Adv. Appl. Clifford Algebras 27 (2017), 1369–1386

0188-7009/021369-18
published online April 1, 2016
DOI 10.1007/s00006-016-0658-x

Advances in Applied Clifford Algebras

On the Geometry of the Hyperbolic Scator Space in 1+2 Dimensions

Artur Kobus and Jan L. Cieśliński*

Abstract. We consider the scator space in 1+2 dimensions—a hypercomplex, non-distributive hyperbolic algebra introduced by Fernández-Guasti and Zaldívar. We find a method for treating scators algebraically by embedding them into a distributive and commutative algebra. A notion of dual scators is introduced and discussed. We also study isometries of the scator space. It turns out that zero divisors cannot be avoided while dealing with these isometries. The scator algebra may be endowed with a nice physical interpretation, although it suffers from lack of some physically demanded important features. Despite that, there arise some open questions, e.g., whether hypothetical tachyons can be considered as usual particles possessing time-like trajectories.

Mathematics Subject Classification. 30G35, 20M14.

Keywords. Scators, Tachyons, Non-distributive algebras, Hypercomplex numbers.

1. Introduction

Following Fernández-Guasti and Zaldívar [1] we consider a commutative, non-distributive $1+2$ dimensional algebra S, which is also associative provided that divisors of zero are excluded. The elements of this algebra will be called $1+2$ dimensional scators [1]. Scators are denoted by $\overset{o}{a} = (a_0; a_1, a_2)$, where components a_1, a_2 are referred to as *director* components, and a_0 is usually called *scalar* component, or, in physical context, *temporal* component. The space of scators possesses the additive structure of the usual vector space and a specific non-distributive product. Scalars form a subset of the scator space closed under addition and multiplication. In this paper we confine ourselves to this definition of a scator, leaving aside earlier concepts of scators as objects generalizing scalars and vectors, see [2,3]. Dealing with scators, one has to be aware that they are defined in a fixed reference frame (although some analogue of the Lorentz transformation can be introduced).

*Corresponding author.

It was shown in [4] that this algebra may be given physical interpretation corresponding to some ideas of special relativity, although the metric in the scator space is different from the standard metric of Minkowski space. This scator metric (defined below) is called a scator-deformed Lorentz metric. It was emphasized that scators and deformed metrics can describe a kinematics of some kind of particles, including hypothetical tachyons.

In our paper we study metric properties of scators, paying special attention to proper definitions for causal realms appearing in this framework, what finally will lead to some convergence with work of Kapuścik [5].

To begin with, we reconsider some main points in reasoning leading to physical interpretation of scators, as introduced in [1]. The idea of describing physical phenomena along lines of the scator framework takes its origin in the fact that multiplication of scators mimics the shape of velocity addition formula known from special relativity (SR). Adapting popular convention $c = 1$ (speed of light being unity), we denote scators by overset o symbol, so that

$$\overset{o}{a} = (a_0; a_1, a_2) = a_0 \left(1; \frac{a_1}{a_0}, \frac{a_2}{a_0}\right), \quad \overset{o}{b} = (b_0; b_1, b_2) = b_0 \left(1; \frac{b_1}{b_0}, \frac{b_2}{b_0}\right), \quad (1.1)$$

since multiplication by a scalar acts on scators in usual component-wise way. Here also we introduce notation $\beta_{a1} = \frac{a_1}{a_0}$, $\beta_{b1} = \frac{b_1}{b_0}$, etc. We will refer to objects of the form $\frac{\overset{o}{a}}{a_0} = (1; \beta_{a1}, \beta_{a2})$, as velocity scators. We should show now that the composition rule for the velocity scators yields the Einstein law of addition of velocities. Moreover, properly applied, it recovers Lorentz transformation of physical measurements from one inertial frame to another.

Hence we define

$$\overset{o}{a}\overset{o}{b} = (a_0; a_1, a_2)(b_0; b_1, b_2) = \left(a_0 b_0 + a_1 b_1 + a_2 b_2 + \frac{a_1 a_2 b_1 b_2}{a_0 b_0}; a_0 b_1 + a_1 b_0 \right.$$
$$\left. + \frac{a_1 a_2 b_2}{a_0} + \frac{a_2 b_1 b_2}{b_0}, a_0 b_2 + a_2 b_0 + \frac{a_1 a_2 b_1}{a_0} + \frac{a_1 b_1 b_2}{b_0}\right), \quad (1.2)$$

and, if we denote $\overset{o}{c} = \overset{o}{a}\overset{o}{b}$, we can rewrite the above formula in the component form

$$\begin{aligned} c_0 &= a_0 b_0 \left(1 + \frac{a_1 b_1}{a_0 b_0}\right)\left(1 + \frac{a_2 b_2}{a_0 b_0}\right), \\ c_1 &= (a_0 b_1 + a_1 b_0)\left(1 + \frac{a_2 b_2}{a_0 b_0}\right), \\ c_2 &= (a_0 b_2 + a_2 b_0)\left(1 + \frac{a_1 b_1}{a_0 b_0}\right), \end{aligned} \quad (1.3)$$

which easily reads for velocity scators $\overset{o}{a}, \overset{o}{b}, \overset{o}{c}$, that, for example

$$\beta_{c1} = \frac{c_1}{c_0} = \frac{\beta_{a1} + \beta_{b1}}{1 + \beta_{a1}\beta_{b1}}, \quad \beta_{c2} = \frac{c_2}{c_0} = \frac{\beta_{a2} + \beta_{b2}}{1 + \beta_{a2}\beta_{b2}}, \quad (1.4)$$

what reminds the formula for velocity addition in SR. Using these definitions we have to require $a_0 \neq 0, b_0 \neq 0$ (this is a crucial assumption).

It is worth stressing that this structure agrees with SR in $1+1$ dimensions only. Results produced by the multiplication formula for higher dimensions can be interpreted as a deformation of the standard Lorentz metric. The latter is obtained in the limit $a_0 \to \infty$, $b_0 \to \infty$ [4]. Therefore, the scator structure is compatible with the standard physical interpretation only in the $1+1$ case. Higher-dimensional generalizations should be treated with care.

We emphasize the fact that multiplication acts in a non-distributive way, what is the hallmark of the structure. Therefore, computing

$$\Delta(\overset{o}{a}, \overset{o}{b}; \overset{o}{c}) := (\overset{o}{a} + \overset{o}{b})\overset{o}{c} - \overset{o}{a}\overset{o}{c} - \overset{o}{b}\overset{o}{c} = \frac{(b_0 a_1 - a_0 b_1)(a_0 b_2 - b_0 a_2)}{a_0 b_0 (a_0 + b_0)} \left(\frac{c_1 c_2}{c_0}; c_2, c_1 \right), \tag{1.5}$$

where $\overset{o}{c} = (c_0; c_1, c_2)$, we easily see that in the generic case the right-hand side does not vanish, i.e., $(\overset{o}{a} + \overset{o}{b})\overset{o}{c} \neq \overset{o}{a}\overset{o}{c} + \overset{o}{b}\overset{o}{c}$ provided that $b_0 a_1 \neq a_0 b_1$ and $a_0 b_2 \neq b_0 a_2$. Hence, in general, the scator product is not distributive. By the way, the scator appearing on the right-hand side of (1.5) will be referred to as dual to $\overset{o}{c}$ (see Definition 2.1 below).

Many properties of the scator product (1.2) were widely investigated in many contexts [1,8,9], also physical [4]. To gain some insight in possible physical interpretation of the scator algebra, we recall here some basic terminology from [1] and [9]:

Definition 1.1. The modulus squared of a scator is given by

$$\|\overset{o}{a}\|^2 = \overset{o}{a}\overset{o}{a}{}^* = a_0^2 \left(1 - \frac{a_1^2}{a_0^2}\right)\left(1 - \frac{a_2^2}{a_0^2}\right) = a_0^2 - a_1^2 - a_2^2 + \frac{a_1^2 a_2^2}{a_0^2}. \tag{1.6}$$

Definition 1.2. We say that a scator is time-like, if

$$a_0^2 > a_1^2, \quad \text{and} \quad a_0^2 > a_2^2, \quad \text{or} \quad a_0^2 < a_2^2, \quad \text{and} \quad a_0^2 < a_1^2, \tag{1.7}$$

it is said to be space-like, if

$$a_0^2 < a_1^2, \quad \text{and} \quad a_0^2 > a_2^2 \quad \text{or} \quad a_0^2 < a_2^2, \quad \text{and} \quad a_0^2 > a_1^2, \tag{1.8}$$

and it is light-like, if

$$a_0^2 = a_1^2, \quad \text{or} \quad a_0^2 = a_2^2. \tag{1.9}$$

The classification proposed above is analogous to what is well known from special relativity. Note that in both cases componentwise addition of scators or vectors does not preserve this classification. However, the norm of the product of two scators is just the product of their norms [9], which is very useful in this context. In this paper we present new results concerning the metric structure of the scator space, extending results of [4,9].

Note the base difference between what is proposed above and what was done in [8]: in the cited paper classification of scators was performed using components of the objects and quite sensible assumption that nothing observable can happen outside of the light bipyramid. As opposite, here we perform a classification in terms of the deformed metric (1.6) only, what has its formal consequences in the appearance of some new causal realms, with potential physical interpretations.

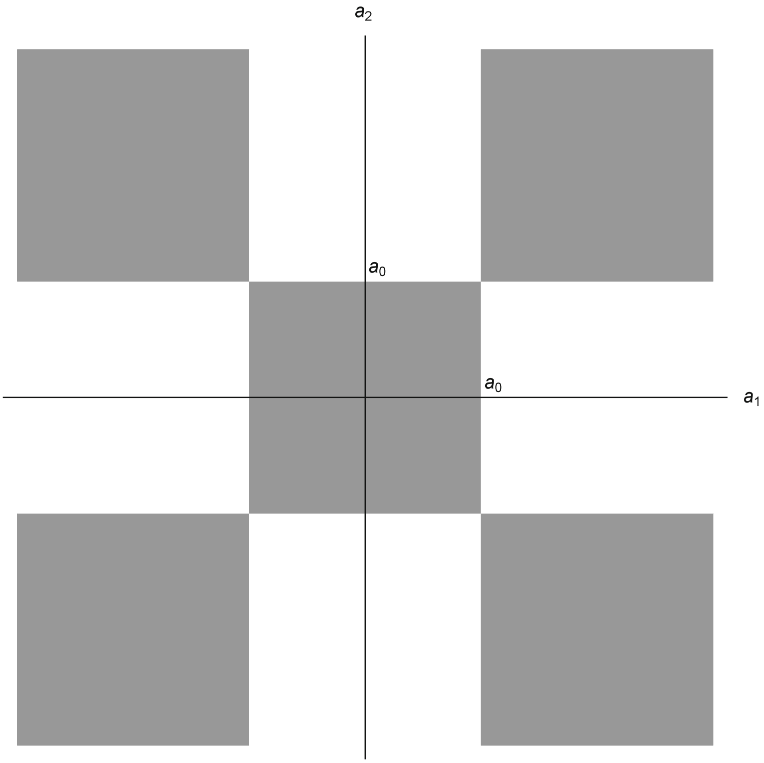

FIGURE 1. Time-like wings at a fixed time a_0 ($a_0 \neq 0$). Duality operations (sect. 2) take us out from the bipyramid (represented in this cross-section by the inner square) to other causal realms, and conversely

For instance, we indicate that the bipyramid considered in [4] has some kind of time-like "wings" around, described by the regime $a_0^2 < a_1^2$ and $a_0^2 < a_2^2$, while, for example, in [9], there were considered mainly time-like events inside the light bipyramid ($a_0^2 > a_1^2$ and $a_0^2 > a_2^2$). Time-like region is marked as dark at Fig. 1.

We point out that the existence of such wings is not mentioned in the paper [4]. Probably this possibility was excluded from considerations by assumption. However, the wings seem to reveal some new aspects of causality in scator-deformed Lorentz metric (see Fig. 1) and in next sections we propose a possible physical interpretation.

A closely related notion, "super-restricted space conditions", was introduced in [9] without a direct relation to the type of considered events. Super-restricted space conditions define either time-like events (in even-dimensional spaces) or space-like events (in odd-dimensional spaces).

Note also important formal consequences of the proposed framework:

- The scator space is not invariant with respect to rotations (see Fig. 1). It may remind an archaic approach of Hamilton, where vector's components were treated as simple scalars.
- In the scator framework we do not expect continuous symmetries properties of any kind and the summation convention have to be dropped. In this paper we introduce and study a number of discrete symmetries, instead.
- In order to not affect physics by dilations, in the scator framework Lorentz boosts must be represented by unit-magnitude elements [4]. Good candidate for this is normalized velocity scator $\overset{o}{U} = \frac{1}{c}\frac{d\overset{o}{x}}{d\tau}$, where the proper time τ is defined in perfect analogy with this of SR, but here we use the scator-deformed Lorentz metric.
- In [4] it was also shown that under a boost we have

$$\overset{o}{x}' = \overset{o}{x}\overset{o}{U}^{*} = (x_0; x_1, x_2, x_3)\gamma_u(1; -\beta_{u1}, -\beta_{u2}, -\beta_{u3}),$$

  where γ_u is deformed Lorentz factor (the norm of the velocity scator corresponding to the normalized scator $\overset{o}{U}$), and the conjugation is needed in order to keep the standard sign convention.
- Note that the above defined $\overset{o}{U}$ must be time-like, so that it excludes superluminal boosts from the game.

The paper is organized as follows. In Sect. 2, we introduce basic objects and transformations responsible for isometries in the scator space S. Then, in Sects. 3 and 4, we propose and develop a new framework in which calculation proceeds in a more natural way, using a distributive product. In Sect. 5, we continue along these lines focusing on isometries. Sect. 6 is entirely devoted to the question of metric properties of scators; in particular, we obtain the closest analogue of a scalar product we can get, although it is not bilinear. The last section contains physical comments and conclusions.

2. Dualities: Phenomenological Treatment

Now we turn our attention to the issue of isometries in the scator space S. We begin by defining some operations and then check their properties.

Definition 2.1. If we have a 3-scator $\overset{o}{a} = (a_0; a_1, a_2)$, then we call the scator of the form

$$\overset{o}{\bar{a}} = \left(\frac{a_1 a_2}{a_0}; a_2, a_1\right) \tag{2.1}$$

its dual (or ordinary dual) scator, and star denotes hypercomplex conjugate:

$$\overset{o}{a}^{*} = (a_0; -a_1, -a_2). \tag{2.2}$$

It follows straightforwardly that we have

Remark 2.2. Operation of duality commutes with hypercomplex conjugation.

Remark 2.3. Operation of duality is a second order involution: $\overline{\overset{o}{(\bar{a})}} = \overset{o}{a}$.

Remark 2.4. Hypercomplex conjugation is a multiplicative homomorphism in S, i.e.,

$$(\overset{o}{a}\overset{o}{b})^* = (\overset{o}{a})^*(\overset{o}{b})^*. \tag{2.3}$$

Remark 2.5. Operation of duality does not provide yet another homomorphic sctructure in S, i.e., $\overset{o}{\overline{ab}} \neq \overset{o}{\bar{a}}\overset{o}{\bar{b}}$.

The above statements follow from considerations similar to direct calculations included in [7]. Soon we will get a better understanding of these facts by applying a new approach which is both faster and simpler.

Proposition 2.6. *Operation of duality preserves the scator product:*

$$\overset{o}{\bar{a}}\overset{o}{\bar{b}} = \overset{o}{a}\overset{o}{b}. \tag{2.4}$$

Proof. We present explicit calculation for the scalar component:

$$(\overset{o}{a}\overset{o}{b})_0 = a_0 b_0 + a_1 b_1 + a_2 b_2 + \frac{a_1 a_2 b_1 b_2}{a_0 b_0}, \tag{2.5}$$

and

$$\begin{aligned}
(\overset{o}{\bar{a}}\overset{o}{\bar{b}})_0 &= \bar{a}_0 \bar{b}_0 + \bar{a}_1 \bar{b}_1 + \bar{a}_2 \bar{b}_2 + \frac{\bar{a}_1 \bar{a}_2 \bar{b}_1 \bar{b}_2}{\bar{a}_0 \bar{b}_0} \\
&= \frac{a_1 a_2 b_1 b_2}{a_0 b_0} + a_2 b_2 + a_1 b_1 + \frac{a_1 a_2 b_1 b_2}{\frac{a_1 a_2 b_1 b_2}{a_0 b_0}} \\
&= a_0 b_0 + a_1 b_1 + a_2 b_2 + \frac{a_1 a_2 b_1 b_2}{a_0 b_0} = (\overset{o}{a}\overset{o}{b})_0.
\end{aligned} \tag{2.6}$$

Similar direct computation can be done for director components. $\square$

Proposition 2.7. *Operation of duality is an isometry in 3-scator space.*

Proof. For ordinary scator $\overset{o}{a}$ we have its norm

$$\|\overset{o}{a}\|^2 = \overset{o}{a}\overset{o}{a}^* = a_0^2 \left(1 - \frac{a_1^2}{a_0^2}\right)\left(1 - \frac{a_2^2}{a_0^2}\right). \tag{2.7}$$

Thus, for the dual scator $\overset{o}{\bar{a}}$, we get

$$\begin{aligned}
\|\overset{o}{\bar{a}}\|^2 = \overset{o}{\bar{a}}\overset{o}{\bar{a}}^* &= \left(\frac{a_1 a_2}{a_0}; a_2, a_1\right)\left(\frac{a_1 a_2}{a_0}; -a_2, -a_1\right) \\
&= \frac{a_1^2 a_2^2}{a_0^2}\left(1 - \frac{a_2^2}{\frac{a_1^2 a_2^2}{a_0^2}}\right)\left(1 - \frac{a_1^2}{\frac{a_1^2 a_2^2}{a_0^2}}\right) \\
&= \frac{a_1^2 a_2^2}{a_0^2} + a_0^2 - a_1^2 - a_2^2 = a_0^2\left(1 - \frac{a_1^2}{a_0^2}\right)\left(1 - \frac{a_2^2}{a_0^2}\right),
\end{aligned} \tag{2.8}$$

which exactly coincides with the norm of the original scator. $\square$

Remark 2.8. Taking into account (1.2), (2.2) and (2.1), we can easily verify that

$$\overset{o}{a}^*\overset{o}{b} + \overset{o}{a}\overset{o}{b}^* = 2\left(a_0 b_0 - a_1 b_1 - a_2 b_2 + \frac{a_1 a_2 b_1 b_2}{a_0 b_0}\right), \tag{2.9}$$

where the right-hand side is proportional to **1** (omitted for simplicity here and in many other places).

Definition 2.9. If we have a 3-scator $\overset{o}{a} = (a_0; a_1, a_2)$, then we call a scator of the form

$$\overset{o}{\bar{a}}_i = \left(a_1; a_0, \frac{a_1 a_2}{a_0}\right) \tag{2.10}$$

its internal dual scator, and a scator of the form

$$\overset{o}{\bar{a}}_e = \left(a_2; \frac{a_1 a_2}{a_0}, a_0\right) \tag{2.11}$$

its external dual.

Lemma 2.10. *Internal and external duality operations anti-commute with hypercomplex conjugation.*

Proof. We have

$$\left(\overset{o}{\bar{a}}_i\right)^* = \left(a_1; a_0, \frac{a_1 a_2}{a_0}\right)^* = \left(a_1; -a_0, -\frac{a_1 a_2}{a_0}\right) \tag{2.12}$$

and

$$\overline{\overset{o}{(a^*)}}_i = \overline{(a_0; -a_1, -a_2)_i} = \left(-a_1; a_0, \frac{a_1 a_2}{a_0}\right), \tag{2.13}$$

so that

$$(\overset{o}{\bar{a}})^*_i + \overline{\overset{o}{(a^*)}}_i = 0. \tag{2.14}$$

Similarly we have

$$(\overset{o}{\bar{a}})^*_e = (a_2; \frac{a_1 a_2}{a_0}, a_0)^* = \left(a_2; -\frac{a_1 a_2}{a_0}, -a_0\right) \tag{2.15}$$

and

$$\overline{\overset{o}{(a^*)}}_e = \overline{(a_0; -a_1, -a_2)_e} = \left(-a_2; \frac{a_1 a_2}{a_0}, a_0\right), \tag{2.16}$$

so that

$$(\overset{o}{\bar{a}}_e)^* + \overline{(a^*)}_e = 0, \tag{2.17}$$

which ends the proof. □

Remark 2.11. Internal and external duality operations are second order involutions.

Remark 2.12. Operation of external and internal duality do not provide yet another homomorphic structure in S, that is

$$\overline{\overset{o}{(ab)}}_i \neq \overset{o}{\bar{a}}_i \overset{o}{\bar{b}}_i, \qquad \overline{\overset{o}{(ab)}}_e \neq \overset{o}{\bar{a}}_e \overset{o}{\bar{b}}_e, \tag{2.18}$$

which follows from straightforward calculation.

Remark 2.13. Operations of internal and external duality preserve scator product.

Definition 2.14. A transformation that exchanges time-like events with space-like events (and conversely) and leaves the type of light-like events unchanged is called a causality swap (or a pseudo-isometry).

Proposition 2.15. *Both internal and external duality operations are causality swaps of 3-scator space.*

Proof. Denoting $\overset{o}{\bar{a}}_i = (\bar{a}_{0i}; \bar{a}_{1i}, \bar{a}_{2i})$, we compute:

$$\begin{aligned}
\|\overset{o}{\bar{a}}_i\|^2 &= \bar{a}_{0i}^2 \left(1 - \frac{\bar{a}_{1i}^2}{\bar{a}_{0i}^2}\right)\left(1 - \frac{\bar{a}_{2i}^2}{\bar{a}_{0i}^2}\right) \\
&= \bar{a}_{0i}^2 \left(1 - \frac{\bar{a}_{1i}^2}{\bar{a}_{0i}^2} - \frac{\bar{a}_{2i}^2}{\bar{a}_{0i}^2} + \frac{\bar{a}_{1i}^2}{\bar{a}_{0i}^2}\frac{\bar{a}_{2i}^2}{\bar{a}_{0i}^2}\right) = a_1^2\left(1 - \frac{a_0^2}{a_1^2}\right)\left(1 - \frac{a_2^2}{a_0^2}\right) \\
&= -a_0^2\left(1 - \frac{a_1^2}{a_0^2}\right)\left(1 - \frac{a_2^2}{a_0^2}\right) = -\|\overset{o}{a}\|^2
\end{aligned} \tag{2.19}$$

Therefore, if original scator represents a time-like event, then its internal dual has to represent a space-like event, and *vice versa*. Light-like scators do not change their type. Analogous computation, with the same consequences, can be done for the external dual. □

Corollary 2.16. *Duality operations commuting with hypercomplex conjugation are isometries, while duality operations anti-commuting with hypercomplex conjugation are causality swaps.*

Finally, we arrive at a very strong theorem providing some kind of translator between different kinds of duals.

Theorem 2.17. *Ordinary, internal, and external duality operations have the following properties:*

$$\begin{aligned}
\overset{o}{\bar{a}}\overset{o}{\bar{b}}_i &= \overset{o}{\bar{a}}_i\overset{o}{\bar{b}} = (\overset{o}{\overline{ab}})_e = \overset{o}{\bar{a}}_e\overset{o}{b} = \overset{o}{a}\overset{o}{\bar{b}}_e, \\
\overset{o}{\bar{a}}\overset{o}{\bar{b}}_e &= \overset{o}{\bar{a}}_e\overset{o}{\bar{b}} = (\overset{o}{\overline{ab}})_i = \overset{o}{\bar{a}}_i\overset{o}{b} = \overset{o}{a}\overset{o}{\bar{b}}_i \\
\overset{o}{\bar{a}}_e\overset{o}{\bar{b}}_i &= \overset{o}{\bar{a}}_i\overset{o}{\bar{b}}_e = \overset{o}{\overline{ab}} = \overset{o}{\bar{a}}\overset{o}{b} = \overset{o}{a}\overset{o}{\bar{b}}.
\end{aligned} \tag{2.20}$$

Proof. One can perform lengthy straightforward calculation. However, applying a new approach of next sections, we will be able to present a very short proof in Theorem 5.3. □

3. Some Further Algebra

Let S' denote the space of hyperbolic 3-scators with non-vanishing scalar component ($\overset{o}{a} \in S'$ if and only if $a_0 \neq 0$). We are going to identify S' with objects in a linear space A of dimension 4, where the fourth component is reserved for some geometric invariant.

Definition 3.1. The fundamental embedding of S' into A is given by the map $F : S' \mapsto \tilde{S} \equiv F(S) \subset A$, defined by

$$S' \ni \overset{o}{a} = (a_0; a_1, a_2) \to F(\overset{o}{a}) := (a_0; a_1, a_2, a_3), \quad a_3 = \frac{a_1 a_2}{a_0}. \tag{3.1}$$

Remark 3.2. F is a bijection between S' and $\tilde{S}$, so it has an inverse $F^{-1}: \tilde{S} \to S'$. We have also a natural projection

$$\pi : A \ni (a_0; a_1, a_2, a_3) \to (a_0; a_1, a_2) \in S. \tag{3.2}$$

Note that $\pi \neq F^{-1}$ but $\pi|_{\tilde{S}} = F^{-1}$.

We denote by $\{\mathbf{1}, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_{12}\}$ a basis in the linear space A such that

$$(a_0; a_1, a_2, a_3) = a_0\mathbf{1} + a_1\boldsymbol{i}_1 + a_2\boldsymbol{i}_2 + a_3\boldsymbol{i}_{12}. \tag{3.3}$$

The first three elements span the scator space S. The basis $\{\mathbf{1}, \boldsymbol{i}_1, \boldsymbol{i}_2\}$ is related to $\{1, \hat{e}_1, \hat{e}_2\}$ used in papers [1,8]. As in these papers, we demand that

$$\mathbf{1}^2 = \boldsymbol{i}_1^2 = \boldsymbol{i}_2^2 = \boldsymbol{i}_{12}^2 = 1, \tag{3.4}$$

but we define

$$\boldsymbol{i}_{12} = \boldsymbol{i}_1\boldsymbol{i}_2 = \boldsymbol{i}_2\boldsymbol{i}_1\ . \tag{3.5}$$

while the authors of [1,8] make *different* assumptions: $\hat{e}_1\hat{e}_2 = \hat{e}_2\hat{e}_1 = 0$. The last equalities can be neatly interpreted in our framework because

$$F^{-1}(\boldsymbol{i}_1\boldsymbol{i}_2) = F^{-1}(\boldsymbol{i}_{12}) = 0. \tag{3.6}$$

Now we make our fundamental assumption about the space A. We assume that this is a commutative, associative and *distributive* algebra, compare [7]. We denote

$$F(\overset{o}{a}) = \left(a_0; a_1, a_2, \frac{a_1a_2}{a_0}\right), \quad F(\overset{o}{b}) = \left(b_0; b_1, b_2, \frac{b_1b_2}{b_0}\right), \tag{3.7}$$

where we have written fourth components of scators explicitly in order to keep track of possible agreement with expected results. We have

$$F(\overset{o}{a})F(\overset{o}{b}) = \left(a_0 + a_1\boldsymbol{i}_1 + a_2\boldsymbol{i}_2 + \frac{a_1a_2}{a_0}\boldsymbol{i}_{12}\right)\left(b_0 + b_1\boldsymbol{i}_1 + b_2\boldsymbol{i}_2 + \frac{b_1b_2}{b_0}\boldsymbol{i}_{12}\right).$$

Next, we use distributivity and (3.4)

$$\begin{aligned} F(\overset{o}{a})F(\overset{o}{b}) &= a_0b_0 + a_1b_1 + a_2b_2 + \frac{a_1a_2b_1b_2}{a_0b_0} + \boldsymbol{i}_1(a_0b_1 + a_1b_0) \\ &\quad + \boldsymbol{i}_{12}\boldsymbol{i}_1\frac{a_1a_2}{a_0}b_1 + \boldsymbol{i}_1\boldsymbol{i}_{12}\frac{b_1b_2}{b_0}a_1 + \boldsymbol{i}_2(a_0b_2 + a_2b_0) + \boldsymbol{i}_{12}\boldsymbol{i}_2\frac{a_1a_2}{a_0}b_2 \\ &\quad + \boldsymbol{i}_2\boldsymbol{i}_{12}\frac{b_1b_2}{b_0}a_2 + \boldsymbol{i}_{12}\left(\frac{a_1a_2}{a_0}b_0 + \frac{b_1b_2}{b_0}a_0\right) + \boldsymbol{i}_1\boldsymbol{i}_2(a_1b_2 + a_2b_1). \end{aligned} \tag{3.8}$$

Then, due to (3.5), we obtain

$$\begin{aligned} F(\overset{o}{a})F(\overset{o}{b}) &= a_0b_0 + a_1b_1 + a_2b_2 + \frac{a_1a_2b_1b_2}{a_0b_0} + \boldsymbol{i}_1(a_0b_1 + a_1b_0) \\ &\quad + \boldsymbol{i}_{12}\boldsymbol{i}_2\frac{a_1a_2}{a_0}b_2 + \boldsymbol{i}_2\boldsymbol{i}_{12}\frac{b_1b_2}{b_0}a_2 + \boldsymbol{i}_2(a_0b_2 + a_2b_0) + \boldsymbol{i}_{12}\boldsymbol{i}_1\frac{a_1a_2}{a_0}b_1 \\ &\quad + \boldsymbol{i}_1\boldsymbol{i}_{12}\frac{b_1b_2}{b_0}a_1 + \boldsymbol{i}_{12}\left(\frac{a_1a_2}{a_0}b_0 + \frac{b_1b_2}{b_0}a_0 + a_1b_2 + a_2b_1\right). \end{aligned} \tag{3.9}$$

We see that the obtained scalar component coincides with the scalar component of (1.2). In order to get the same director components we have to assume

$$\boldsymbol{i}_1\boldsymbol{i}_{12} = \boldsymbol{i}_{12}\boldsymbol{i}_1 = \boldsymbol{i}_2, \quad \boldsymbol{i}_2\boldsymbol{i}_{12} = \boldsymbol{i}_{12}\boldsymbol{i}_2 = \boldsymbol{i}_1, \tag{3.10}$$

which follows from (3.4) and (3.5). Finally, we get

$$\begin{aligned} F(\overset{o}{a})F(\overset{o}{b}) = a_0b_0 + a_1b_1 + a_2b_2 + \frac{a_1a_2b_1b_2}{a_0b_0} \\ + \boldsymbol{i}_1\left(a_0b_1 + a_1b_0 + \frac{a_1a_2}{a_0}b_2 + \frac{b_1b_2}{b_0}a_2\right) \\ + \boldsymbol{i}_2\left(a_0b_2 + a_2b_0 + \frac{a_1a_2}{a_0}b_1 + \frac{b_1b_2}{b_0}a_1\right) \\ + \boldsymbol{i}_{12}\left(\frac{a_1a_2}{a_0}b_0 + \frac{b_1b_2}{b_0}a_0 + a_1b_2 + a_2b_1\right). \end{aligned} \tag{3.11}$$

Corollary 3.3. *From* (3.11) *we immediately see that* $\overset{o}{a}\overset{o}{b} = \pi(F(\overset{o}{a})F(\overset{o}{b}))$.

Theorem 3.4. *The fundamental embedding* (3.1) *is a multiplicative homomorphism of S and $\tilde{S}$:*

$$F(\overset{o}{a})F(\overset{o}{b}) = F(\overset{o}{a}\overset{o}{b}), \quad \textit{hence} \quad \overset{o}{a}\overset{o}{b} = F^{-1}(F(\overset{o}{a})F(\overset{o}{b})). \tag{3.12}$$

Proof. We may check it by tedious straightforward computation, multiplying scalar and fourth component of (3.11) and comparing it with the product of director components. However, calculations can be avoided when we take into account the following factorization:

$$F(\overset{o}{a}) = a_0\left(1 + \frac{a_1}{a_0}\boldsymbol{i}_1\right)\left(1 + \frac{a_2}{a_0}\boldsymbol{i}_2\right), \quad F(\overset{o}{b}) = b_0\left(1 + \frac{b_1}{b_0}\boldsymbol{i}_1\right)\left(1 + \frac{b_2}{b_0}\boldsymbol{i}_2\right),$$

$$F(\overset{o}{a})F(\overset{o}{b}) = a_0b_0\left(1 + \frac{a_1b_1}{a_0b_0} + \left(\frac{a_1}{a_0} + \frac{b_1}{b_0}\right)\boldsymbol{i}_1\right)\left(1 + \frac{a_2b_2}{a_0b_0} + \left(\frac{a_2}{a_0} + \frac{b_2}{b_0}\right)\boldsymbol{i}_2\right).$$

We see immediately that $F(\overset{o}{a})F(\overset{o}{b}) \in \tilde{S}$ which completes the proof. □

Remark 3.5. We can easily check that

$$F(\overset{o}{a^*}) = (F(\overset{o}{a}))^*, \quad F(\lambda\overset{o}{a}) = \lambda F(\overset{o}{a}), \quad F^{-1}(\lambda F(\overset{o}{a})) = \lambda\overset{o}{a}, \tag{3.13}$$

where λ is a real constant.

4. Formula for Distributive Multiplication

A crucial point in our analysis is to express the difference (1.5) in terms of the fundamental embedding. We take into account distributivity of the algebra A assumed in the previous section. Note that F is a multiplicative homomorphism but is not additive, i.e., in general

$$F(\overset{o}{a}) + F(\overset{o}{b}) \neq F(\overset{o}{a} + \overset{o}{b}). \tag{4.1}$$

Therefore, multiplication of scators has a geometric interpretation, while addition of such objects cannot be treated geometrically. In particular,

$$(F(\overset{o}{a}+\overset{o}{b}))F(\overset{o}{c}) \neq F(\overset{o}{a})F(\overset{o}{c})+F(\overset{o}{b})F(\overset{o}{c}). \tag{4.2}$$

But then we surely have

$$F(\overset{o}{a})F(\overset{o}{c})+F(\overset{o}{b})F(\overset{o}{c}) = (F(\overset{o}{a})+F(\overset{o}{b}))F(\overset{o}{c}), \tag{4.3}$$

because the product in $\tilde{S}$ *is* distributive. We will try to express $F(\overset{o}{a}+\overset{o}{b})$ in terms of $F(\overset{o}{a})$ and $F(\overset{o}{b})$. First, we compute

$$\begin{aligned}
F(\overset{o}{a}+\overset{o}{b})F(\overset{o}{c}) &= \left(a_0; a_1, a_2, \frac{a_1a_2}{a_0}\right)\left(c_0; c_1, c_2, \frac{c_1c_2}{c_0}\right) \\
&\quad + \left(b_0; b_1, b_2, \frac{b_1b_2}{b_0}\right)\left(c_0; c_1, c_2, \frac{c_1c_2}{c_0}\right) \\
&= \Bigg(c_0(a_0+b_0)+c_1(a_1+b_1)+c_2(a_2+b_2)+\frac{(a_1+b_1)(a_2+b_2)}{a_0+b_0} \\
&\quad \times\frac{c_1c_2}{c_0}; c_1(a_0+b_0)+c_0(a_1+b_1)+\frac{c_1c_2}{c_0}(a_2+b_2) \\
&\quad \times+\frac{(a_1+b_1)(a_2+b_2)}{a_0+b_0}c_2, c_2(a_0+b_0)+c_0(a_2+b_2) \\
&\quad +\frac{c_1c_2}{c_0}(a_1+b_1)+\frac{(a_1+b_1)(a_2+b_2)}{a_0+b_0}c_1, c_1(a_2+b_2) \\
&\quad +c_2(a_1+b_1)+\frac{c_1c_2}{c_0}(a_0+b_0)+\frac{(a_1+b_1)(a_2+b_2)}{a_0+b_0}c_0\Bigg).
\end{aligned} \tag{4.4}$$

On the other hand

$$\begin{aligned}
F(\overset{o}{a})F(\overset{o}{c})+F(\overset{o}{b})F(\overset{o}{c}) &= \left(a_0; a_1, a_2, \frac{a_1a_2}{a_0}\right)\left(c_0; c_1, c_2, \frac{c_1c_2}{c_0}\right) \\
&\quad + \left(b_0; b_1, b_2, \frac{b_1b_2}{b_0}\right)\left(c_0; c_1, c_2, \frac{c_1c_2}{c_0}\right) \\
&= (c_0(a_0+b_0)+c_1(a_1+b_1)+c_2(a_2+b_2) \\
&\quad + \left(\frac{a_1a_2}{a_0}+\frac{b_1b_2}{b_0}\right)\frac{c_1c_2}{c_0}; c_1(a_0+b_0)+c_0(a_1+b_1) \\
&\quad + \frac{c_1c_2}{c_0}(a_2+b_2)+\left(\frac{a_1a_2}{a_0}+\frac{b_1b_2}{b_0}\right)c_2, c_2(a_0+b_0) \\
&\quad + c_0(a_2+b_2)+\frac{c_1c_2}{c_0}(a_1+b_1)+\left(\frac{a_1a_2}{a_0}+\frac{b_1b_2}{b_0}\right)c_1, \\
&\quad c_1(a_2+b_2)+c_2(a_1+b_1)+\frac{c_1c_2}{c_0}(a_0+b_0) \\
&\quad + \left(\frac{a_1a_2}{a_0}+\frac{b_1b_2}{b_0}\right)c_0\Bigg).
\end{aligned} \tag{4.5}$$

Therefore

$$F(\overset{o}{a}+\overset{o}{b})F(\overset{o}{c})-F(\overset{o}{a})F(\overset{o}{c})-F(\overset{o}{b})F(\overset{o}{c}) = \left(\frac{(a_1+b_1)(a_2+b_2)}{a_0+b_0} - \frac{a_1a_2}{a_0} - \frac{b_1b_2}{b_0}\right) \times \left(\frac{c_1c_2}{c_0}; c_2, c_1, c_0\right) \tag{4.6}$$

and this is exactly $F(\Delta(a,b;c))$, as defined in (1.5)! Thus we have

$$F((\overset{o}{a}+\overset{o}{b})\overset{o}{c} - \overset{o}{a}\overset{o}{c} - \overset{o}{b}\overset{o}{c}) = F(\overset{o}{a}+\overset{o}{b})F(\overset{o}{c}) - F(\overset{o}{a})F(\overset{o}{c}) - F(\overset{o}{b})F(\overset{o}{c}), \tag{4.7}$$

which looks suspiciously homomorphic. Equation (4.6) can be rewritten as

$$F(\overset{o}{a}+\overset{o}{b})F(\overset{o}{c}) - F(\overset{o}{a})F(\overset{o}{c}) - F(\overset{o}{b})F(\overset{o}{c}) = \kappa(\overset{o}{a},\overset{o}{b})F(\overset{o}{\bar{c}}) \tag{4.8}$$

where $\kappa(\overset{o}{a},\overset{o}{b})$ is the scalar function standing before the dual of $\overset{o}{c}$ in formulas (1.5) and (4.6).

Remark 4.1. From (1.6) it follows that the inverse of a scator (with respect to the scator product) is given by

$$(\overset{o}{c})^{-1} = \frac{(\overset{o}{c})^*}{\|\overset{o}{c}\|^2}, \tag{4.9}$$

and light-like scators are not invertible.

Theorem 4.2.

$$F(\overset{o}{a}+\overset{o}{b}) - F(\overset{o}{a}) - F(\overset{o}{b}) = \kappa(\overset{o}{a},\overset{o}{b})\boldsymbol{i}_{12}. \tag{4.10}$$

Proof. We multiply both sides of (4.8) by $F((\overset{o}{c})^{-1})$ and, taking into account that F is a homomorphism, we obtain the left-hand side of (4.10). Then we observe that

$$F(\overset{o}{\bar{c}}) = \frac{c_1c_2}{c_0} + c_2\boldsymbol{i}_1 + c_1\boldsymbol{i}_2 + c_0\boldsymbol{i}_{12} = \boldsymbol{i}_{12}F(\overset{o}{c}). \tag{4.11}$$

Therefore

$$F(\overset{o}{\bar{c}})F((\overset{o}{c})^{-1}) = \frac{\boldsymbol{i}_{12}F((\overset{o}{c})^*)F(\overset{o}{c})}{\|\overset{o}{c}\|^2} = \boldsymbol{i}_{12} \tag{4.12}$$

which ends the proof. □

Remark 4.3. As a direct consequence of (4.12) we get the following strange result

$$\overset{o}{\bar{a}}(\overset{o}{a})^{-1} = 0, \tag{4.13}$$

which manifestly shows that the scator algebra has numerous zero divisors.

Corollary 4.4. *Theorem 4.2 implies that for a given set of scators,* $\overset{o}{a}_i = (a_{0i}; a_{1i}, a_{2i})$, *where* $i = 1, \ldots, n$, *we have:*

$$F(\overset{o}{a}_1+\cdots+\overset{o}{a}_n) = F(\overset{o}{a}_1)+F(\overset{o}{a}_2)+\cdots+F(\overset{o}{a}_n)+\kappa_n(\overset{o}{a}_1,\overset{o}{a}_2,\ldots,\overset{o}{a}_n)\boldsymbol{i}_{12}. \tag{4.14}$$

where κ_n (a symmetric scalar function) can be expressed by κ:

$$\kappa_n(\overset{o}{a}_1, \overset{o}{a}_2, \ldots, \overset{o}{a}_n) = \sum_{j=2}^{n} \kappa(\overset{o}{a}_1 + \overset{o}{a}_2 + \cdots + \overset{o}{a}_{j-1}, \overset{o}{a}_j). \tag{4.15}$$

Corollary 4.5. *From Theorem 4.2 it follows that the inverse of the fundamental embedding is an additive homomorphism, i.e.,*

$$F^{-1}(F(\overset{o}{a}) + F(\overset{o}{b})) = F^{1}(F(\overset{o}{a}) + F^{-1}(F(\overset{o}{b})) \equiv \overset{o}{a} + \overset{o}{b}, \tag{4.16}$$

because $F^{-1}(\boldsymbol{i}_{12}) = 0$.

We point out that F^{-1} is not a multiplicative homomorphism. Counterexamples will be given in the next section.

5. Dualities: Systematic Treatment

Here we also provide some back-up for what we have done earlier: at first, we see why we call duality operation a *duality* operation. First, we note that multiplication by the bivector $\boldsymbol{i}_{12}$ acts on any element of the algebra A in a way similar to the Hodge operator producing its complementary element with respect to the "maximal form" $\boldsymbol{i}_{12}$. We have two pairs of complementary basis elements: the first one is $\mathbf{1}$ and $\boldsymbol{i}_{12}$ and the second one is $\boldsymbol{i}_1$ and $\boldsymbol{i}_2$.

Definition 5.1. Duality operations are naturally given by:

$$\overset{o}{\bar{a}} = F^{-1}(\boldsymbol{i}_{12}F(\overset{o}{a})), \quad \overset{o}{\bar{a}}_i = F^{-1}(\boldsymbol{i}_1 F(\overset{o}{a})), \quad \overset{o}{\bar{a}}_e = F^{-1}(\boldsymbol{i}_2 F(\overset{o}{a})), \tag{5.1}$$

and, of course, $\overset{o}{a} = F^{-1}(\mathbf{1}F(\overset{o}{a}))$.

Therefore, we have

$$F(\overset{o}{\bar{a}}) = \boldsymbol{i}_{12}F(\overset{o}{a}), \quad F(\overset{o}{\bar{a}}_i) = \boldsymbol{i}_1 F(\overset{o}{a}), \quad F(\overset{o}{\bar{a}}_e) = \boldsymbol{i}_2 F(\overset{o}{a}). \tag{5.2}$$

Thus, taking into account $F((\overset{o}{a})^{-1}) = (F(\overset{o}{a}))^{-1}$ and (3.12), we get:

$$\overset{o}{\bar{a}}\overset{o}{a}^{-1} = 0, \quad \overset{o}{\bar{a}}_i \overset{o}{a}^{-1} = \boldsymbol{i}_1, \quad \overset{o}{\bar{a}}_e \overset{o}{a}^{-1} = \boldsymbol{i}_2. \tag{5.3}$$

In order to unify proofs it is convenient to introduce the following notation, generalizing (5.1) and (5.2):

$$\delta_{\boldsymbol{d}}(\overset{o}{a}) = F^{-1}(\boldsymbol{d}F(\overset{o}{a})), \quad F(\delta_{\boldsymbol{d}}(\overset{o}{a})) = \boldsymbol{d}F(\overset{o}{a}), \quad F(\overset{o}{a}) = \boldsymbol{d}F(\delta_{\boldsymbol{d}}(\overset{o}{a})), \tag{5.4}$$

where $\delta_{\boldsymbol{d}}$ denotes one of duality operations and $\boldsymbol{d}$ denotes the corresponding basis element ($\boldsymbol{i}_1$, $\boldsymbol{i}_2$ or $\boldsymbol{i}_{12}$). Note that $\boldsymbol{d}^2 = 1$.

Lemma 5.2. *Any duality operation commutes with inversion, i.e.,*

$$\delta_{\boldsymbol{d}}((\overset{o}{a})^{-1}) = (\delta_{\boldsymbol{d}}(\overset{o}{a}))^{-1}. \tag{5.5}$$

Proof. Using $F(\overset{o}{a}) = \boldsymbol{d}F(\delta_{\boldsymbol{d}}(\overset{o}{a}))$ and $1 = F(\overset{o}{a})F((\overset{o}{a})^{-1})$ (because F is a multiplicative homomorphism) we obtain

$$1 = \boldsymbol{d}F(\delta_{\boldsymbol{d}}(\overset{o}{a}))F((\overset{o}{a})^{-1}) = F(\delta_{\boldsymbol{d}}(\overset{o}{a}))\boldsymbol{d}F((\overset{o}{a})^{-1}) = F(\delta_{\boldsymbol{d}}(\overset{o}{a}))F(\delta_{\boldsymbol{d}}((\overset{o}{a})^{-1})).$$

On the other hand, we have $1 = F(\delta_{\boldsymbol{d}}(\overset{o}{a}))F((\delta_{\boldsymbol{d}}(\overset{o}{a}))^{-1})$. Therefore,

$$F(\delta_{\boldsymbol{d}}((\overset{o}{a})^{-1})) = F((\delta_{\boldsymbol{d}}(\overset{o}{a}))^{-1})$$

and, taking into account that F is a bijection, we obtain (5.5). □

Now, we can give a simple proof for Theorem 2.17. We rewrite this theorem in a more convenient way, using the unification of dualities formulated in this section.

Theorem 5.3. *Duality operations have the following properties:*

$$\begin{aligned} \delta_{\boldsymbol{p}}(\overset{o}{a}\overset{o}{b}) &= \delta_{\boldsymbol{p}}(\overset{o}{a})\overset{o}{b} = \overset{o}{a}\delta_{\boldsymbol{p}}(\overset{o}{b}), \\ \delta_{\boldsymbol{p}}(\overset{o}{a})\delta_{\boldsymbol{q}}(\overset{o}{b}) &= \delta_{\boldsymbol{q}}(\overset{o}{a})\delta_{\boldsymbol{p}}(\overset{o}{b}) = \delta_{\boldsymbol{pq}}(\overset{o}{a}\overset{o}{b}), \end{aligned} \tag{5.6}$$

where $\boldsymbol{p}, \boldsymbol{q} \in \{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_{12}\}$.

Proof. Using (5.4) and (3.12) we obtain:

$$\delta_{\boldsymbol{p}}(\overset{o}{a}\overset{o}{b}) = F^{-1}(\boldsymbol{p}F(\overset{o}{a}\overset{o}{b})) = F^{-1}(\boldsymbol{p}F(\overset{o}{a})F(\overset{o}{b})) = F^{-1}(F(\overset{o}{a})\boldsymbol{p}F(\overset{o}{b})). \tag{5.7}$$

$$\begin{aligned} F^{-1}(\boldsymbol{p}F(\overset{o}{a})F(\overset{o}{b})) &= F^{-1}(F(\delta_{\boldsymbol{p}}(\overset{o}{a}))F(\overset{o}{b})) = \delta_{\boldsymbol{p}}(\overset{o}{a})\overset{o}{b}, \\ F^{-1}(F(\overset{o}{a})\boldsymbol{p}F(\overset{o}{b})) &= F^{-1}(F(\overset{o}{a})F(\delta_{\boldsymbol{p}}(\overset{o}{b}))) = \overset{o}{a}\delta_{\boldsymbol{p}}(\overset{o}{b}), \end{aligned} \tag{5.8}$$

which ends the proof. □

Corollary 5.4. *The preservation of the scator product by any duality is a special case of Theorem 5.3 for* $\boldsymbol{q} = \boldsymbol{p}$*, namely:*

$$\delta_{\boldsymbol{p}}(\overset{o}{a})\delta_{\boldsymbol{p}}(\overset{o}{b}) = \overset{o}{a}\overset{o}{b}, \tag{5.9}$$

because $\delta_{\mathbf{1}}$ *is the identity operation.*

Using (3.12) we can express the scator product in terms of duality operations. Expanding $\overset{o}{a}$ with respect to the standard basis of A, we have

$$\overset{o}{a}\overset{o}{b} = F^{-1}((a_0 + a_1\boldsymbol{i}_1 + a_2\boldsymbol{i}_2 + \frac{a_1a_2}{a_0}\boldsymbol{i}_{12})F(\overset{o}{b})). \tag{5.10}$$

Then

$$\overset{o}{a}\overset{o}{b} = F^{-1}(a_0F(\overset{o}{b}) + a_1\boldsymbol{i}_1F(\overset{o}{b}) + a_2\boldsymbol{i}_2F(\overset{o}{b}) + \frac{a_1a_2}{a_0}\boldsymbol{i}_{12}F(\overset{o}{b})), \tag{5.11}$$

and, finally, using (5.1), we get

$$\overset{o}{a}\overset{o}{b} = F^{-1}(a_0F(\overset{o}{b}) + a_1F(\overset{o}{\bar{b}}_i) + a_2F(\overset{o}{\bar{b}}_e) + \frac{a_1a_2}{a_0}F(\overset{o}{\bar{b}})). \tag{5.12}$$

Thus we see that the formula for multiplication takes the form

$$\overset{o}{a}\overset{o}{b} = a_0\overset{o}{b} + a_1\overset{o}{\bar{b}}_i + a_2\overset{o}{\bar{b}}_e + \frac{a_1a_2}{a_0}\overset{o}{\bar{b}}. \tag{5.13}$$

6. Metric Properties of Scators

We begin with a simple fact.

Lemma 6.1. *Squared modulus of a scator satisfies:*

$$\|\overset{o}{a}\overset{o}{b}\|^2 = \|\overset{o}{a}\|^2\|\overset{o}{b}\|^2, \quad \|\lambda\overset{o}{a}\| = |\lambda|\|\overset{o}{a}\|. \tag{6.1}$$

where λ is a real constant.

Proof. Using Definition (1.6) we have

$$\|\overset{o}{a}\overset{o}{b}\|^2 = (\overset{o}{a}\overset{o}{b})(\overset{o}{a}\overset{o}{b})^* = (\overset{o}{a}\overset{o}{a}^*)(\overset{o}{b}\overset{o}{b}^*) = \|\overset{o}{a}\|^2\|\overset{o}{b}\|^2, \tag{6.2}$$

by virtue of commutativity and associativity of the scator product. The second equality follows directly from (1.6). $\square$

Therefore, the squared modulus of the scator product factorizes like the product of complex numbers. Unfortunately, the squared modulus is not a quadratic form of the scator components, which means that there is not a corresponding bilinear form. However, we will introduce an analogue of the scalar product postulating the formula obeyed by quadratic forms.

Definition 6.2. Scalar product of scators is defined by

$$\overset{o}{a}\cdot\overset{o}{b} = \frac{1}{2}\left(\|\overset{o}{a}+\overset{o}{b}\|^2 - \|\overset{o}{a}\|^2 - \|\overset{o}{b}\|^2\right). \tag{6.3}$$

Corollary 6.3.

$$\overset{o}{a}\cdot\overset{o}{a} = \|\overset{o}{a}\|^2. \tag{6.4}$$

Theorem 6.4. *The scalar product in S is explicitly given by*

$$\overset{o}{a}\cdot\overset{o}{b} = a_0b_0 - a_1b_1 - a_2b_2 + \frac{(a_1+b_1)^2(a_2+b_2)^2}{2(a_0+b_0)^2} - \frac{a_1^2a_2^2}{2a_0^2} - \frac{b_1^2b_2^2}{2b_0^2}. \tag{6.5}$$

Proof. We start from

$$\|\overset{o}{a}+\overset{o}{b}\|^2 = (\overset{o}{a}+\overset{o}{b})(\overset{o}{a}+\overset{o}{b})^* = (\overset{o}{a}+\overset{o}{b})(\overset{o}{a}^*+\overset{o}{b}^*), \tag{6.6}$$

and, since we cannot safely proceed because of lack of distributivity in S, we move on to $\tilde{S}$, i.e.,

$$\|\overset{o}{a}+\overset{o}{b}\|^2 = F^{-1}(F(\overset{o}{a}+\overset{o}{b})F(\overset{o}{a}^*+\overset{o}{b}^*)), \tag{6.7}$$

and from the formula (4.10) we get

$$\|\overset{o}{a}+\overset{o}{b}\|^2 = F^{-1}((F(\overset{o}{a})+F(\overset{o}{b})+\kappa(\overset{o}{a},\overset{o}{b})\boldsymbol{i}_{12})(F(\overset{o}{a}^*)+F(\overset{o}{b}^*)+\kappa(\overset{o}{a}^*,\overset{o}{b}^*)\boldsymbol{i}_{12})).$$

Hence, using (5.2) and Theorem 3.4, we obtain

$$\begin{aligned}\|\overset{o}{a}+\overset{o}{b}\|^2 &= \overset{o}{a}\overset{o}{a}^* + \overset{o}{b}\overset{o}{b}^* + \overset{o}{a}\overset{o}{b}^* + \overset{o}{a}^*\overset{o}{b} + \kappa(\overset{o}{a},\overset{o}{b})(\bar{\overset{o}{a}}^* + \bar{\overset{o}{b}}^*) + \kappa(\overset{o}{a}^*,\overset{o}{b}^*)(\bar{\overset{o}{a}} + \bar{\overset{o}{b}}) \\ &\quad + \kappa(\overset{o}{a},\overset{o}{b})\kappa(\overset{o}{a}^*,\overset{o}{b}^*).\end{aligned} \tag{6.8}$$

To proceed further we need Remark 2.8 and the following properties of the hypercomplex conjugation and κ:

$$\overset{o}{a}{}^* + \overset{o}{\bar{a}} = \frac{2a_1a_2}{a_0}\ , \tag{6.9}$$

$$\kappa(\overset{o}{a}{}^*, \overset{o}{b}{}^*) = \kappa(\overset{o}{a}, \overset{o}{b}) = \frac{(a_1+b_1)(a_2+b_2)}{(a_0+b_0)} - \frac{a_1a_2}{a_0} - \frac{b_1b_2}{b_0}. \tag{6.10}$$

Then from (6.8) and Definition 6.2 we have

$$\begin{aligned} 2\overset{o}{a}\cdot\overset{o}{b} &= \kappa^2 + 2\kappa\left(\frac{a_1a_2}{a_0} + \frac{b_1b_2}{b_0}\right) + 2\left(a_0b_0 - a_1b_1 - a_2b_2 + \frac{a_1a_2b_1b_2}{a_0b_0}\right) \\ &= \left(\kappa + \frac{a_1a_2}{a_0} + \frac{b_1b_2}{b_0}\right)^2 - \frac{a_1^2a_2^2}{a_0^2} - \frac{b_1^2b_2^2}{b_0^2} + 2(a_0b_0 - a_1b_1 - a_2b_2), \end{aligned}$$

where $\kappa = \kappa(\overset{o}{a}, \overset{o}{b})$. We complete the proof by substituting (6.10). □

Remark 6.5. We easily see that

$$(\lambda\overset{o}{a})\cdot(\lambda\overset{o}{b}) = \lambda^2(\overset{o}{a}\cdot\overset{o}{b}). \tag{6.11}$$

but, in general, $\lambda(\overset{o}{a}\cdot\overset{o}{b}) \neq (\lambda\overset{o}{a})\cdot\overset{o}{b} \neq \overset{o}{a}\cdot(\lambda\overset{o}{b}) \neq \lambda(\overset{o}{a}\cdot\overset{o}{b})$ and

$$(\overset{o}{a}+\overset{o}{b})\cdot\overset{o}{c} \neq \overset{o}{a}\cdot\overset{o}{c} + \overset{o}{b}\cdot\overset{o}{c}. \tag{6.12}$$

We point out that this scalar product is non-defined if a scalar component of any factor is zero.

7. Underlying Physics and Conclusions

The scator metric (1.6) approaches the flat Minkowski metric of special relativity space-time in the limit $a_0 \to \infty$. In the context of doubly-special relativity (DSR) [6], this limit means the restriction to times much larger than the Planck's time scale. Therefore the subject is not only of purely mathematical interest but may have interesting physical points, as well.

The first question is, can we think of tachyons as para-particles possessing space-like trajectories? In the scator framework this would mean that at each instance of time tachyons have to be described by scators with negative squared modulus. Thus understood tachyons, in order to remain tachyons, need to experience sub-luminal Lorentz boosts at that each instance of time, since otherwise the sign of the squared modulus would get reverted. It is hard to imagine an inertial, super-luminal observer that could be turned into another one with some kind of sub-luminal Lorentz-like transformation preserving scator metric. Hence it seems to us that a more natural way of understanding tachyons is that they are ordinary particles in a different causal domain and they cannot reach us because of the infinite-energy requirement to pass the bipyramidal light-barrier, inside which we are capable of taking the measurements.

To sum up: we may suppose that tachyons are being super-luminal in a sense of belonging to a different sub-luminal causal realm, although then we look exactly the same way to them. Note that this hypothesis is surprisingly in accordance with recent remarks on the nature of tachyons [5].

Thus, some interesting formal effects arise in a quite natural fashion in the scator framework. Convergences with recent physical concepts [4,5], even if accidental, is intriguing, all the more so that this framework does not show any rotational symmetry properties we could expect on scientific and intuitive grounds.

The next question that appears in the physical context is whether the approach proposed in this paper works in the case of $1 + 3$ dimensions? The answer is *yes*. It may be easily shown that in physical space-time of scators there is

$$F(\overset{o}{a} + \overset{o}{b}) = F(\overset{o}{a}) + F(\overset{o}{b}) + c_{1,2}\boldsymbol{i}_1\boldsymbol{i}_2 + c_{1,3}\boldsymbol{i}_1\boldsymbol{i}_3 + c_{2,3}\boldsymbol{i}_2\boldsymbol{i}_3 + c_{1,2,3}\boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3, \quad (7.1)$$

as opposed to (4.10), opening all doors needed [7]. In the above expression we have

$$c_{i,j}(\overset{o}{a}, \overset{o}{b}) = \frac{(a_i + b_i)(a_j + b_j)}{a_0 + b_0} - \frac{a_i a_j}{a_0} - \frac{b_i b_j}{b_0} \quad (7.2)$$

and

$$c_{1,2,3}(\overset{o}{a}, \overset{o}{b}) = \frac{(a_1 + b_1)(a_2 + b_2)(a_3 + b_3)}{(a_0 + b_0)^2} - \frac{a_1 a_2 a_3}{a_0^2} - \frac{b_1 b_2 b_3}{b_0^2}. \quad (7.3)$$

We see that the generalization, although simple in principle, may lead to cumbersome calculations. This also implies existence of more dualities, associated with the basis of the A-space, now 8-dimesional. Here also dualities introduced for $1 + 2$ dimensional case find their interpretation: the ordinary duality takes us into the realm of wings around the dipyramid, while internal and external dualities carry us out of the light bipyramid.

Finally, we point out that the possibility for scators to be physically interpreted is strongly suppressed by the fact that the scator algebra does not possess rotational invariance. Fortunately, the considered dynamics does not provoke the appearence of absolute-rest frame [4], which leaves some hope for potential applications.

References

[1] Fernández-Guasti, M., Zaldívar, F.: A hyperbolic non-distributive algebra in 1+2 dimensions. Adv. Appl. Clifford Algebras **23**, 639–656 (2013)

[2] Choban, PJ.: An introduction to scators, M.Sc. Thesis, Oregon State Univ. (1964)

[3] Tauber, S.: Lumped parameters and scators. Math. Model. **2**, 227–232 (1981)

[4] Fernández-Guasti, M.: Time and space transformations in a scator deformed Lorentz metric. Eur. Phys. J. Plus **129**, 10 (195) (2014)

[5] Kapuścik, E.: On fatal error in Tachyonic physics. Int. J. Theor. Phys. **54**(11), 4041–4045 (2015)

[6] Amelino-Camelia, G.: Doubly-special relativity: facts, myths and some key open questions. Symmetry **2**, 230–271 (2010)

[7] Cieśliński, J.L., Kobus, A.: A distributive interpretation of the scator algebra (**in preparation**)

[8] Fernández-Guasti, M., Zaldívar, F.: Multiplicative representation of a hyperbolic non distributive algebra. Adv. Appl. Clifford Algebras **24**(3), 661–674 (2014)

[9] Fernández-Guasti, M., Zaldívar, F.: Hyperbolic superluminal scator algebra. Adv. Appl. Clifford Algebras. **25**, 321–335 (2015)

A. Kobus
Politechnika Białostocka
Wydział Budownictwa i Inżynierii Środowiska
ul. Wiejska 45E
15-351
Białystok
Poland

Jan L. Cieśliński
Uniwersytet w Białymstoku
Wydział Fizyki
ul. Ciołkowskiego 1L
15-245
Białystok
Poland
e-mail: j.cieslinski@uwb.edu.pl

Received: June 16, 2015.
Accepted: March 16, 2016.