



Reflection Principle for the Complex Monge–Ampère Equation and Plurisubharmonic Functions

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To Professor Sirkka-Liisa Eriksson on her 60th birthday.

Abstract. We study reflection principle for several central objects in pluripotential theory. First we show that the odd reflected function gives an extension for pluriharmonic functions over a flat boundary. Then we show that the even reflected function gives an extension for nonnegative plurisubharmonic functions. In particular cases odd and/or even reflected functions give extensions for classical solutions of the homogeneous complex Monge–Ampère equation. Finally, we state reflection principle for the generalized complex Monge–Ampère equation and maximal plurisubharmonic functions.

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1. Introduction

Reflection is a method to extend functions and, in particular, solutions of homogeneous equations across a flat boundary. Classically, it is applied to some strong type equations but later on also to several weak type equations. The original reflection principle states that an analytic function given in the upper half unit disk can be extended to the whole unit disk by reflection. This result originates with Schwarz. A similar principle holds for harmonic functions in the space, see [2].

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In higher real dimensions Martio and Rickman [16] introduced the reflection principle for quasiregular mappings. Later on Martio [14] showed that the reflection principle holds for solutions of certain elliptic partial differential equations, and he also treated further the reflection principle for quasiregular mappings, see also [9]. Moreover, Martio [15] studied equivalent principle for quasiminimizers in \mathbb{R}^n . Recently, Koskenoja [12, 13] considered the reflection principle for both classical and viscosity solutions of the homogeneous real Monge–Ampère equation.

In several complex variables many authors have studied the reflection principle which is well understood for holomorphic mappings and related Cauchy–Riemann equations, see expository surveys by Coupet and Sukhov [6] and by Diederich and Pinchuk [7]. We concentrate on pluriharmonic and plurisubharmonic functions and the homogeneous complex Monge–Ampère equation that are central objects in pluripotential theory, see [3, 8, 10, 11].

2. Basic Properties of the Reflection

We first set central notation connected to the reflection in \mathbb{C}^n . Let G_+ be a domain in $\mathbb{C}_+^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : \text{Im } z_n > 0\}$. Let $P: \mathbb{C}^n \rightarrow \mathbb{C}^n$ be the *reflection* with respect to $\partial\mathbb{C}_+^n$, that is, $P(z) = P(z_1, \dots, z_n) = (z_1, \dots, z_{n-1}, \bar{z}_n)$. Suppose that there is a non-empty set $G_0 \subset \partial\mathbb{C}_+^n$ open in ∂G_+ . Set $G = G_+ \cup G_0 \cup G_-$ where $G_- = PG_+$. Then G is a domain (open and connected set) in \mathbb{C}^n . Suppose that a function $u: G_+ \rightarrow \mathbb{R}$ satisfies the following boundary condition on G_0 :

$$\lim_{z \rightarrow w} u(z) = 0 \text{ for all } w \in G_0. \tag{2.1}$$

We define the *odd reflected function* $\tilde{u}: G \rightarrow \mathbb{R}$,

$$\tilde{u}(z) = \begin{cases} u(z), & z \in G_+, \\ 0, & z \in G_0, \\ -u(P(z)), & z \in G_-. \end{cases} \tag{2.2}$$

Correspondingly, the *even reflected function* $\hat{u}: G \rightarrow \mathbb{R}$ is given by

$$\hat{u}(z) = \begin{cases} u(z), & z \in G_+, \\ 0, & z \in G_0, \\ u(P(z)), & z \in G_-. \end{cases} \tag{2.3}$$

In [12] Koskenoja studied differentiability properties of the reflected functions in the real n -space. Most of these results and examples can be adopted straightforward to the complex n -space. It is remarkable that if we reflect a differentiable function, then it may happen that the differentiability gets broken in the reflection boundary G_0 , see examples in [12].

Recall next some standard terminology. Let $\Omega \subset \mathbb{C}^n$ be open. A mapping $f: \Omega \rightarrow \mathbb{R}$ is said to be *differentiable* (or \mathbb{R} -differentiable) in Ω if it is differentiable in Ω with respect to the real coordinates. This means that the first order real partial derivatives of f exist at each point of Ω . Correspondingly, we say that a mapping is *twice differentiable* in Ω meaning that the

second order real partial derivatives exist at each point of Ω . It is obvious that if $u \in C^1(G_+)$, then $\tilde{u}, \hat{u} \in C^1(G_-)$, and if $u \in C^2(G_+)$, then $\tilde{u}, \hat{u} \in C^2(G_-)$, see [12, Lemmas 3.1 and 3.7].

We start with giving reflection formulas for the complex partial differential operators

$$\frac{\partial}{\partial z_k} = \frac{1}{2} \left(\frac{\partial}{\partial x_k} - i \frac{\partial}{\partial y_k} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_k} = \frac{1}{2} \left(\frac{\partial}{\partial x_k} + i \frac{\partial}{\partial y_k} \right), \quad k = 1, \dots, n,$$

and for the complex Hesse matrix, see [17]. These formulas are complex counterparts of the formulas given in [12, Lemmas 3.1 and 3.7 and Theorem 3.14]. We omit many calculations since the methods are rather evident and often similar to those of given earlier in the proofs.

Lemma 2.1. *Let a point $z \in G_-$ be such that u is differentiable at $P(z) \in G_+$. Then*

$$\frac{\partial \tilde{u}}{\partial z_k}(z) = \begin{cases} -\frac{\partial u}{\partial z_k}(P(z)), & k = 1, \dots, n-1, \\ -\frac{\partial u}{\partial \bar{z}_k}(P(z)), & k = n, \end{cases} \tag{2.4}$$

$$\frac{\partial \tilde{u}}{\partial \bar{z}_k}(z) = \begin{cases} -\frac{\partial u}{\partial \bar{z}_k}(P(z)), & k = 1, \dots, n-1, \\ -\frac{\partial u}{\partial z_k}(P(z)), & k = n, \end{cases} \tag{2.5}$$

$$\frac{\partial \hat{u}}{\partial z_k}(z) = \begin{cases} \frac{\partial u}{\partial z_k}(P(z)), & k = 1, \dots, n-1, \\ \frac{\partial u}{\partial \bar{z}_k}(P(z)), & k = n, \end{cases} \tag{2.6}$$

$$\frac{\partial \hat{u}}{\partial \bar{z}_k}(z) = \begin{cases} \frac{\partial u}{\partial \bar{z}_k}(P(z)), & k = 1, \dots, n-1, \\ \frac{\partial u}{\partial z_k}(P(z)), & k = n, \end{cases} \tag{2.7}$$

In particular,

$$\frac{\partial \tilde{u}}{\partial z_k}(z) = -\frac{\partial \hat{u}}{\partial z_k}(z) \tag{2.8}$$

and

$$\frac{\partial \tilde{u}}{\partial \bar{z}_k}(z) = -\frac{\partial \hat{u}}{\partial \bar{z}_k}(z) \tag{2.9}$$

for every $k = 1, \dots, n$.

Proof. The reflection $P = (P_1, \dots, P_n): \mathbb{C}^n \rightarrow \mathbb{C}^n$ is given coordinately by

$$P_l(z) = \begin{cases} z_l, & l = 1, \dots, n-1, \\ \bar{z}_l, & l = n, \end{cases}$$

and therefore

$$\bar{P}_l(z) = \begin{cases} \bar{z}_l, & l = 1, \dots, n-1, \\ z_l, & l = n, \end{cases}$$

where we denote $\bar{P}_l: \mathbb{C}^n \rightarrow \mathbb{C}$, $\bar{P}_l(z) = \overline{P_l(z)}$ whenever $z \in \mathbb{C}^n$. Therefore we have

$$\frac{\partial P_l}{\partial z_k}(z) = \frac{\partial \bar{P}_l}{\partial \bar{z}_k}(z) = \begin{cases} 1, & k = l = 1, \dots, n-1, \\ 0, & \text{otherwise,} \end{cases} \tag{2.10}$$

and

$$\frac{\partial P_l}{\partial \bar{z}_k}(z) = \frac{\partial \bar{P}_l}{\partial z_k}(z) = \begin{cases} 1, & k = l = n, \\ 0, & \text{otherwise.} \end{cases} \tag{2.11}$$

Hence by the complex chain rules, see [11, (1.2.10), (1.2.11)],

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial z_k}(z) &= \frac{\partial(-(u \circ P))}{\partial z_k}(z) \\ &= -\frac{\partial(u \circ P)}{\partial z_k}(z) = -\sum_{l=1}^n \left(\frac{\partial u}{\partial z_l}(P(z)) \frac{\partial P_l}{\partial z_k}(z) + \frac{\partial u}{\partial \bar{z}_l}(P(z)) \frac{\partial \bar{P}_l}{\partial z_k}(z) \right) \\ &= \begin{cases} -\frac{\partial u}{\partial z_k}(P(z)), & k = 1, \dots, n-1, \\ -\frac{\partial u}{\partial \bar{z}_k}(P(z)), & k = n, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial \bar{z}_k}(z) &= \frac{\partial(-(u \circ P))}{\partial \bar{z}_k}(z) \\ &= -\frac{\partial(u \circ P)}{\partial \bar{z}_k}(z) = -\sum_{l=1}^n \left(\frac{\partial u}{\partial z_l}(P(z)) \frac{\partial P_l}{\partial \bar{z}_k}(z) + \frac{\partial u}{\partial \bar{z}_l}(P(z)) \frac{\partial \bar{P}_l}{\partial \bar{z}_k}(z) \right) \\ &= \begin{cases} -\frac{\partial u}{\partial \bar{z}_k}(P(z)), & k = 1, \dots, n-1, \\ -\frac{\partial u}{\partial z_k}(P(z)), & k = n. \end{cases} \end{aligned}$$

In the same way, we obtain formulas (2.6) and (2.7). Finally, Eq. (2.8) follows from (2.4) and (2.6), and (2.9) follows from (2.5) and (2.7). \square

Remark 2.2. We could prove formulas (2.4), (2.6), (2.5) and (2.7) also without complex chain rules just by using the corresponding formulas for the first order partial differential operators given in [12]. For example, since by [12, Lemma 3.1]

$$D\tilde{u}(z) = \left(-\frac{\partial u}{\partial x_1}(P(z)), -\frac{\partial u}{\partial y_1}(P(z)), \dots, -\frac{\partial u}{\partial x_n}(P(z)), \frac{\partial u}{\partial y_n}(P(z)) \right),$$

we have

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial z_k}(z) &= \frac{1}{2} \left(\frac{\partial \tilde{u}}{\partial x_k}(z) - i \frac{\partial \tilde{u}}{\partial y_k}(z) \right) \\ &= \begin{cases} \frac{1}{2} \left(-\frac{\partial u}{\partial x_k}(P(z)) + i \frac{\partial u}{\partial y_k}(P(z)) \right) = -\frac{\partial u}{\partial z_k}(P(z)), & k = 1, \dots, n-1, \\ \frac{1}{2} \left(-\frac{\partial u}{\partial x_k}(P(z)) - i \frac{\partial u}{\partial y_k}(P(z)) \right) = -\frac{\partial u}{\partial \bar{z}_k}(P(z)), & k = n. \end{cases} \end{aligned}$$

The *complex Hesse matrix* (or the *complex Hessian*) of a twice differentiable function u at a point z is the $n \times n$ matrix

$$Hu(z) = \left[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) \right],$$

where the entries are given by the second order complex partial differential operator

$$\frac{\partial^2}{\partial z_j \partial \bar{z}_k} = \frac{1}{4} \left(\frac{\partial^2}{\partial x_j \partial x_k} + i \frac{\partial^2}{\partial x_j \partial y_k} - i \frac{\partial^2}{\partial y_j \partial x_k} + \frac{\partial^2}{\partial y_j \partial y_k} \right). \tag{2.12}$$

In particular, if a function u is twice continuously differentiable, then by (2.12) the trace of the complex Hesse matrix of u at z satisfies

$$4 \sum_{j=1}^n \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j}(z) = \sum_{j=1}^n \left(\frac{\partial^2 u}{\partial x_j^2}(z) + \frac{\partial^2 u}{\partial y_j^2}(z) \right) = \Delta u(z).$$

Lemma 2.3. *Let a point $z \in G_-$ be such that u is twice differentiable at $P(z) \in G_+$. Then*

$$H\tilde{u}(z) = - \begin{bmatrix} \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_1}(P(z)) & \cdots & \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_{n-1}}(P(z)) & \frac{\partial^2 u}{\partial z_1 \partial z_n}(P(z)) \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial^2 u}{\partial z_{n-1} \partial \bar{z}_1}(P(z)) & \cdots & \frac{\partial^2 u}{\partial z_{n-1} \partial \bar{z}_{n-1}}(P(z)) & \frac{\partial^2 u}{\partial z_{n-1} \partial z_n}(P(z)) \\ \frac{\partial^2 u}{\partial \bar{z}_n \partial \bar{z}_1}(P(z)) & \cdots & \frac{\partial^2 u}{\partial \bar{z}_n \partial \bar{z}_{n-1}}(P(z)) & \frac{\partial^2 u}{\partial z_n \partial \bar{z}_n}(P(z)) \end{bmatrix} \tag{2.13}$$

and

$$H\hat{u}(z) = \begin{bmatrix} \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_1}(P(z)) & \cdots & \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_{n-1}}(P(z)) & \frac{\partial^2 u}{\partial z_1 \partial z_n}(P(z)) \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial^2 u}{\partial z_{n-1} \partial \bar{z}_1}(P(z)) & \cdots & \frac{\partial^2 u}{\partial z_{n-1} \partial \bar{z}_{n-1}}(P(z)) & \frac{\partial^2 u}{\partial z_{n-1} \partial z_n}(P(z)) \\ \frac{\partial^2 u}{\partial \bar{z}_n \partial \bar{z}_1}(P(z)) & \cdots & \frac{\partial^2 u}{\partial \bar{z}_n \partial \bar{z}_{n-1}}(P(z)) & \frac{\partial^2 u}{\partial z_n \partial \bar{z}_n}(P(z)) \end{bmatrix}. \tag{2.14}$$

In particular,

$$H\tilde{u}(z) = -H\hat{u}(z). \tag{2.15}$$

Proof. The complex chain rules yield now together with formulas (2.5), (2.7), (2.10) and (2.11) that

$$\begin{aligned} \frac{\partial^2 \tilde{u}}{\partial z_j \partial \bar{z}_k}(z) &= \frac{\partial}{\partial z_j} \left(\frac{\partial \tilde{u}}{\partial \bar{z}_k}(z) \right) \\ &= \begin{cases} \frac{\partial}{\partial z_j} \left(-\frac{\partial u}{\partial \bar{z}_k}(P(z)) \right) = -\frac{\partial}{\partial z_j} \left(\left(\frac{\partial u}{\partial \bar{z}_k} \circ P \right)(z) \right), & k = 1, \dots, n-1, \\ \frac{\partial}{\partial z_j} \left(-\frac{\partial u}{\partial z_k}(P(z)) \right) = -\frac{\partial}{\partial z_j} \left(\left(\frac{\partial u}{\partial z_k} \circ P \right)(z) \right), & k = n, \end{cases} \\ &= \begin{cases} -\sum_{l=1}^n \left(\frac{\partial^2 u}{\partial z_l \partial \bar{z}_k}(P(z)) \frac{\partial P_l}{\partial z_j}(z) + \frac{\partial^2 u}{\partial \bar{z}_l \partial \bar{z}_k}(P(z)) \frac{\partial \bar{P}_l}{\partial z_j}(z) \right), & k = 1, \dots, n-1, \\ -\sum_{l=1}^n \left(\frac{\partial^2 u}{\partial z_l \partial z_k}(P(z)) \frac{\partial P_l}{\partial z_j}(z) + \frac{\partial^2 u}{\partial \bar{z}_l \partial z_k}(P(z)) \frac{\partial \bar{P}_l}{\partial z_j}(z) \right), & k = n, \end{cases} \\ &= \begin{cases} -\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(P(z)), & j, k = 1, \dots, n-1, \\ -\frac{\partial^2 u}{\partial \bar{z}_j \partial \bar{z}_k}(P(z)), & j = n, k = 1, \dots, n-1, \\ -\frac{\partial^2 u}{\partial z_j \partial z_k}(P(z)), & j = 1, \dots, n-1, k = n, \\ -\frac{\partial^2 u}{\partial \bar{z}_j \partial z_k}(P(z)), & j, k = n. \end{cases} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 \hat{u}}{\partial z_j \partial \bar{z}_k}(z) &= \frac{\partial}{\partial z_j} \left(\frac{\partial \hat{u}}{\partial \bar{z}_k}(z) \right) \\ &= \begin{cases} \frac{\partial}{\partial z_j} \left(\frac{\partial u}{\partial \bar{z}_k}(P(z)) \right) = \frac{\partial}{\partial z_j} \left(\left(\frac{\partial u}{\partial \bar{z}_k} \circ P \right)(z) \right), & k = 1, \dots, n-1, \\ \frac{\partial}{\partial z_j} \left(\frac{\partial u}{\partial z_k}(P(z)) \right) = \frac{\partial}{\partial z_j} \left(\left(\frac{\partial u}{\partial z_k} \circ P \right)(z) \right), & k = n, \end{cases} \\ &= \begin{cases} \sum_{l=1}^n \left(\frac{\partial^2 u}{\partial z_l \partial \bar{z}_k}(P(z)) \frac{\partial P_l}{\partial z_j}(z) + \frac{\partial^2 u}{\partial \bar{z}_l \partial \bar{z}_k}(P(z)) \frac{\partial \bar{P}_l}{\partial z_j}(z) \right), & k = 1, \dots, n-1, \\ \sum_{l=1}^n \left(\frac{\partial^2 u}{\partial z_l \partial z_k}(P(z)) \frac{\partial P_l}{\partial z_j}(z) + \frac{\partial^2 u}{\partial \bar{z}_l \partial z_k}(P(z)) \frac{\partial \bar{P}_l}{\partial z_j}(z) \right), & k = n, \end{cases} \end{aligned}$$

$$= \begin{cases} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(P(z)), & j, k = 1, \dots, n-1, \\ \frac{\partial^2 u}{\partial \bar{z}_j \partial \bar{z}_k}(P(z)), & j = n, k = 1, \dots, n-1, \\ \frac{\partial^2 u}{\partial z_j \partial z_k}(P(z)), & j = 1, \dots, n-1, k = n, \\ \frac{\partial^2 u}{\partial \bar{z}_j \partial z_k}(P(z)), & j, k = n. \end{cases}$$

Therefore, we have verified formulas (2.13) and (2.14) from which formula (2.15) follows immediately. \square

Remark 2.4. We observed in Remark 2.2 that for the first order formulas of the reflected functions we can avoid the use of the complex chain rules in the proofs. It is similarly possible now for the second order formulas (2.13) and (2.14). For example, it follows from [12, formula (3.8) of Lemma 3.7] that

$$\begin{aligned} \frac{\partial^2 \tilde{u}}{\partial z_j \partial \bar{z}_k}(z) &= \frac{1}{4} \left(\frac{\partial^2 \tilde{u}}{\partial x_j \partial x_k}(z) + i \frac{\partial^2 \tilde{u}}{\partial x_j \partial y_k}(z) - i \frac{\partial^2 \tilde{u}}{\partial y_j \partial x_k}(z) + \frac{\partial^2 \tilde{u}}{\partial y_j \partial y_k}(z) \right) \\ &= \begin{cases} \frac{1}{4} \left(-\frac{\partial^2 u}{\partial x_j \partial x_k}(P(z)) - i \frac{\partial^2 u}{\partial x_j \partial y_k}(P(z)) \right. \\ \quad \left. + i \frac{\partial^2 u}{\partial y_j \partial x_k}(P(z)) - \frac{\partial^2 u}{\partial y_j \partial y_k}(P(z)) \right), & j, k = 1, \dots, n-1, \\ \frac{1}{4} \left(-\frac{\partial^2 u}{\partial x_j \partial x_k}(P(z)) - i \frac{\partial^2 u}{\partial x_j \partial y_k}(P(z)) \right. \\ \quad \left. - i \frac{\partial^2 u}{\partial y_j \partial x_k}(P(z)) + \frac{\partial^2 u}{\partial y_j \partial y_k}(P(z)) \right), & j = n, k = 1, \dots, n-1, \\ \frac{1}{4} \left(-\frac{\partial^2 u}{\partial x_j \partial x_k}(P(z)) + i \frac{\partial^2 u}{\partial x_j \partial y_k}(P(z)) \right. \\ \quad \left. + i \frac{\partial^2 u}{\partial y_j \partial x_k}(P(z)) + \frac{\partial^2 u}{\partial y_j \partial y_k}(P(z)) \right), & j = 1, \dots, n-1, k = n \\ \frac{1}{4} \left(-\frac{\partial^2 u}{\partial x_j \partial x_k}(P(z)) + i \frac{\partial^2 u}{\partial x_j \partial y_k}(P(z)) \right. \\ \quad \left. - i \frac{\partial^2 u}{\partial y_j \partial x_k}(P(z)) - \frac{\partial^2 u}{\partial y_j \partial y_k}(P(z)) \right), & j, k = n. \end{cases} \\ &= \begin{cases} -\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(P(z)), & j, k = 1, \dots, n-1, \\ -\frac{\partial^2 u}{\partial \bar{z}_j \partial \bar{z}_k}(P(z)), & j = n, k = 1, \dots, n-1, \\ -\frac{\partial^2 u}{\partial z_j \partial z_k}(P(z)), & j = 1, \dots, n-1, k = n, \\ -\frac{\partial^2 u}{\partial \bar{z}_j \partial z_k}(P(z)), & j, k = n. \end{cases} \end{aligned}$$

3. Reflection Principle for Pluriharmonic Functions

Let $\Omega \subset \mathbb{C}^n$ be open. A C^2 -function $u: \Omega \rightarrow \mathbb{R}$ is *pluriharmonic* in Ω if for all $j, k = 1, \dots, n$,

$$\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) = 0 \tag{3.1}$$

at every $z \in \Omega$. Equivalently, a function u is pluriharmonic in Ω if and only if for every $z \in \Omega$ and $w \in \mathbb{C}^n \setminus \{0\}$ the function $\lambda \mapsto u(z + \lambda w)$ is harmonic on $\{\lambda \in \mathbb{C} : z + \lambda w \in \Omega\}$. This means that a pluriharmonic function is harmonic when restricted to any complex line inside Ω .

Pluriharmonic functions form an invariant class under biholomorphic mappings. Therefore, in several complex variables pluriharmonic functions is a more important class of functions than harmonic functions which are not invariant under holomorphic mappings or even under complex linear mappings.

Pluriharmonic functions are harmonic in the sense of real coordinates. Hence the classical reflection principle for harmonic functions imply that a

pluriharmonic function given in an open $G_+ \subset \mathbb{C}_+^n$ has a harmonic extension \tilde{u} to the reflected domain G but it is not straightforward that \tilde{u} is pluriharmonic in G . However, using the classical reflection principle for harmonic functions, it is simple to prove that the reflection principle holds for pluriharmonic functions. The crucial observation in the proof is that the reflected function \tilde{u} is C^∞ in G , and hence the second order derivatives exist in G_0 and they are limits of the second order derivatives in $G_+ \cup G_-$. In general, if a function u is C^2 in G_- , it may happen that \tilde{u} is not differentiable in a point $z_0 \in G_0$, see Example 5.1.

Theorem 3.1. *If $u: G_+ \rightarrow \mathbb{R}$ is pluriharmonic and the boundary condition (2.1) holds, then the odd reflected function \tilde{u} is pluriharmonic in G .*

Proof. Suppose that $u: G_+ \rightarrow \mathbb{R}$ is pluriharmonic and the boundary condition (2.1) holds. Since u is harmonic in G , the classical reflection principle for harmonic functions implies that \tilde{u} is harmonic in G . It follows that \tilde{u} is C^∞ in G .

Let $z \in G_-$. Since u is pluriharmonic in $P(z)$, formula (2.13) yields

$$\frac{\partial^2 \tilde{u}}{\partial z_j \partial \bar{z}_k}(z) = -\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(P(z)) = 0$$

for every $j, k = 1, \dots, n - 1$. Since \tilde{u} is C^∞ in G , we may change the order of the partial differentiation, and we have again by formula (2.13) that

$$\frac{\partial^2 \tilde{u}}{\partial z_n \partial \bar{z}_n}(z) = -\frac{\partial^2 u}{\partial \bar{z}_n \partial z_n}(P(z)) = -\frac{\partial^2 u}{\partial z_n \partial \bar{z}_n}(P(z)) = 0.$$

Moreover, since u is C^∞ and harmonic in G_+ , formulas in Remark 2.4 yield

$$\frac{\partial^2 \tilde{u}}{\partial z_j \partial \bar{z}_k}(z) = -\frac{\partial^2 u}{\partial \bar{z}_j \partial \bar{z}_k}(P(z)) = -\frac{\partial^2 u}{\partial z_j \partial z_k}(P(z)) = 0$$

whenever $j = n, k = 1, \dots, n - 1$ or $j = 1, \dots, n - 1, k = n$. Hence the Eq. (3.1) holds for all $j, k = 1, \dots, n$ at every $z \in G_+ \cup G_-$, in other words, \tilde{u} is pluriharmonic in $G_+ \cup G_-$.

Let $z_0 \in G_0$. Since \tilde{u} is C^∞ in G , we obtain

$$\frac{\partial^2 \tilde{u}}{\partial z_j \partial \bar{z}_k}(z_0) = \lim_{\substack{z \rightarrow z_0 \\ z \in G}} \frac{\partial^2 \tilde{u}}{\partial z_j \partial \bar{z}_k}(z) = 0$$

for all $j, k = 1, \dots, n$. Hence the Eq. (3.1) holds in G for all $j, k = 1, \dots, n$, and thus \tilde{u} is pluriharmonic in G by definition. \square

Remark 3.2. Armitage [1] showed that the classical reflection principle for h harmonic in G_+ holds when one assumes (instead of the boundary condition (2.1), that is, h tending to 0 at each point of $G_0 \subset \partial \mathbb{R}_+^n$) that h converges locally in mean to 0 on G_0 , that is, for all $(x, 0) \in G_0$ there exists $r > 0$ such that

$$\lim_{t \rightarrow 0^+} \int_{|y-x| < r} h(y, t) dy = 0. \tag{3.2}$$

It is clear that the boundary condition (2.1) is stronger than the boundary condition (3.2). Therefore, in Theorem 4.1, it is sufficient to assume that the boundary condition (3.2) holds.

The classical reflection principle for harmonic functions can be proved by using the mean value principle of harmonic functions, see [2, Proof of Theorem 1.3.6]. It is remarkable that for the proof of the reflection principle for pluriharmonic functions (Theorem 4.1) the mean value principle can not be applied in G_0 . This is because if we take points $z_0 \in G_0$ and $z \in G_+$, then the reflected point $P(z) \in G_-$ is not usually in an arbitrary complex line passing through the points z_0 and z .

4. Reflection Principle for Plurisubharmonic Functions

Let Ω be an open set in \mathbb{C}^n . An upper semicontinuous function $u: \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$ which is not identically $-\infty$ on any component of Ω is said to be *plurisubharmonic* in Ω if for each $z \in \Omega$ and $w \in \mathbb{C}^n$, the function $\lambda \mapsto u(z + \lambda w)$ is subharmonic or identically $-\infty$ on every component of the set $\{\lambda \in \mathbb{C}: z + \lambda w \in \Omega\}$. A function v is called *plurisuperharmonic* in Ω if $-v$ is plurisubharmonic in Ω .

We need to restrict our considerations here to plurisubharmonic functions having nonnegative values only. The set of nonnegative reals is denoted by $\mathbb{R}_+ = \{x \in \mathbb{R}: x \geq 0\}$.

Theorem 4.1. *If $u: G_+ \rightarrow \mathbb{R}_+$ is plurisubharmonic and the boundary condition (2.1) holds, then the even reflected function \hat{u} is plurisubharmonic in G .*

Proof. Upper semicontinuity of u in G_+ implies that \hat{u} is upper semicontinuous in G_- . Therefore \hat{u} is upper semicontinuous in $G = G_+ \cup G_0 \cup G_-$ since \hat{u} is continuous in a neighbourhood of G_0 . If $z \in G_-$ and $w \in \mathbb{C}^n$ are such that $\{z + \lambda w: \lambda \in \mathbb{C}, |\lambda| \leq 1\} \subset G_-$, then

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \hat{u}(z + e^{it}) dt &= \frac{1}{2\pi} \int_0^{2\pi} u(P(z + e^{it})) dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(P(z) + e^{it}) dt \\ &\geq u(P(z)) = \hat{u}(z) \end{aligned} \tag{4.1}$$

because $P(z) \in G_+$, $\{P(z) + \lambda w: \lambda \in \mathbb{C}, |\lambda| \leq 1\} \subset G_+$ and u is plurisubharmonic in G_+ . Hence the plurisubmean value principle [11, Theorem 2.9.1] holds for \hat{u} in $G_+ \cup G_-$, since it is clear that it holds in G_+ where $\hat{u} = u$ is plurisubharmonic. It holds also for every $z_0 \in G_0$ since

$$\hat{u}(z_0) = 0 \leq \frac{1}{2\pi} \int_0^{2\pi} \hat{u}(z_0 + e^{it}) dt$$

because $\hat{u} \geq 0$ in $\{z + \lambda w: \lambda \in \mathbb{C}, |\lambda| \leq 1\}$ whenever it is inside G . Hence \hat{u} is plurisubharmonic in G . □

Remark 4.2. Since the even reflection preserves upper semicontinuity, it follows from (4.1) that the following more general observation is valid: If u is plurisubharmonic in an open set $U \subset \mathbb{C}_+^n$, then \hat{u} is plurisubharmonic in $PU \subset \mathbb{C}_-^n$.

5. Reflection Principle for the Classical Homogeneous Complex Monge–Ampère Equation

In this section we study reflection principle for the classical homogeneous complex Monge–Ampère equation

$$\det \left[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right] = 0$$

whenever u is twice differentiable. However, we immediately meet several principal difficulties which are not present in the corresponding theory of the homogeneous Laplace equation. Studies of the reflection principle for the homogeneous Laplace equation and for harmonic functions coincide, but in case of the homogeneous complex Monge–Ampère equation and pluriharmonic functions the situation is different and more complicated.

If a plurisubharmonic function u is a solution to the homogeneous complex Monge–Ampère equation in G_+ , then the odd reflected function \tilde{u} is plurisuperharmonic in G_- . Hence it is worthwhile that we first restrict our study to the case where the plurisubharmonic solution of the homogeneous complex Monge–Ampère equation is in the class C^2 in G_+ . Therefore we consider the strong pointwise (i.e. classical) form of the complex Monge–Ampère equation.

Example 5.1. Consider the functions $v, w: \mathbb{C}_+^2 \rightarrow \mathbb{R}$, $v(z) = v(z_1, z_2) = \frac{z_2 - \bar{z}_2}{2i} = \text{Im } z_2$ and $w(z) = w(z_1, z_2) = z_1 \bar{z}_1$ where $z = (z_1, z_2) \in \mathbb{C}_+^2$. Then

$$\det \left[\frac{\partial^2 v}{\partial z_j \partial \bar{z}_k} (z) \right] = \det \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

and

$$\det \left[\frac{\partial^2 w}{\partial z_j \partial \bar{z}_k} (z) \right] = \det \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

at every $z \in \mathbb{C}_+^2$, hence both v and w are classical solutions of the homogeneous complex Monge–Ampère equation in \mathbb{C}_+^2 . Moreover, the boundary condition (2.1) holds for v and w .

For the even reflected function \hat{v} we have for each $z \in \mathbb{C}_-^2$ that

$$\hat{v}(z) = v(P(z)) = v(z_1, \bar{z}_2) = \frac{\bar{z}_2 - z_2}{2i} = -\text{Im } z_2,$$

and hence the first order derivative $\frac{\partial \hat{v}}{\partial z_2}(z) = -\frac{1}{2i}$. But since for each $z \in \mathbb{C}_+^2$ we have $\frac{\partial \hat{v}}{\partial z_2}(z) = \frac{1}{2i}$, the first order derivative $\frac{\partial \hat{v}}{\partial z_2}(z_0)$ does not exist in any $z_0 \in \partial \mathbb{C}_+^2$. Consequently, \hat{v} is not C^1 in any neighbourhood of $\partial \mathbb{C}_+^2$.

For the odd reflected function \tilde{w} we have for each $z \in \mathbb{C}_-^2$ that

$$\det \left[\frac{\partial^2 \tilde{w}}{\partial z_j \partial \bar{z}_k} (z) \right] = \det \left[\frac{\partial^2 (-w)}{\partial z_j \partial \bar{z}_k} (P(z)) \right] = \det \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

Therefore the limit

$$\lim_{z \rightarrow z_0} \det \left[\frac{\partial^2 \tilde{w}}{\partial z_j \partial \bar{z}_k} (z) \right] = 0 \quad \text{at every } z_0 \in \partial \mathbb{C}_+^2.$$

However, the second order partial derivative $\frac{\partial^2 \tilde{w}}{\partial z_1 \partial \bar{z}_1} (z_0)$ does not exist in any $z_0 \in \partial \mathbb{C}_+^2$, because the limit 1 when approaching to z_0 from above does not equal to the limit -1 from below. Consequently, \tilde{w} is not C^2 in any neighbourhood of $\partial \mathbb{C}_+^2$.

The next theorem states our main result of this section.

Theorem 5.2. *Suppose that $u \in C^2(G_+)$ satisfies the homogeneous Monge–Ampère equation $\det \left[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right] = 0$ in G_+ and the boundary condition (2.1) holds.*

- (i) *The odd reflected function \tilde{u} satisfies the equation $\det \left[\frac{\partial^2 \tilde{u}}{\partial z_j \partial \bar{z}_k} \right] = 0$ in G if and only if the second order derivatives $\frac{\partial^2 \tilde{u}}{\partial z_j \partial \bar{z}_k}$ exist at every $z_0 \in G_0$.*
- (ii) *The even reflected function \hat{u} satisfies the equation $\det \left[\frac{\partial^2 \hat{u}}{\partial z_j \partial \bar{z}_k} \right] = 0$ in G if and only if the second order derivatives $\frac{\partial^2 \hat{u}}{\partial z_j \partial \bar{z}_k}$ exist at every $z_0 \in G_0$.*

Proof. (i) Since $\tilde{u}(z) = u(z)$ for every $z \in G_+$, it is clear that \tilde{u} satisfies the homogeneous Monge–Ampère equation in G_+ . If $z \in G_-$, then

$$\det \left[\frac{\partial^2 \tilde{u}}{\partial z_j \partial \bar{z}_k} (z) \right] = \det \left[\frac{\partial^2 (-u)}{\partial z_j \partial \bar{z}_k} (P(z)) \right] = - \det \left[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} (P(z)) \right] = 0,$$

because $P(z) \in G_+$. Suppose that $z_0 \in G_0$. Then the limit from above or below as well as from the left or the right is

$$\lim_{\substack{z \rightarrow z_0 \\ z \in G_+ \cup G_-}} \det \left[\frac{\partial^2 \tilde{u}}{\partial z_j \partial \bar{z}_k} (z) \right] = \lim_{\substack{z \rightarrow z_0 \\ z \in G_0}} \det \left[\frac{\partial^2 \tilde{u}}{\partial z_j \partial \bar{z}_k} (z) \right] = 0,$$

since $\tilde{u} \equiv 0$ in G_0 . Therefore

$$\lim_{z \rightarrow z_0} \det \left[\frac{\partial^2 \tilde{u}}{\partial z_j \partial \bar{z}_k} (z) \right] = 0$$

and the assertion (i) holds. The proof of (ii) is similar. □

Remark 5.3. If the assumptions of Theorem 5.2 are valid, it is possible due to Example 5.1 that \tilde{u} and \hat{u} are not C^2 in any neighbourhood of G_0 .

6. Reflection Principle for the Generalized Complex Monge–Ampère Equation

Finally, we consider reflection principle for the homogeneous generalized complex Monge–Ampère equation, analogously to solutions of certain elliptic partial differential equations, see [9] and [14]. A plurisubharmonic and locally bounded function $u: \Omega \rightarrow [-\infty, \infty)$ can be operated by the *generalized complex Monge–Ampère operator* $(dd^c)^n$, see [11, Section 3.4]. If $u \in C^2(\Omega)$, then

$$(dd^c u)^n = 4^n n! \det \left[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right] dV, \tag{6.1}$$

where dV is the volume form in \mathbb{C}^n .

The primary definition of the generalized complex Monge–Ampère operator was given by Bedford and Taylor [4]. Later on, Cegrell [5] introduced a slightly more general and in some sense optional definition of $(dd^c)^n$. Then the plurisubharmonic function u is not required to be locally bounded but the definition is still not valid for all plurisubharmonic functions given in a general open set $\Omega \subset \mathbb{C}^n$. If u is plurisubharmonic and locally bounded in Ω , then $(dd^c u)^n$ gives a nonnegative Borel measure on Ω . Contrary to this, if u is plurisuperharmonic and locally bounded in Ω , then $(dd^c u)^n$ gives a non-positive Borel measure on Ω .

A plurisubharmonic and locally bounded function $u: \Omega \rightarrow [-\infty, \infty)$ is said to satisfy the *homogeneous generalized complex Monge–Ampère equation* $(dd^c u)^n = 0$ in Ω if

$$\int_{\Omega} (dd^c u)^n = 0. \tag{6.2}$$

In other words, the *Monge–Ampère mass* of u in Ω is zero if u satisfies the Eq. (6.2). Our first result is an easy observation concerning the reflection principle for the homogeneous generalized complex Monge–Ampère equation. Note that every pluriharmonic function satisfies the Eq. (6.2), but the converse is not true.

Theorem 6.1. *If u is pluriharmonic in G_+ such that the boundary condition (2.1) holds, then the odd reflected function \tilde{u} satisfies the homogeneous generalized complex Monge–Ampère equation $(dd^c \tilde{u})^n = 0$ in G .*

Proof. By Theorem 4.1, \tilde{u} is pluriharmonic in G . Hence \tilde{u} satisfies the homogeneous generalized complex Monge–Ampère equation $(dd^c \tilde{u})^n = 0$ in G . □

A plurisubharmonic function $u: \Omega \rightarrow \mathbb{R}$ is said to be *maximal* if for each open set $D \Subset \Omega$ and for each upper semicontinuous function v on \bar{D} ,

$$v \text{ is plurisubharmonic in } D \text{ and } v \leq u \text{ on } \partial D \implies v \leq u \text{ in } D.$$

It is equivalent to require that

$$\begin{aligned} &v \text{ is plurisubharmonic in } \Omega, \ D \Subset \Omega \text{ is open and } v \leq u \text{ on } \partial D \\ &\implies v \leq u \text{ in } D, \end{aligned}$$

see [11, Proposition 3.1.1].

The notion of maximal plurisubharmonic functions is due to Sadullæv [18]. In one complex variable, the maximal plurisubharmonic functions are precisely the harmonic functions, hence solutions to the Laplace equation $\Delta u = 0$ and consequently they belong to $C^\infty(\Omega)$. In more than one variable, the class contains, for example, all plurisubharmonic functions which depend on $n - 1$ variables only.

It is known that a locally bounded plurisubharmonic function u in Ω satisfies the homogeneous generalized complex Monge–Ampère equation (6.2) if and only if u is maximal [11, Theorem 4.4.2]. Moreover, we observe that if a locally bounded plurisubharmonic function u in G_+ is nonnegative and satisfies the boundary property (2.1), then it is plurisuperharmonic in Ω_- . Therefore our main result regarding the reflection principle for the generalized complex Monge–Ampère equation can be stated for the even reflected functions only.

Theorem 6.2. *Let u be a nonnegative, locally bounded and plurisubharmonic function in G_+ such that the boundary condition (2.1) holds. If u is maximal in G_+ , then \hat{u} is maximal in G .*

Proof. Let u be maximal in G_+ . By Theorem 4.1 the even reflected function \hat{u} is plurisubharmonic in G . It is clear that the restriction $\hat{u}|_{G_-}$ is plurisubharmonic and maximal in G_- .

Suppose by contradiction that \hat{u} is not maximal in G . Then there exist a plurisubharmonic function v in G , an open set $D \Subset G$ and a point $z_0 \in D$ such that $v \leq \hat{u}$ on ∂D but $v(z_0) > \hat{u}(z_0)$. If $z_0 \in G_+$, then $u = \hat{u}|_{G_+}$ is not maximal in G_+ , and if $z_0 \in G_-$, then $\hat{u}|_{G_-}$ is not maximal in G_- , which are both contradictions. Finally, suppose that $z_0 \in G_0$. Since v is upper semicontinuous, there is a point $z_1 \in G_+ \cap D$ such that $v(z_1) > \hat{u}(z_1) = u(z_1)$. By considering the function

$$w(z) = \max\{v(z), u(z)\}, \quad z \in G_+,$$

which is plurisubharmonic in G_+ , we see that u is not maximal in G_+ . This is again a contradiction. \square

Corollary 6.3. *Let u be a nonnegative, locally bounded and plurisubharmonic function in G_+ such that the boundary condition (2.1) holds. If u satisfies the homogeneous generalized complex Monge–Ampère equation $(dd^c u)^n = 0$ in G_+ , then the even reflected function \hat{u} satisfies the same equation $(dd^c \hat{u})^n = 0$ in G .*

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