Mark A. Reynolds | *The Geometer's Angle*
Marriages of Incommensurables

Introduction

Finding relationships between incommensurable systems of geometry can be likened to the alchemist's search for a chemical process whereby a base metal can be changed to a precious one. A true alchemist knows that the search and transmutation isn't so much a physical one, but rather, a transformation of consciousness and spirit where a union takes place between his spirit and matter, in much the same way that prayer and ritual – hopefully – work for the religious faithful. The “philosopher's stone”, that praised and elusive elixir, can be compared to the geometer's quest to draw a construction that will link him to a more spiritual existence, a deeper understanding of the universe of potentialities. The Seal of Solomon (the Star of David) in the Middle East, the horizontal and vertical axes of the Cross worldwide, and the symbol of the Great Goddess Yin-Yang in the East are but three examples of this concept. The *mandala* (Sanskrit for circle) is a universally found device that seeks to unite the circle with the square – what is above with what is below. Symbolically and traditionally, the *mandala* serves as a blueprint for mindfulness and proper actions. The circle is equated with heaven or the cosmos, and the square with the earth. The square usually suggests the four directions, the four seasons, or the four elements, all of which have four equal magnitudes. In geometry, one of the crowning glories of an effective transformation is *quadratura circuli*, or, in more modern terms, the squaring of the circle. The question it posits is this: Is there a construction, with compasses, a straightedge with no calibrations, and drawing tools alone, that can yield either the areas or the perimeters of the two shapes equal?¹ Although proven to be impossible to do in the 19th century (a solution nonetheless!) because of the irrationality of π , a search on the Internet will show that squaring the circle is still alive and well. The issue is that, as a rule, incommensurable systems – irrationals² – cannot be mixed with other irrational or rational systems into a cohesive whole, in much the same way that J.S. Bach cannot be fully appreciated while listening to the Beastie Boys

or Perry Como. This is not to say that all incommensurables cannot be joined together. One need only look to the square root rectangle system as an example. This system is generated by the incommensurability of the side and diagonal of the square. Using the Pythagorean Theorem, these two lengths cannot be rectified, for when the side is 1, which is rational, the diagonal will be $\sqrt{2}$, the decimal places being eternally long and whose sequence is impossible to predict. This diagonal can be used to generate a new rectangle with sides in this ratio: 1 : 1.4142... The system then continues to use the diagonal of the previous rectangle to make a new rectangle, so that we have an arithmetic progression under the radical signs: $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, ... $\sqrt{n}$, $\sqrt{(n+1)}$, ... But when attempts are made to use this 'root system' with, say, the whole number ratios (e.g., 1:1, 1:2, 2:3, 3:4, etc.), inherent difficulties usually arise with alignment, structure, and eurythmy. In his *Ten Books on Architecture*, Vitruvius states that eurythmy is,

... beauty and fitness in the adjustments of the members. This is found when the members of a work are of a height suited to their breadth, of a breadth suited to their length, and, in a word, when they all correspond symmetrically...³

(the emphasis on symmetrically is mine).

I emphasize symmetry because there are various types of symmetry operations in geometry, and they follow different geometric paths. Very frequently, these systems clash...they just don't fit together without discord to both. However, a skilled geometer⁴ can make the necessary adjustments to overcome some of these issues, but a marriage of incommensurables is, for all intents and purposes, a union where an entire geometric structure is generated into another that is totally unrelated to it. Now I realize that a mathematician will argue that an irrational like π or the golden section is only an "approximation of the true magnitude", and because of this, any claims made to a fitting together of these measures to a rational measure isn't really truthful, mathematically. For me, it is the issue of what the geometry is suggesting to me when I am constructing the drawing, and what it may be indicating symbolically and philosophically.⁵ If I am constructing a Tibetan *mandala*, finding a center that I determine is that center, and I draw a circle with a predetermined radius, I do not concern myself with the irrationality of π when I begin circumscribing it with a square. My task is to do the best job I can possibly do to make sure it is the best circle and square I can do today. My focus is on technique and process, symbolic and philosophical meaning and aesthetics. Another geometer may be more concerned with the shape's power and intensity.⁶ Still another may be using the shapes as a contemplative device. In my work, I concern myself with exactitude, and I frequently use a calculator. I always use very sharp graphite and expensive tools, i.e., I don't buy compasses in drug stores. If I'm onto something important, like a new discovery, I increase the scale 5 to 10 times in order for the line thickness to be less influential in the final calculations. I do try to have scholarly approaches and professional concerns; and yet I also know that we live in an earthbound plane, complete with numerous limitations, including mathematically imperfect compasses. I still focus on what the geometric configurations may suggest to me. In the following group of drawings, my hope is that the beauty of the constructions may not be lost or under appreciated even under the scrutiny of the most brilliant of our mathematical readers.

For me, discovering new relationships between incommensurables, finding connections that are hidden from view, is deeply rewarding. It fascinates me that for all of the hard edged, technical aspects of geometry, its almost restricting and cold objectivity, all of these things give way to the organic nature that lies hidden under the appearances and preconceived notions about it. Some of us aren't aware of this living quality that geometry has. Nonetheless, geometry has this almost canny ability to bend and be flexible to our will. An open mind will also find that geometry can work with us, as well as the other way around. A symbiotic relationship develops. It responds to the creative energy that the geometer brings to it, and it seems to me that geometric structure resonates on a very deep level with human consciousness. When I work to find these unions between and among geometric systems, I sense that the relationship between the mind and geometry is almost instinctual. Geometry teaches us to look for visual relationships, to explore possibilities of what may potentially exist in its shapes, forms, and ratios. Geometry is the visual relationships between numbers. It is also a bridge between what is and what may be. Our civilization left two dimensional, Euclidean geometry long ago on its quest to understand the universe. Now there is geometry for spacetime, for quantum physics, and 0 – dimension space. It is interesting to note that it is still geometry, but now, it is all dressed up. However, I trust the column will demonstrate that 2-dimensional geometry is still alive, and that it hasn't been fully explored and exhausted. There is still fertile ground. I hope that the constructions I show and discuss will motivate some of our readers to go exploring for themselves.

Drawing 1: The Three Triangles in the Great Pyramid

As I was preparing for my trip to Egypt last year, I took some time the night before to do a drawing of the Great Pyramid of Khufu at Giza. As I developed my sketch into a rough two point perspective, I realized that there are actually three triangles in the half base of the elevation, but we see only one; the other two are overlapped by the first.

Most of us are aware of the elevation of the pyramid, but what are we actually seeing when we look at this profile? In the following drawings, I will demonstrate the relationships of the three triangles, and show a triangle that I've never seen in any of my readings on the subject. For the purposes of clarity in the column, I will call this third triangle, "MR Triangle1". This arrangement of ratios, even though I found them, has physically existed at least back to the builders' drawings for Khufu's Pyramid. Potentially, the relationship has existed for a very long time, as long as the universe has. Geometric relationships were present in the cosmos before human life, as we know it on the Earth, began and developed. As such, in this construction, there is a natural "marriage of incommensurables". They are: ϕ , $\sqrt{\phi}$, $\sqrt{2}$, and $1/\phi$.

1. Figure 1.1 shows the construction of the golden section rectangle using the half diagonal of the square, $\sqrt{5} \div 2$. Therefore, $T_2G : GO :: \phi : 1$. GT_2 is then rotated to the opposite side of the golden section rectangle to become $\sqrt{\phi}$ at T_2a . O becomes the half base of the pyramid. The right, scalene triangle that results from this rotation has sides: a.) 1 (the half base); b.) $\sqrt{\phi}$ (the cathetus, the vertical, or the altitude), and ϕ (the hypotenuse). This triangle is sometimes referred to as

the “Triangle of Price” (TP), after Professor Price, and is the only right, scalene triangle whose sides are in a geometric progression, based on $\sqrt{\phi}$, 1.2720196...

2. Figure 1.1a shows the plan of the pyramid projected into a two point perspective. This view provides us with the three triangles. They are:
 - I. Triangle TP, GT_2O ($\sqrt{\phi}$);
 - II. Triangle MR1, ZT_2O (this is the triangle that I stumbled upon.); and,
 - III. Triangle MR2, ZT_2N (ϕ , as half of a golden section rectangle, cut along its diagonal).
3. Figure 1.2 shows the separation of the three right, scalene triangles from the pyramid and their three different ratios. Basically, however, the marriage is between the two systems: the ϕ family and $\sqrt{2}$.
4. Figure 1.3 shows each of the three triangles, their ratios, and their measures.

The important things to note are that physically, when we observe the sloped elevation of the pyramid – the triangular plane that conceals the other two triangles behind it – we are seeing the three components of the golden section rectangle, i.e., short side, 1; long side, ϕ ; the diagonal, 1.902..., or $\sqrt{\phi} + 2$. This diagonal is the actual sloping edge of the pyramid. The Triangle of Price is concealed by both the ϕ half-rectangle in front, and the diagonally positioned MR1

Figure 1.1.

Figure 1.1a.

triangle. This is the third, sandwiched triangle whose base runs along the diagonal of the square in the half base. When working out the ratios, this third rectangle shares the diagonal of the ϕ rectangle, the 1.902..., which is the sloping edge of the pyramid. Instead of using the sides of the square in the half base, as the ϕ and $\sqrt{\phi}$ ratios do, it uses the diagonal of a square, or, $\sqrt{2}$. Two of the three major divisions of the square are exhibited in this marriage: 1) the half-diagonal for the golden section; 2) the whole diagonal for the $\sqrt{2}$; 3) whole numbers/fractional parts. What we have here is an example of a natural, geometric relationship where nothing has been built, forced, bent, or removed. Perhaps this is why the marriage has lasted for at over 2500 years, a bit longer than even the successful ones today!

Drawing 2: The Coincidence of a Harmonic Sequence with the Square Root Rectangle Progression

The last construction to present in this issue's column involves a marriage between a simple harmonic sequence that is converted into the square root rectangle progression. I have never seen this construction anywhere, and I invite readers to let me know if they have, and to also invite some of the mathematical readership to offer a proof for why this construction is mathematically sound.

Figure 1.2.

Figure 1.3.

As part of the preview to the procedure, I include the Rule of Thales. It states that in a semicircle, any point on the circumference, when drawn to the ends of the diameter, will be a 90° angle. It was this rule, and the simple fact that a square can be inscribed within the boundary of a circle using two diameters that intersect at 90° that led me to discovering this marriage.

This construction is rich in quality of content. Along with the harmonic progression and the square root square rectangle sequence, it also contains the diagonal / reciprocal relationship as well as two geometric progressions. Whenever we have the diagonal / reciprocal relationship, we also have the “occult center”, so called because it is “hidden from the eye”.

Lastly, the construction also demonstrates the “geometric mean between two extremes”. As defined: when a semicircle is drawn, and the diameter is divided into any two measures – the two extremes – the perpendicular to this diameter, at the junction of these two measures, will yield the geometric mean between the two extremes.

PROCEDURE

1. As in Figure 2.1, draw any rectangle, AKMZ, and bisect it (as in, RP, by q_1 and q_2), to obtain the midpoint, O. For this construction, I have chosen the double square for purely aesthetic reasons. What is of importance is the midpoint, O, as it will be converted to a diameter of a circle.

Figure 2.1.

Figure 2.2.

Figure 2.2a.

-

Figure 2.3b.

4. Next, in order to have more clarity on the development of this harmonic progression, see Figure 2.3. (I have selected a square for the demonstration, but again, this procedure applies to all rectangles.) By starting with the two diagonals, AM and ZK, and the half-diagonal, ZG, we can begin the harmonic progression: $1/1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, \dots, 1/n, 1/n + 1 \dots$ Z is the radiating point for the progression, and it generates points, p_1, p_2, p_3, p_4, p_5 , etc. along the diagonal, AM, so that by drawing through each of these 'p' points along the diagonal, AM, we generate points, G, N, B, H, J, I, L, etc., along the side, KM. This harmonic progression is infinite, and bound only by the medium used in the rendering. Figure 2.3a, and Figure 2.3b show this technique applied to the rectangle with which we started.
5. In Figure 2.4, a circle is drawn with center, O, and diameters, KM and RP. In Figure 2.1, KM was the side, KM, and the perpendicular, RP. Here, both have been converted into diameters. There are times in geometry where one element can become several things simultaneously. For our purposes, we will concentrate on the semicircle, KPM, and that the diameter, KM, that was a side of the original rectangle, and that now, it will also become the diagonal of all of the square root rectangles in the sequence. KM is a constant.
6. Figure 2.5 shows square, RKPR, inscribed within the circle. This simple fact holds the key to the marriage. It is vital to recognize that point, O, is $1/2$ of KM. We now have a rectangle with side = 1, the $\sqrt{1}$ rectangle. Thus begins the

Figure 2.4.

Figure 2.5.

sequence of the root rectangles. The Rule of Thales can be seen here four times, at R, K, P, and M, as a semicircle can begin and end at any of the infinite points of the circumference, and the application of any of the infinite number of diameters within the circle. In the sequence of the 'root' rectangles, the number under the radical sign will progress arithmetically, from 1 to infinity, with the $\sqrt{2}$ coming next, the $\sqrt{3}$ to follow, ad infinitum. In addition to this, one of the most outstanding qualities of the construction is this principle: the number of the denominator in the fractional harmonic progression below is always 1 more than the number under the radical sign in the square root sequence above! So, we have, $1/2 = \sqrt{1}$, $1/3 = \sqrt{2}$, $1/4 = \sqrt{3}$..., etc. A complex and beautiful marriage indeed!

7. As the harmonic sequence of the rectangle is developed, as shown in Figure 2.6, the measures of $1/2$, $1/3$, $1/4$, and so on, are drawn up to intersect the circle at P, P₁, P₂, P₃, P₄, etc. These 'p' points will become 90° angles for the root rectangle system.
8. Figure 2.7 demonstrates the Rule of Thales along the semicircle, KPM. The 'p' points have been replaced by the radical signs of the beginning of the root rectangle sequence, which is also infinite. The harmonic progression is shown at the bottom of the rectangle, along, AZ. Remember that, KM, the diameter/diagonal length, is the constant, and it serves to render the fractional parts and the diagonal constant for the rectangles. Also, as noted in the

Figure 2.6.

Figure 2.7.

-

although I mention and use some of the rules and propositions of Pythagoras, Euclid and other classical geometers, I also want to demonstrate another marriage: that of the creative mind with the power and the possibilities that geometry possesses.

First published in the NNJ online October 2000

Notes

1. The other two classical problems are the trisection of the angle, and the doubling of the cube. See Thomas Heath, *A History of Greek Mathematics*, 2 vols. (New York: Dover, 1981), for an in depth study of these and other classical mathematical problems.
2. So called because the numbers in the decimal sequence are infinitely long and cannot be predicted. Even though $1/3$ is infinitely long, 0.3333333333..., we know that the next number is another 3.
3. Vitruvius. *Ten Books on Architecture*. Trans. Morris Hickey Morgan, 1914 (rpt. New York: Dover, 1960).
4. Andrea Palladio had seven ratios that he suggested be used in building. All but one were whole number ratios. The seventh was a square and its diagonal. I refer the reader to a new edition and an outstanding translation of Palladio's great treatise, *The Four Books on Architecture*. Trans. Robert Tavernor and Richard Schofield (Cambridge MA: MIT Press, 1997).
5. I never stray far from humanist, Pythagorean/neo-Platonic, and Egyptian/Greek thought regarding the study of Geometry.
6. One need only look at the effects, beliefs, and consequences of a cruciform, a Star of David, a swastika, a crescent moon, or a pentagram to know the power of geometric shapes.

Figure 2.10.

The Author

Mark A. Reynolds is a visual artist who works primarily in drawing, printmaking and mixed media. He received his Bachelor's and Master's Degrees in Art and Art Education at Towson University in Maryland. He was also awarded the Andelot Fellowship to do post-graduate work in drawing and printmaking at the University of Delaware. He also an educator who teaches sacred geometry, linear perspective, drawing, and printmaking to both graduate and undergraduate students at the Academy of Art College in San Francisco, California. He was voted Outstanding Educator of the Year by the students in 1992. Additionally, he is a geometer, and his specialties in this field include doing geometric analyses of architecture, paintings, and design. A sample of his work is available on the Internet at NextMonet.com. For the past decade, he has been at work on an extensive body of drawings, paintings and prints that incorporate and explore the ancient science of sacred, or contemplative, geometry. He is widely exhibited, showing his work in group competitions and one person shows, especially in California. His work is in corporate, public, and private collections, and he is represented by the Mill & Short Gallery in San Francisco. He is a member of the California Society of Printmakers, the Los Angeles Printmaking Society, and the Marin Arts Council. He is the author of "A Comparative Geometric Analysis of the Heights and Bases of the Great Pyramid of Khufu and the Pyramid of the Sun at Teotihuacan", *NNJ* 1 (1999), pp. 23-38.