \phi(\alpha^k(x_1), \ldots, \widehat{\alpha^k(x_i)}, \ldots, \\
&\qquad \widehat{D(x_l)}, \ldots, \alpha^k(x_n)) \left[\alpha^k(x_i), D(x_l)\right] \\
&\qquad + \sum_{l=1}^{n} \sum_{l=i<j} (-1)^{j+l+1} \phi\Big(\alpha^k(x_1), \ldots, \widehat{D(x_l)}, \ldots, \\
&\qquad \widehat{\alpha^k(x_j)}, \ldots, \alpha^k(x_n)\Big) \left[D(x_l), \alpha^k(x_j)\right].
\end{aligned}
$$

If D is an α^k-derivation, then $D([x_1, \ldots, x_n]_\phi) = \sum_{l=1}^{n} [\alpha^k(x_1), \ldots, \alpha^k(x_{l-1}),$ $D(x_l), \ldots, \alpha^k(x_{l+1}), \ldots, \alpha^k(x_n)]_\phi$, which gives

$$
\begin{aligned}
&\sum_{i<j\ ;\ i,j \neq l} (-1)^{i+j+1} \left(\sum_{l=1}^{n} \phi\Big(\alpha^k(x_1), \ldots, \widehat{\alpha^k(x_i)}, \ldots,\right. \\
&\qquad \left. D(x_l), \ldots, \widehat{\alpha^k(x_j)}, \ldots, \alpha^k(x_n)\Big)\right) \left[\alpha^k(x_i), \alpha^k(x_j)\right] = 0.
\end{aligned}
$$

Finally, if we fix (i, j), we have

$$
\sum_{l=1}^{n-2} \phi\Big(\alpha^k(x_1), \ldots, D(x_l), \ldots, \alpha^k(x_{n-2})\Big) = 0.
$$

Proposition 2.7. *Let* $(\mathfrak{g}, [\ ,\ ], \alpha)$ *be a hom–Lie algebra and* $D \in QDer(\mathfrak{g})$ *be an* α^k*-quasiderivation and* $D' : \mathfrak{g} \to \mathfrak{g}$ *be the endomorphism associated with* D *such that*

$$\sum_{i=1}^{n-2} \phi(x_1, \dots D(x_i), \dots, x_{n-2}) = 0.$$

Then, D *is an* α^k*-quasiderivation of the* n*-hom–Lie algebra* $(\mathfrak{g}, [\ , \dots,\ ]_\phi, \alpha)$ *with the same associated endomorphism* D'.

Proposition 2.8. *Let* $(\mathfrak{g}, [\ ,\ ], \alpha)$ *be a hom–Lie algebra and* $\theta : \mathfrak{g} \to \mathfrak{g}$ *be an* α^k*-centroid such that*

$$\phi(\theta(x_1), \dots x_i, \dots, x_{n-2})\left[\alpha^k(x), y\right] = \phi(x_1, \dots x_i, \dots, x_{n-2})[\theta(x), y].$$

Then, D *is an* α^k*-centroid on the* n*-hom–Lie algebra* $(\mathfrak{g}, [\ , \dots,\ ]_\phi, \alpha)$.

Proof. If $x_1, \dots, x_n \in \mathfrak{g}$, we have

$$\begin{aligned}\theta([x_1, \dots, x_n]_\phi) &= \sum_{i<j}^{n} (-1)^{i+j+1} \phi(x_1, \dots, \hat{x_i}, \dots, \hat{x_j}, \dots, x_n)\theta([x_i, x_j]) \\ &= \sum_{i<j}^{n} (-1)^{i+j+1} \phi(x_1, \dots, \hat{x_i}, \dots, \hat{x_j}, \dots, x_n)\left[\theta(x_i), \alpha^k(x_j)\right].\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\left[\theta(x_1), \alpha^k(x_2), \dots, \alpha^k(x_n)\right]_\phi \\ &= \sum_{i<j}^{n} (-1)^{i+j+1} \phi(\theta(x_1), \alpha^k(x_2), \dots, \hat{x_i}, \dots, \hat{x_j}, \dots, \alpha^k(x_n))\left[\alpha^k(x_i), \alpha^k(x_j)\right] \\ &= \sum_{i<j}^{n} (-1)^{i+j+1} \phi(x_1, \dots, \hat{x_i}, \dots, \hat{x_j}, \dots, x_n)\left[\theta(x_i), \alpha^k(x_j)\right] \\ &= \theta([x_1, \dots, x_n]_\phi).\end{aligned}$$

3. hom–Lie n-Tuple Systems

3.1. hom–Lie Triple Systems

In this section, we start by recalling the definitions of Lie triple systems and hom–Lie triple systems.

Definition 3.1. [20]
A vector space T together with a trilinear map $(x, y, z) \to [x, y, z]$ is called a Lie triple system (LTS) if

1. $[x, x, z] = 0$,
2. $[x, y, z] + [y, z, x] + [z, x, y] = 0$,
3. $[u, v, [x, y, z]] = [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]]$,

for all $x, y, z, u, v \in T$.

Definition 3.2. [25] A hom–Lie triple system (hom-LTS for short) is denoted by $(T, [\cdot,\cdot,\cdot], \alpha)$, which consists of a $\mathbb{K}$-vector space T, a trilinear product $[\cdot,\cdot,\cdot] : T \times T \times T \to T$, and a linear map $\alpha : T \to T$, called the twisted map, such that α preserves the product and for all $x, y, z, u, v \in T$,

1. $[x, x, z] = 0$,
2. $[x, y, z] + [y, z, x] + [z, x, y] = 0$,
3. $[\alpha(u), \alpha(v), [x, y, z]] = [[u, v, x], \alpha(y), \alpha(z)] + [\alpha(x), [u, v, y], \alpha(z)] + [\alpha(x), \alpha(y), [u, v, z]]$.

Remark 3.3. When the twisted map α is equal to the identity map, a hom-LTS is an LTS. So LTS are special examples of hom-LTS.

Definition 3.4. A hom–Lie triple system $(T, [\cdot,\cdot,\cdot], \alpha)$ is called multiplicative if $\alpha([x, y, z]) = [\alpha(x), \alpha(y), \alpha(z)]$, for all $x, y, z \in T$.

Theorem 3.5. [25]
Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a multiplicative hom–Lie algebra. Then

$$\mathfrak{g}_T = (\mathfrak{g}, [\cdot,\cdot,\cdot] = [\cdot,\cdot] \circ ([\cdot,\cdot] \otimes \alpha), \alpha^2),$$

is a multiplicative hom–Lie triple system.

3.2. hom–Lie n-Tuple System

In this section, we introduce the definitions of Lie n-tuple systems and multiplicative hom–Lie n-tuple systems. We give the analogue of Theorem 3.5 in the hom–Lie n-tuple systems case.

Definition 3.6. A vector space $\mathcal{G}$ together with a n-linear map $(x_1, \ldots, x_n) \to [x_1, \ldots, x_n]$ is called a Lie n-tuple system if

1. $[x, x, y_1, \ldots, y_{n-2}] = 0$, for all $x, y_1, \ldots, y_{n-2} \in \mathcal{G}$.
2. $\circlearrowleft_{x_1,x_2,x_3} [x_1, \ldots, x_n] = 0$, for all $x_1, \ldots, x_n \in \mathcal{G}$.
3. $\big[x_1, \ldots, x_{n-1}, [y_1, \ldots, y_n]\big] = \sum_{i=1}^{n} \big[y_1, \ldots, y_{i-1}, [x_1, \ldots, x_{n-1}, y_i], y_{i+1}, \ldots, y_n\big]$,

for all $x_1, \ldots, x_{n-1}, y_1, \ldots, y_n \in \mathcal{G}$.

Definition 3.7. A vector space $\mathcal{G}$ together with a n-linear map $(x_1, \ldots, x_n) \to [x_1, \ldots, x_n]$ and a family $\widetilde{\alpha} = (\alpha_i)_{1\leq i\leq n-1}$ of linear maps $\alpha_i : \mathcal{G} \longrightarrow \mathcal{G}$ is called a hom–Lie n-tuple system if

1. $[x, x, y_1, \ldots, y_{n-2}] = 0$, for all $x, y_1, \ldots, y_{n-2} \in \mathcal{G}$.
2. $\circlearrowleft_{x_1,x_2,x_3} [x_1, \ldots, x_n] = 0$, for all $x_1, \ldots, x_n \in \mathcal{G}$.
3. $\big[\alpha_1(x_1), \ldots, \alpha_{n-1}(x_{n-1}), [y_1, \ldots, y_n]\big]$
$$= \sum_{i=1}^{n} \big[\alpha_1(y_1), \ldots \alpha_{i-1}(y_{i-1}), [x_1, \ldots, x_{n-1}, y_i] \alpha_i(y_{i+1}), \ldots, \alpha_{n-1}(y_n)\big],$$
for all $x_1, \ldots, x_{n-1}, y_1, \ldots, y_n \in \mathcal{G}$.

Definition 3.8. A hom–Lie n-tuple system $(\mathcal{G}, [\ ,\ldots,\ ], \widetilde{\alpha})$ is called a multiplicative hom–Lie n-tuple system if $\alpha_1 = \cdots = \alpha_{n-1} = \alpha$ and $\alpha([x_1, \ldots, x_n]) = [\alpha(x_1), \ldots, \alpha(x_n)]$ for all $x_1, \ldots, x_n \in \mathcal{G}$.

Remark 3.9. When the twisted maps α_i are equal to the identity map, hom–Lie n-tuple systems are Lie n-tuple systems. So Lie n-tuple systems are special examples of hom–Lie n-tuple systems.

The following result gives a way to construct hom–Lie n-tuple systems starting from classical Lie n-tuple systems and algebra endomorphisms.

Proposition 3.10. *Let $(\mathcal{G}, [\ ,\dots,\ ])$ be a Lie n-tuple system and $\alpha : \mathcal{G} \to \mathcal{G}$ be a linear map such that $\alpha([x_1,\dots,x_n]) = [\alpha(x_1),\dots,\alpha(x_n)]$. Then, $(\mathcal{G}, [\ ,\dots,\ ]_\alpha, \alpha)$ is a hom–Lie n-tuple system, where $[x_1,\dots,x_n]_\alpha = [\alpha(x_1),\dots,\alpha(x_n)]$, for all $x_1,\dots,x_n \in \mathcal{G}$.*

Let $(\mathfrak{g}, [\ ,\ ], \alpha)$ be a hom–Lie algebra. We define the following n-linear map:

$$\begin{aligned}&[\ ,\dots,\ ]_n : \mathfrak{g}^{\otimes n} \longrightarrow \mathfrak{g}\\ &(x_1,\dots,x_n) \longmapsto \Big[x_1,\dots,x_n\Big]_n = \Big[[[\dots[x_1,x_2],\alpha(x_3)],\alpha^2(x_4)]\dots\alpha^{n-3}(x_{n-1})],\alpha^{n-2}(x_n)\Big].\end{aligned} \tag{3.1}$$

For $n = 2$, $[x_1,x_2]_2 = [x_1,x_2]$ and for $n \geq 3$ we have $[x_1,\dots,x_n]_n = [[x_1,\dots,x_{n-1}]_{n-1}, \alpha^{n-2}(x_n)]$.

Theorem 3.11. *Let $(\mathfrak{g}, [\ ,\ ], \alpha)$ be a multiplicative hom–Lie algebra. Then*

$$\mathfrak{g}_n = (\mathfrak{g}, [\ ,\dots,\ ]_n, \alpha^{n-1})$$

is a multiplicative hom–Lie n-tuple system.

When $n = 3$ we obtain the multiplicative hom–Lie triple system constructed in Theorem 3.5. To prove this theorem, we need the following lemma.

Lemma 3.12. *Let $(\mathfrak{g}, [\ ,\ ], \alpha)$ be a multiplicative hom–Lie algebra, and ad^2 the adjoint map defined by*

$$ad^2_x(y) = ad_x(y) = [x,y].$$

Then, we have

$$ad^2_{\alpha^{n-1}(x)}[y_1,\dots,y_n]_n = \sum_{k=1}^{n} \Big[\alpha(y_1),\dots,\alpha(y_{k-1}), ad^2_x(y_k), \alpha(y_{k+1}),\dots,\alpha(y_n)\Big]_n,$$

where $x \in \mathfrak{g}, y \in \mathfrak{g}$ and $(y_1,\dots,y_n) \in \mathfrak{g}^n$.

Proof. For $n = 2$, using the hom–Jacobi identity we have

$$\begin{aligned}\mathrm{ad}^2_{\alpha(x)}[y,z] &= [\alpha(x),[y,z]] = [[x,y],\alpha(z)] + [\alpha(y),[x,z]]\\ &= \Big[\mathrm{ad}^2_x(y),\alpha(z)\Big] + \Big[\alpha(y),\mathrm{ad}^2_x(z)\Big].\end{aligned}$$

Assume that the property is true up to order n, that is

$$\begin{aligned}&\mathrm{ad}^2_{\alpha^{n-1}(X)}[y_1,\dots,y_n]_n\\ &\quad= \sum_{k=1}^{n} [\alpha(y_1),\dots,\alpha(y_{k-1}), \mathrm{ad}^2_X(y_k), \alpha(y_{k+1}),\dots,\alpha(y_n)]_n.\end{aligned}$$

If $x \in \mathfrak{g}$ and $(y_1,\dots,y_{n+1}) \in \mathfrak{g}^{n+1}$, we have

$$\mathrm{ad}^2_{\alpha^n(x)}[y_1,\dots,y_{n+1}]$$

$$
\begin{aligned}
&= \mathrm{ad}^2_{\alpha^n(x)}[[y_1,\ldots,y_n]_n,\alpha^{n-1}(y_{n+1})]_2\\
&= \Big[\mathrm{ad}^2_{\alpha^{n-1}(x)}[y_1,\ldots,y_n]_n,\alpha^n(y_{n+1})\Big]_2\\
&\quad+\Big[[\alpha(y_1),\ldots,\alpha(y_n)]_n,\mathrm{ad}^2_{\alpha^{n-1}(x)}(\alpha^{n-1}(y_{n+1}))\Big]_2\\
&= \sum_{k=1}^{n}\Big[[\alpha(y_1),\ldots,\alpha(y_{k-1}),\mathrm{ad}^2_x(y_k),\alpha(y_{k+1}),\ldots,\alpha(y_n)]_n,\alpha^n(y_{n+1})\Big]\\
&\quad+\Big[[\alpha(y_1),\ldots,\alpha(y_n)]_n,\alpha^{n-1}(\mathrm{ad}^2_x(y_{n+1}))\Big]_2\\
&= \sum_{k=1}^{n}\Big[\alpha(y_1),\ldots,\alpha(y_{k-1}),\mathrm{ad}^2_x(y_k),\alpha(y_{k+1}),\ldots,\alpha(y_n),\alpha(y_{n+1})\Big]_{n+1}\\
&\quad+\Big[\alpha(y_1),\ldots,\alpha(y_n),\mathrm{ad}^2_x(y_{n+1})\Big]_{n+1}\\
&= \sum_{k=1}^{n+1}\Big[\alpha(y_1),\ldots,\alpha(y_{k-1}),\mathrm{ad}^2_x(y_k),\alpha(y_{k+1}),\ldots,\alpha(y_{n+1})\Big]_{n+1}.
\end{aligned}
$$

The lemma is proved. □

Proof. (**Proof of Theorem** 3.11) Let $X=(x_1,\ldots,x_{n-1})\in\mathfrak{g}^{n-1}$ and $Y=(y_1,\ldots,y_n)\in\mathfrak{g}^n$.

(i) It is easy to see that $[x_1,x_1,x_2,\ldots,x_{n-1}]_n=[[\ldots[[x_1,x_1]_2,\alpha(x_2)]_2,\alpha^2(x_3)]_2,\ldots]_2,\alpha^{n-2}(x_{n-1})]_2=0$

(ii) Using the hom–Jacobi condition, it is easy to prove $\circlearrowleft_{x_1,x_2,x_3}[x_1,\ldots,x_n]=0$, for all $x_1,\ldots,x_n\in\mathcal{G}$.

(iii) Using Lemma (3.12), we have

$$
\begin{aligned}
&\Big[\alpha^{n-1}(x_1),\ldots,\alpha^{n-1}(x_{n-1}),[y_1,\ldots,y_n]_n\Big]_n\\
&\quad= \Big[[\alpha^{n-1}(x_1),\ldots,\alpha^{n-1}(x_{n-1})]_{n-1},[\alpha^{n-2}(y_1),\ldots,\alpha^{n-2}(y_n)]_n\Big]_2\\
&\quad= \mathrm{ad}^2_{\alpha^{n-1}[x_1,\ldots,x_{n-1}]}([\alpha^{n-2}(y_1),\ldots,\alpha^{n-2}(y_n)]_n)\\
&\quad= \sum_{k=1}^{n}\Big[\alpha^{n-1}(y_1),\ldots,\mathrm{ad}^2_{[x_1,\ldots,x_{n-1}]}(\alpha^{n-2}(y_k)),\ldots,\alpha^{n-1}(y_n)\Big]_n\\
&\quad= \sum_{k=1}^{n}\Big[\alpha^{n-1}(y_1),\ldots,[[x_1,\ldots,x_{n-1}],\alpha^{n-2}(y_k)]_2,\ldots,\alpha^{n-1}(y_n)\Big]_n\\
&\quad= \sum_{k=1}^{n}\Big[\alpha^{n-1}(y_1),\ldots,[x_1,\ldots,x_{n-1},y_k]_n,\ldots,\alpha^{n-1}(y_n)\Big]_n.
\end{aligned}
$$

Example 3.13. Using Example 1.5 and Theorem 3.11, for $\lambda\in\mathbb{R}^*$, we have the following.

For $n=3$, $(\mathfrak{sl}_2(\mathbb{R}),[\ ,\ ,\ ]_3,\alpha^2_\lambda)$ is a hom–Lie triple system. The different brackets are as follows:

$$[H,X,Y]_3=[[H,X]_{\alpha_\lambda},\alpha_\lambda(Y)]_{\alpha_\lambda}=2H;\ \ [H,X,H]_3=-4\lambda^4X;$$
$$[H,Y,X]_3=4H.$$
$$[H,Y,H]_3=-\frac{4}{\lambda^4}Y;\ \ [X,Y,Y]_3=-\frac{2}{\lambda^4}Y;\ \ [X,Y,X]_3=2\lambda^4X.$$

Each of the other brackets is equal to zero.

For $n=4$, $(\mathfrak{sl}_2(\mathbb{R}),[\ ,\ ,\ ,\ ]_4,\alpha_\lambda^3)$ is a hom–Lie 4-uplet system. The different brackets are defined as follows:

$$[H,X,H,H]_4=[[H,X,H]_3,\alpha^2(H)]_{\alpha_\lambda}=-4\lambda^4[X,H]_{\alpha_\lambda}=8\lambda^6X;$$
$$[H,X,H,Y]_4=-4H;$$
$$[H,Y,H,H]_4=-\frac{8}{\lambda^6}Y;\quad [H,Y,H,X]_4=4H;$$
$$[H,X,Y,X]_4=4\lambda^6X;\quad [H,X,Y,Y]_4=-\frac{2}{\lambda^6}Y;$$
$$[H,Y,X,X]_4=8\lambda^6X;\quad [H,Y,X,Y]_4=-\frac{8}{\lambda^6}Y;$$
$$[X,Y,X,Y]_4=2H;$$
$$[X,Y,X,H]_4=-4\lambda^6X;\quad [X,Y,Y,X]_4=2H;$$
$$[X,Y,Y,H]_4=-\frac{4}{\lambda^6}Y.$$

Each of the other brackets is equal to zero.

Proposition 3.14. *Let $(\mathfrak{g},[\ ,\ ],\alpha)$ be a multiplicative hom–Lie algebra and $D:\mathfrak{g}\to\mathfrak{g}$ be an α^k-derivation of $\mathfrak{g}$ for an integer k. Then, D is an α^k-derivation of $\mathfrak{g}_n$.*

Proof. By recurrence

Fix $n=3$. For $x,y,z\in\mathfrak{g}$, we have

$$\begin{aligned}D([x,y,z])&=D([[x,y],\alpha(z)])\\&=\Big[D([x,y]),\alpha^{k+1}(z)\Big]+\Big[\Big[\alpha^k(x),\alpha^k(y)\Big],D(\alpha(z))\Big]\\&=\Big[\Big[D(x),\alpha^k(y)\Big],\alpha^{k+1}(z)\Big]+\Big[\Big[\alpha^k(x),D(y)\Big],\alpha^{k+1}(z)\Big]\\&\quad+\Big[\Big[\alpha^k(x),\alpha^k(y)\Big],\alpha(D(z))\Big]\\&=\Big[D(x),\alpha^k(y),\alpha^k(z)\Big]+\Big[\alpha^k(x),D(y),\alpha^k(z)\Big]\\&\quad+\Big[\alpha^k(x),\alpha^k(y),D(z)\Big].\end{aligned}$$

Now, suppose that the property is true to order $n-1$, i.e:

$$D([x_1,\ldots,x_{n-1}]_{n-1})=\sum_{i=1}^{n}\Big[\alpha^k(x_1),\ldots,D(x_k),\ldots,\alpha^k(x_{n-1})\Big]_{n-1}.$$

If $(x_1,\ldots,x_n)\in\mathfrak{g}^n$, then

$$\begin{aligned}
D([x_1,\dots,x_n]_n) &= D\Big(\Big[[x_1,\dots,x_{n-1}]_{n-1},\alpha^{n-2}(x_n)\Big]\Big)\\
&= \Big[D([x_1,\dots,x_{n-1}]_{n-1}),\alpha^{n+k-2}(x_n)\Big]\\
&\quad+\Big[[\alpha^k(x_1),\dots,\alpha^k(x_{n-1})]_{n-1},D(\alpha^{n-2}(x_n))\Big]\\
&= \Big[D([x_1,\dots,x_{n-1}]_{n-1}),\alpha^{n-2}\Big(\alpha^k(x_n)\Big)\Big]\\
&\quad+\Big[\Big[\alpha^k(x_1),\dots,\alpha^k(x_{n-1})\Big]_{n-1},\alpha^{n-2}(D(x_n))\Big]\\
&= \sum_{i=1}^{n-1}\Big[\Big[\alpha^k(x_1),\dots,D(x_i),\dots,\alpha^k(x_{n-1})\Big]_{n-1},\alpha^{n-2}\Big(\alpha^k(x_n)\Big)\Big]\\
&\quad+\Big[\alpha^k(x_1),\dots,\alpha^k(x_{n-1}),D(x_n)\Big]_n\\
&= \sum_{i=1}^{n-1}\Big[\alpha^k(x_1),\dots,D(x_i),\dots,\alpha^k(x_{n-1}),\alpha^k(x_n)\Big]_n\\
&\quad+\Big[\alpha^k(x_1),\dots,\alpha^k(x_{n-1}),D(x_n)\Big]_n\\
&= \sum_{i=1}^{n}\Big[\alpha^k(x_1),\dots,D(x_i),\dots,\alpha^k(x_{n-1}),\alpha^k(x_n)\Big]_n.
\end{aligned}$$

Proposition 3.15. *Let $(\mathfrak{g},[\ ,\ ],\alpha)$ be a multiplicative hom–Lie algebra and $D,D',\dots,D^{(n-1)}$ be endomorphisms of $\mathfrak{g}$ such that $D^{(i)}$ is α^k-quasiderivation with associated endomorphism $D^{(i+1)}$ for $0\le i\le n-2$. Then, the $(n+1)$-tuple $(D,D,D',D'',\dots,D^{(n-1)})$ is an $(n+1)$-ary α^k-derivation of $\mathfrak{g}_n$.*

Proof. If $x_1,\dots,x_n\in\mathfrak{g}$, then

$$\begin{aligned}
D^{(n-1)}([x_1,\dots,x_n]_n) &= D^{(n-1)}([[x_1,\dots,x_{n-1}]_{n-1},\alpha^{n-2}(x_n)])\\
&= [D^{(n-2)}([x_1,\dots,x_{n-1}]_{n-1}),\alpha^k(x_n)]\\
&\quad+\Big[\Big[\alpha^k(x_1),\dots,\alpha^k(x_{n-1})\Big]_{n-1},D^{(n-2)}(\alpha^{n-2}(x_n))\Big]\\
&\ \ \vdots\\
&= \Big[D(x_1),\alpha^k(x_2),\dots,\alpha^k(x_n)\Big]_n\\
&\quad+\Big[\alpha^k(x_1),D(x_2),\dots,\alpha^k(x_n)\Big]_n\\
&\quad+\Big[\alpha^k(x_1),\alpha^k(x_2),D'(x_3),\dots,\alpha^k(x_n)\Big]_n\\
&\quad+\dots+\Big[\alpha^k(x_1),\dots,\alpha^k(x_{n-1}),D^{(n-2)}(x_n)\Big]_n.
\end{aligned}$$

Therefore, the $(n+1)$-tuple $(D,D,D',D'',\dots,D^{(n-1)})$ is an $(n+1)$-ary α^k-derivation of $\mathfrak{g}_n$. □

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