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RESEARCH

Nautilus Spirals and the Meta-Golden Ratio Chi

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Abstract
The Nautilus shell is the popular iconic image for a logarithmic spiral. It is also frequently cited as an example of a golden ratio logarithmic spiral in nature. Evidently, this not the case. Contrarian studies have proposed that the Nautilus spiral is actually in the 4:3 ratio. Yet, these recommendations are based on one, or just a few shells. In this research, to compare the mean aspect ratio of Nautilus shells to the 4:3 ratio and the meta-golden ratio Chi, eighty Nautilus shells were measured in the Smithsonian collection. The results show that the *Nautilus* genus is clearly not the widely quoted 4:3 (1.333), but averaged 1.310. However, there was one species that was remarkably different, the Crusty Nautilus averaging 1.356 which is an excellent match for the Meta-golden ratio Chi.

Keywords Meta-golden Ratio Chi · Nautilus spiral · Logarithmic spirals · Golden ratio spiral · Myth · Proportion · Diagonals · Geometry

The Myth Perpetuated

The exquisite chambered Nautilus shell has long been a source of inspiration for artists and architects. According to Richard Padovan (1999: 314), "by the early years of this century, the notions of 'beauty of proportion' and 'the golden section' seemed to have become virtually synonymous." So, it's not surprising that the striking natural beauty of the Nautilus shell has been easily linked to the golden ratio. Examples on the Internet and in literature blindly perpetuate the common fallacy that the Nautilus logarithmic spiral shape is also a golden ratio or Fibonacci logarithmic spiral. It is not. The facts are also misrepresented in blogs, by museums,

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and by influential authors, even including mathematicians and scientists. These sources have stated or inferred that the Nautilus spiral is an excellent natural example of a logarithmic spiral and as such must also be a notable example of the golden ratio spiral. This assumption is faulty logic. Certainly, a logarithmic spiral, as a special case, can be drawn in golden ratio proportions, but it also can be drawn in various other proportions.[1] Even at a cursory glance, any one of the myriad images of the shell compared or superimposed over a golden ratio spiral demonstrates that it simply does not match.

Recent examples demonstrate how deep this fallacy penetrates. Respected mathematician, Keith Devlin sets the stage. Writing in ‘The Myth That Will Not Go Away’ (Devlin 2007), he candidly acknowledged that while debunking various other applications of the golden ratio, he inadvertently claimed that the Nautilus shell growth was governed by the golden ratio. Novelist Dan Brown (2003: 94) famously popularized this myth in ‘The Da Vinci Code’. Astrophysicist Livio (2003: 9) in ‘The Golden Ratio’ would also lead us to believe this false notion. Ironically, ‘Geometry for Dummies’ (Ryan 2016: 371) instructs us that “the golden rectangle spiral…happens to be the same shape as the spiraling shell of the Nautilus…”. Even Smithsonian’s own science Facebook page (Ocean Portal) describes the Nautilus as “a natural example of Fibonacci’s sequence…”. Although graphic designers might claim some innocence for their role in using the visual appeal of the Nautilus shell to grace the covers and pages of publications, their authors may not be able to avoid responsibility so easily.

Debunking the Myth

One strong bulwark to the still prevalent golden ratio myth is Falbo’s (2005: 127) paper that concludes “Anyone with access to such a shell can see immediately that the ratio is somewhere around 4 to 3… not phi”. Even as long ago as 1883, Rev. Moseley had observed that the whorls of the Nautilus spiral grow by about a factor of three (Thompson 1992: 770). British mathematician, John Sharp (2002: 79) also found that the ratio of the distance between two consecutive curves (whorls) of the spiral on radius vectors for four different 360° rotations of his Nautilus specimen spiral ranged from 2.83 to 3.02 and averaged 2.94 or about 3. His average of 2.94 would indicate the aspect ratio of his individual shell was 1.31 ($1.31^4 \approx 2.94$).[2] Sharp gave the tangent angle as about 80.08°, agreeing with 80°5′ set forth by Thompson (1992: 791), a far cry from a φ ratio spiral with a radius vector ratio of about 7, and a tangent angle of 72.97°.

Meisner (2014) alternatively suggests that the Nautilus shell spiral expands by a golden ratio proportion, not by a full revolution as the common myth claims but by 1.618 every 180°. This implies that his specific Nautilus specimen has an aspect ratio of 1.272 (the square root of the golden ratio). For every 90°, the shell

[1] Markowsky (2005: 347) also made this point in his review of Livio (2003).
[2] Cf. the Nautilus mean aspect ratio of 1.31 in Table 1. (The meta-golden ratio χ^4 is equal to 3.38).

spiral would have grown logarithmically by 1.272. Therefore, the expansion over 180° would be $\sqrt{\varphi^2}$ or 1.618 as he asserts. Over a complete revolution of 360°, the growth factor would be 1.272^4 or 2.618. If the mean value for the aspect ratio of Nautilus shells is actually 1.310, the growth ratio for a half rotation would be 1.716. For a full rotation, it would be 2.94 as indicated above, not 2.618. Nonetheless, the range of ratios noted in Table 1, for all sampled *Nautilus* species, is between 1.261 and 1.348. Meisner's specimen falls just within that range and so his hypothesis based on his single shell may have some validity.

Another interpretation employing the $\sqrt{\varphi}$ (1.272) ratio was advocated by Rachel Fletcher (1988: 36–51: Fig. 8).[3] In discussing her analysis, Fonseca (1993) points out that her shell's spiral is inscribed in the $\sqrt{\varphi}$ rectangle with only three sides at a tangent. Four tangents are necessary to establish the correct aspect ratio. Fonseca calculated that her specimen actually averaged a growth ratio of radius vectors for $\theta = 90°$ in the order of 1.316, not 1.272, thus yielding a constant angle β of about 80°. Belgian mathematician Chris Impens (2016), in disputing similar claims, proposed a 4:3 ratio for the Nautilus shell shape. His shell specimen, though, according to his own Geogebra results gave the growth ratio as 0.7598 that would have a reciprocal of 1.316, agreeing with Fonseca's calculation for Fletcher's shell.

According to McMahon and Bonner (1983: 49), as the shell develops each chamber is enlarged by about 6.3%. Thus, the aspect ratio would be 1.277 (similar to Meisner's specimen of 1.272). Unfortunately, the 6.3% number is now repeated in books, particularly in math study handbook problems, as the *volume*. The number is not correct as the volume expansion of Nautilus shell chambers. Perhaps, the genesis for this misunderstanding is the authors' use of the phrase "in every dimension" in their segment on isometry, but in fact it was the planar cross section that they measured. I did my own volume expansion experiment using a glass dropper graduated in 0.05 mls to measure the volume of water in each chamber for three bisected leveled shells. The average volume expansion was 20–27% and not conclusive. However, Tajika et al. (2015) using micro-computed tomographies (CT scans) measured the volumes of chambers for 30 *N. pompilius* shells as well as the widths of shell chambers in three of the specimens. The authors note that "chamber widths expanded at a constant pace irrespective of the change in chamber volume". From their raw data, my calculation of the expansion rate for the width of 101 chambers in the selected specimens averaged 6.5%. This suggests it was the chamber *width* that was the key measure for 6.3% above. The chamber volume expansion rate averaged about 24% (a factor of three with the radius vector).

For a BBC series, Du Sautoy (2011) measured the radius vectors of his specimen revealing that each new chamber was proportionally larger to the preceding one by about 1.08 or 8%. This indicates the aspect ratio would be 1.360, which is beyond the high end of the aspect ratio range for *N. Pompilius* according to this study (see Table 1). Since the specimen did not appear to be a Crusty Nautilus, his larger 'half' of the inaccurate bisection evident in the video might explain the higher growth percentage.

[3] $\sqrt{\varphi}$ is a significant ratio in geometric pictorial composition where the four 'eyes' are in golden ratio proportions, vertically, horizontally and diagonally to all sides of the canvas. (Bartlett 2007: 103).

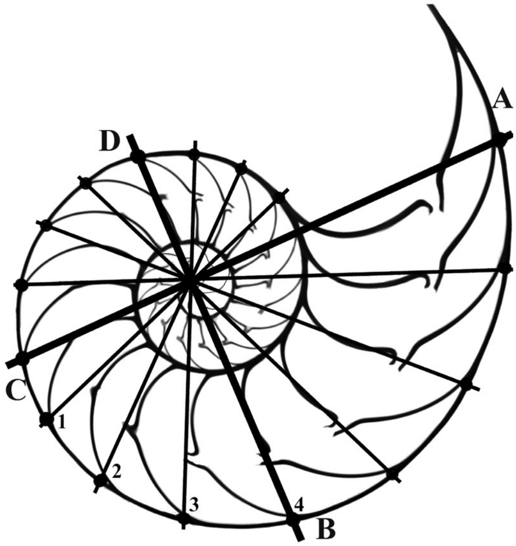

Fig. 1 *N. pompilius* (cropped) indicating four chamber spokes growing 31% over 90° or 7% each. Note the final enclosed chamber is smaller (indicating terminal growth or maturity)

To picture *radial growth* in relation to the *overall exponential expansion rate* imagine radius vectors as 'spokes' from the center axis to the join of the chamber walls at the peripheral edge of the shell. Each chamber spoke gets longer as the spiral gets bigger. The diagram over a *N. pompilius* nanotom scan in Fig. 1 illustrates four spokes expanding 31% or 7% each over a quarter turn. Spoke A is 1.310 times longer than Spoke B as is B:C and C:D. Thus, 16 chamber spokes expand by a factor of 2.94 (1.31^4) in a 360° rotation (cf. Sharp above). In a Crusty Nautilus Chi spiral the ratio of the lengths of two radius vectors spokes 90° apart would be 1.356 reflecting its aspect ratio (see Fig. 10). With the radius vector expanding at 7.9%, the Crusty Nautilus shell spiral enlarges by a factor of 3.38 (1.356^4) over a complete rotation.

A New Myth

In repudiating the golden ratio Nautilus myth, there is a disconcerting trend in persuasively imposing a new mathematical number on Nautilus proportions. In particular, in his aforementioned paper, Falbo (2005) describes his process as: "In 1999, I measured shells of *Nautilus pompilius*, the chambered nautilus, in the collection at the California Academy of Sciences (CAS) in San Francisco… The ratios ranged from 1.24 to 1.43, and the average was 1.33, not phi (which is approximately 1.618). Using Markowsky's ± 2% allowance for to be as small as 1.59 we see that 1.33 is quite far from this expanded value of phi". Undisputed, Falbo's measures have become part of a definitive new mythology. So much so, he is directly quoted as fact on Wikipedia. Typical of many other comments, but implicating other sources that are in agreement, Strogatz (2012) also quotes this new assertion that "nautilus shells…use ratios that average around 1.33, as discussed by Akkana Peck and by Ivars Peterson, using Clement Falbo's measurements of a nautilus shell collection". Yet, the Collections Curator of Invertebrate Zoology and Geology at CAS reasoned that since no record of formal access in 1999 for Dr. Clement Falbo could be found, "he may well not have measured all of the shells available to him at CAS…it could have only been a couple of shells from our Teaching Collection or otherwise" (private comm.).

Therefore, this study theorized that a larger sample might reveal quite different results. Within the range of individual shell ratios measured by Falbo, the meta-golden Chi ratio of 1.356 would fit and might be worth considering.

The Golden Ratio Spiral

The golden ratio is symbolically represented by φ (Phi), where $\varphi = (1+\sqrt{5})/2 \approx 1.618$. It is closely related to the two-term recursive Fibonacci sequence, where the ratio between a number and its predecessor tends to the golden ratio. If a line is divided into two parts, such that the smaller is to the larger as the larger is to the whole, it will have a golden ratio division. A rectangle in golden ratio proportions has the shorter side of 1 to the longer side of 1.618. This φ rectangle subdivides into a square and another golden ratio rectangle infinitely. An approximate logarithmic golden spiral is typically drawn in a golden rectangle or a similar rectangle using consecutive numbers from the Fibonacci sequence. The curve of the logarithmic spiral has a constant angle between a line (radius vector) from the origin or pole to any point on the curve and the tangent at that point. Thus, it is often called an 'equiangular' spiral. A logarithmic spiral maintains its shape as its size increases, a feature of self-similarity.

If a logarithmic spiral is in golden ratio proportions the radial vector ratios would increase in a progression of $\varphi^4 \approx 6.853$ or about seven for a full revolution. For every quarter turn, it would expand by a factor of 1.618. The golden ratio rectangle can be constructed with straightedge and compass starting with a square as outlined in Fig. 2. The interior divisions are found quite simply with a diagonal and one intersecting at right angles to it.[4] The approximate logarithmic spiral is then formed with arcs constructed within each rotating square. It is this familiar 'whirling squares' spiral that is universally compared to the Nautilus shell spiral (Fig. 9).

The Meta-golden Ratio Chi (χ)

Related to the golden ratio is an irrational mathematical constant, the meta-golden ratio Chi (χ). It is the positive solution of $x^2 - x/\varphi = 1$ and corresponds to $[1+\sqrt{(4\varphi+5)}]/(2\varphi)$ with a value of 1.355674…. It is a pleasing mathematical

[4] Starting with a Phi ratio rectangle, using what Le Corbusier called *l'angle droit*; the partition created divides the rectangle into a similar Phi proportioned rectangle and a square. This procedure can be used with any rectangle. If the rectangle, for example, were in a 1:1.310 ratio then the intersecting diagonals would subdivide it into a similar smaller 1.310 rectangle and its derived rectangle. The meta-golden ratio Chi rectangle (1:1.356) is unique in that it divides into another Chi rectangle and a Phi rectangle.

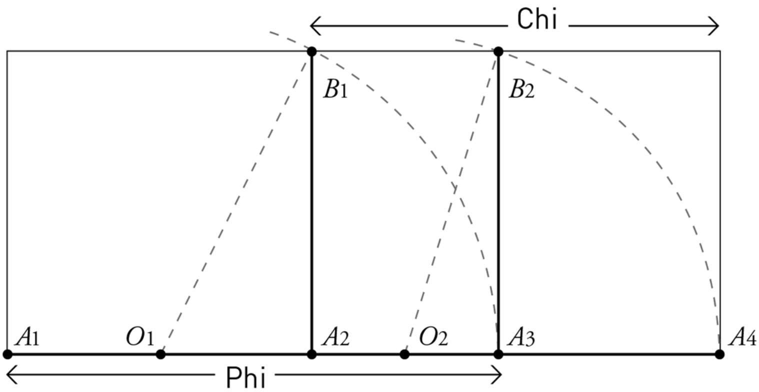

Fig. 2 Construction of a Chi ratio rectangle

number with notable generative geometric properties just as the golden ratio. However, instead of partitioning into a square and a φ-rectangle, the Chi rectangle of width 1 and a length of χ subdivides into a φ-rectangle and another smaller χ-rectangle (Huylebrouck 2014; Bartlett and Huylebrouck 2013). The number χ can be constructed using a compass and straightedge because it belongs to a quadratic closure of the field of rational numbers. This method also illustrates the geometric connection between χ and φ. The starting point is the familiar construction of the golden ratio rectangle (Fig. 1). Start with a segment A_1A_2 of length b. Now build a perpendicular to A_1A_2 segment A_2B_1 of length 1, bisect the segment A_1A_2 by the point O_1. Find the point A_3 of the intersection of the circle centered at O_1 with the radius $|O_1B_1|$ and the line containing A_1A_2. A simple calculation shows that

$$|A_1A_3| = \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + 1} \text{ and } |A_2A_3| = \sqrt{\left(\frac{b}{2}\right)^2 + 1} - \frac{b}{2}.$$

So if we begin with the line segment A_1A_2 of unit length, then the length of A_2A_3 will be $1/\varphi$. Repeating the same procedure starting with the segment A_2A_3 gives the point A_4 such that the segment A_2A_4 has the length χ. The Chi rectangle so formed, $B_1A_2A_4$ is thus divided into a φ-rectangle and another χ-rectangle.

The Nautilus… Facts and Phylogeny

The chambered Nautilus is the sole living cephalopod (octopus) that has an external shell (Fig. 3). It shows the same body and shell arrangement as ancient ammonoid cephalopods and as such, could be considered a ‘living fossil’. Found today exclusively in the depths of the Indo-Pacific, it has defied Darwinian evolutionary odds to survive essentially unchanged for as long as 500 million years, even outliving the Dinosaurs.

It is prized for its characteristic orange-brown tiger-striped shell. First appearing in Europe in the sixteenth century, Nautilus shells were treated as rare

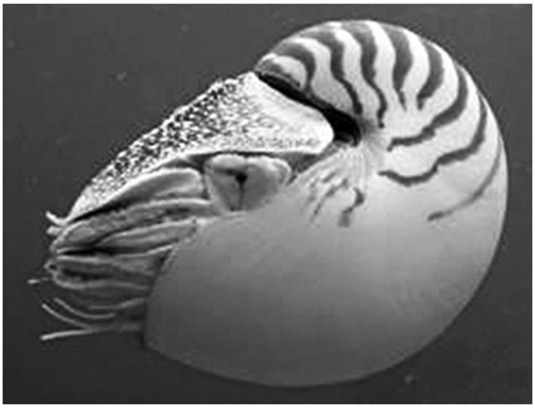

Fig. 3 *Nautilus pompilius* (Photos: Dr. Greg Barord)

and exotic objects to be lavishly mounted for pitchers, cups and table decorations. Polished, engraved or carved with patterns and scenes, they were also artistically embellished with elaborate sculptural elements adorned with bronze, copper, silver, and precious gems. They became status symbols and found their way into the 'cabinets of curiosities' or *wunderkammer* collections of royalty and the rich. Many now can be seen in museums across the world.

Malacologists have generally agreed there are six valid extant species: *N. pompilius*, *N. repertus, N. belauensis, N. macromphalus, N. stenomphalus, and N. scrobiculatus*. Nautilus eggs hatch after an embryonic period of almost a year as an already formed miniature Nautilus. It has a protective seven-chambered spiraled shell about one inch in diameter. As it grows it accretes mother-of-pearl to close off each smaller preceding chamber with a wall or septum. It occupies only the outermost chamber but connects to preceding chambers through a central duct or siphuncle. By osmotically filling or evacuating gas and fluid in the chambers via this duct the Nautilus can regulate buoyancy, allowing it to float neutrally (Saunders and Landman 1987). The support (septal neck) for the duct on each wall is important to mathematical studies of Nautilus shells. A visible bi-section of it signifies an accurate cut of the shell and consequently reasonably exact measurements (see Fig. 7). Distinct from the octopus and squid with eight and ten tentacles, the Nautilus has up to 90 small tentacles or cirri, encircling its mouth. The adult shell is around 150–200 mm in max. diameter (typically twice the max. shell width) and has 33–36 spiraling chambers.

One should keep in mind that each Nautilus specimen is a natural form with inherent biological variability and will not conform perfectly to strict mathematical parameters. Although Nautilus chamber growth follows overall exponential trends, some allometric discontinuities occur due to adverse ecological conditions (e.g. water temperature and pH) as well as in very early and late ontogeny (e.g. the last chamber is reduced in volume). There are also intraspecies and individual specimen deviations.[5] Growth rates from a single shell should not be considered as reliable. The Nautilus simply follows its genetic imperative growing gracefully larger without changing shape.

[5] E.g. aspect ratios over the *Nautilus* and *Allonautilus* species ranged from 1.261 to 1.348.

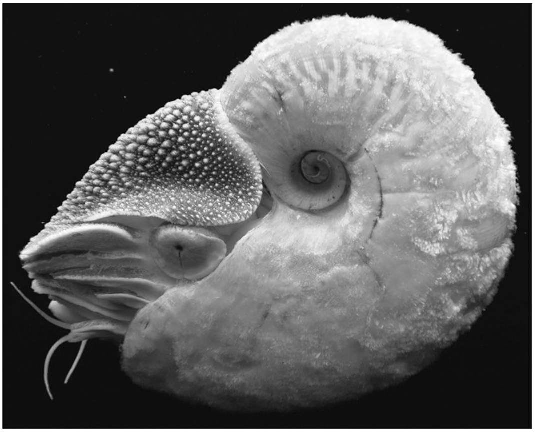

Fig. 4 *A. scrobiculatus* or Crusty Nautilius

The Crusty Nautilus

Requiring a separate description, there is a rare living species of Nautilus that used to be known as *Nautilus scrobiculatus* Lightfoot, 1786. Until recently, *Nautilus* taxa have been recognized as restricted to a single genus, *Nautilus* Linnaeus, 1758. However, it has been established (Ward and Saunders 1997) that there are differences between *Nautilus scrobiculatus* and other *Nautilus* species significant enough for it to be renamed as a sister taxon of Nautilus with its own genus, *Allonautilus* and called *Allonautilus scrobiculatus* Saunders, 1997. First seen live in 1984, the elusive *A. scrobiculatus* was not seen again alive until 31 years later in 2015. Until then, it was feared extinct.[6] Because, when alive the *A. scrobiculatus* is 'encrusted' with a thick fuzzy skin over the shell, it has been commonly referred to as the Crusty Nautilus. The shell typically reaches about 180 mm in maximum diameter (the largest in this study was 177.5 mm.) Unlike *N. pompilius*, the umbilicus (corresponding to the apex or pole of the spiral) is not covered by a callus and so the origin of the spiral formation is much more evident on the exterior of the shell (Figs. 4, 11).

Methods and Materials

Previous research studies were usually based on a single specimen. It seems only Falbo's study measured more than a one shell. But, his published paper gave no details about the specimens or how the shells were measured.

So this study is very different; it took a large sample size, examining 80 shells from the Smithsonian National Museum of Natural History (NMNH) in Washington D.C. The museum specimens had also been identified by world renowned specialists.

[6] Nautilidae populations are endangered because of commercial over-fishing for its shell. "Save the Nautilus Foundation" leads conservation efforts.

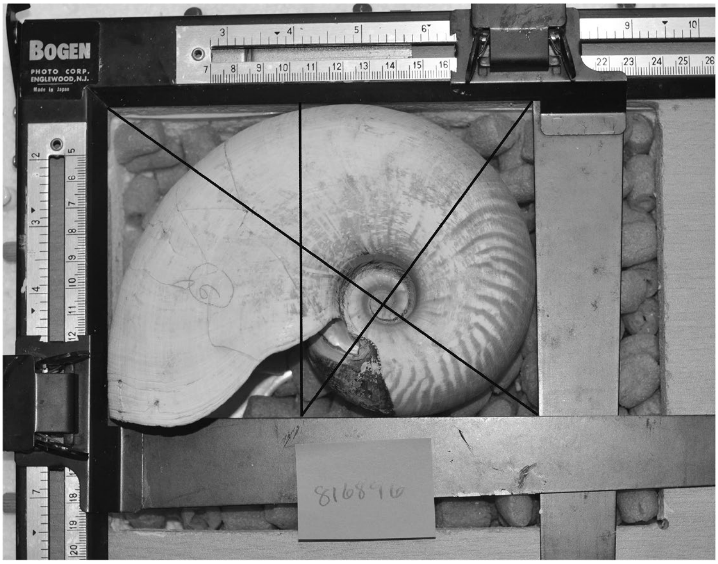

Fig. 5 Crusty Nautilus in measuring easel with Photoshop overlay

The spiral inscribed in the familiar golden ratio or a Fibonacci rectangle is by far the predominant comparison to the Nautilus. Therefore the aim was to duplicate that comparison by measuring the rectangle that envelopes the shell (Fig. 5). The orientation of the shell within a rectangle is critical to correctly measure the aspect ratio because if the spiral is rotated it can conform to varying geometric interpretations. For example, it fits perfectly into a square box, but then the spiral is only tangential to three sides (Fig. 6). The pole is centered vertically. As pointed out earlier, it is necessary to have the spiral tangential to all four sides of a rectangle for an accurate measurement of its aspect ratio.

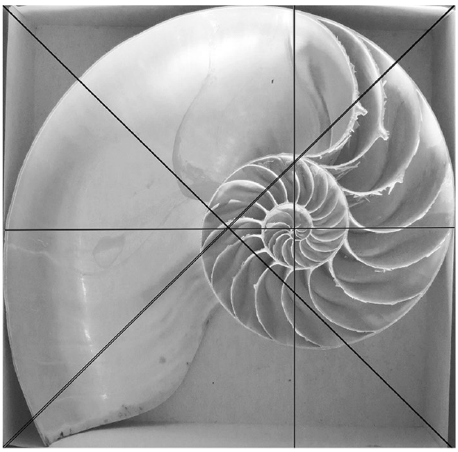

Fig. 6 In a square

Figure 5 is a prime example of a Crusty Nautilus specimen (NMNH #816896) in the easel showing a measurement of 160 × 118 mm. (an aspect ratio of 1:1.356). A Photoshop linear overlay corroborates accurate diagonals through the umbilicus. On the left side, the central terminal end of the shell's aperture is braced into the easel to represent a bisected spiral. Thus, the edges of the apertural lip (peristome) overlap the easel. It took two full days at the Smithsonian's NMNH Dept. of Invertebrate Zoology to carefully measure all accessible, whole, dry Nautilus shells.

Results

Statistically, the distributions were symmetric. The more prevalent *N. pompilius* (46 shells) had aspect ratios ranging from 1.261 to 1.348, a mean of 1.310 and a median of 1.309. For *N. repertus, N. stenomphalus, N. belauensis, and N. macromphalus,* a reliable statistical analysis would have required a sample size of at least 10 each but were not available. However, if the ratios of those species were considered together, adding the individual ratios of the 19 shells, the mean was also 1.310. All five species of Nautilus, if taken as a whole (65 shells), with ratios added individually, not as aggregates, had the same range and also a mean of 1.310 (Table 1).

Although results for the smaller sample sizes of species, other than *N. pompilius*, were not inconsistent, analyzing the five species of *Nautilus* in total was statistically more reliable (Table 2). It was also a method supported by recent genetic research. Disputing traditional taxonomy, new studies suggest that there is only one species of Allonautilus, the Crusty Nautilus (Grulke 2016: 72), while everything else is *Nautilus pompilius* (Vandepas et al. 2016).

Table 1 Summary of statistical analyses

Species	n	min. AS / max. AS	Mean
N. pompilius	46	1.261 / 1.348	1.310
N. repertus	7	1.288 / 1.328	1.309
N. stenomphalus	6	1.296 / 1.348	1.323
N. belauensis	4	1.291 / 1.320	1.312
N. macromphalus	2	1.262 / 1.291	1.276
Total	65	1.261 / 1.348	**1.310**
A. scrobilicatus	15	1.341 / 1.372	**1.356**

AS = Aspect Ratio

Table 2 Statistical overview

Nautilus pompilius		All *Nautilus* species		*Allonautilus scrobilicatus*	
Minimum	1.261	Minimum	1.261	Minimum	1.341
Maximum	1.348	Maximum	1.348	Maximum	1.372
STD	0.018	STD	0.018	STD	0.010
Mean	1.310	Mean	1.310	Mean	1.356

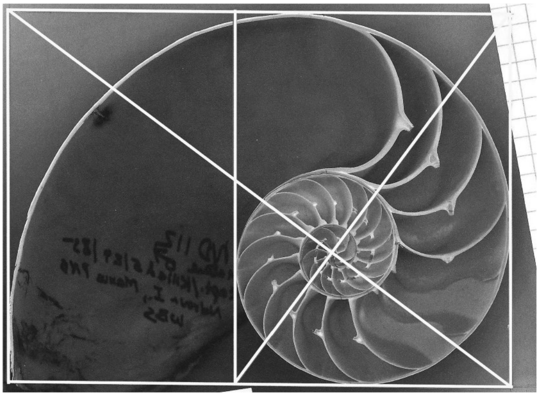

Fig. 7 Scan of bisected *A. scrobiculatus*, AMNH #94891 (provided by Dr. Neil Landman)

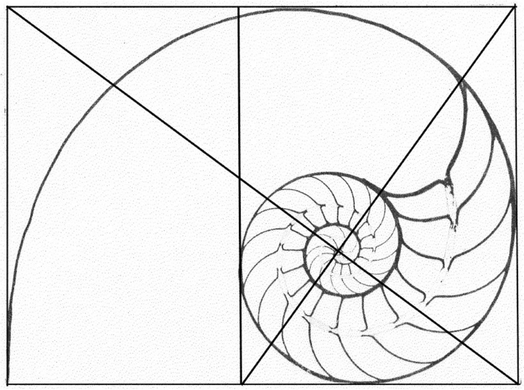

Fig. 8 Nanotom scan of *A. scrobiculatus* (provided by Dr. René Hoffman)

Unpredictably, results for *A. scrobiculatus* or the Crusty Nautilus were significantly different. Remarkably, the mean aspect ratio of the Crusty Nautilus before rounding was 1.35566, an extraordinary correspondence to the meta-golden ratio Chi at 1.35567.[7]

Crusty Nautilus Chi spiral

To examine the spiral itself, a scanned image of a professionally bisected shell of the Crusty Nautilus (*A. scrobiculatus)* was obtained from the American Museum of Natural History (AMNH) in New York (Fig. 7). As one of only three rare bisections in their collection, although two septa are missing, it is otherwise whole. AMNH put slips of graph paper next to the shell so that I could verify the scan was not distorted.[8]. It was not. I also obtained another image, a nanotom CT scan from

[7] With mean aspect ratios of 1.310 and 1.356 respectively (Table 2), the *Nautilus* and *Allonautilus* genera show a morphological variance in spiral expansion rate of about 0.045 or 3.46%. Some corroboration of this was actually recorded by paleobiologists, Shapiro and Saunders (1987: 533 Table II). Respective measures of the whorl expansion rates between these two genera on a small sample were 2.975 and 3.016, a variance of 0.041 and similar to 0.045 observed here.

[8] It also shows an accurate septal neck bisection, critical for accuracy, which is missing in some studies.

Ruhr-University Bochum, Germany as a virtual transverse median section of *A. scrobiculatus* (Fig. 8)

For purposes of comparison and verification, both spirals were oriented and inscribed in a rectangle diagram exactly as the whole shells were clamped into the photo easel device (Figs. 7, 8). Both rectangles are in the meta-golden ratio proportions of 1:1.356. These are individual natural specimens and were not expected to be a perfect match, but both images confirm that the spiral accommodates itself well to the Chi ratio rectangle. The shell's pole or axis of coiling occurs precisely at the intersection of the rectangle's diagonal and one drawn at right angles to it. It is also tangential to the primary whorls of the shell's spiral (see also Fig. 10).

The Golden Ratio vs. the Meta-golden Ratio Spirals

Since the Crusty Nautilus shell spiral has been found to be in meta-golden proportions, a direct comparison with the golden ratio spiral might be appropriate (Figs. 9, 10). The rectangle in Fig. 9 is in a ratio of 1:1.618 while in Fig. 10 it is 1:1.356. Just comparing the curve of the Allonautilus to the golden ratio curve visually, it is persuasively obvious that the Nautilus spiral is not even close to the golden ratio rectangle shape. However, it is interesting to note that each rectangle is subdivided by a golden ratio rectangle. In Fig. 9 it is on the right side, and in Fig. 10 on the left side. If overlapped by the extent of these Phi rectangles, the resulting diagram would then echo Fig. 2. It is also evident that the natural spiral of the Crusty Nautilus is a fairly accurate fit into the 'whirling' meta-golden rectangle (Fig. 10).

Art and architecture

The appearance of the meta-golden ratio Chi (1.356) in art or architecture is not that controversial (Fathauer et al. 2018: 10). After all, the Chi ratio was discovered in the process of analyzing the geometry of composition (Bartlett and

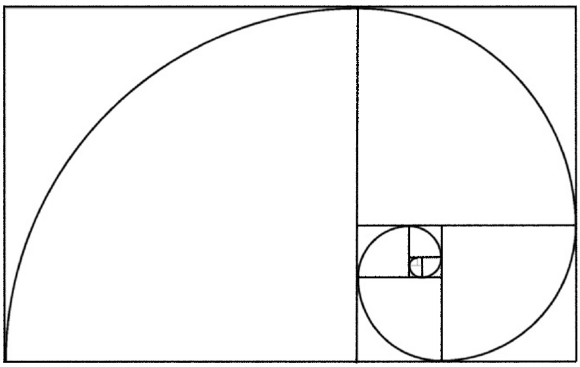

Fig. 9 Golden ratio rectangle with inscribed approximate golden ratio spiral

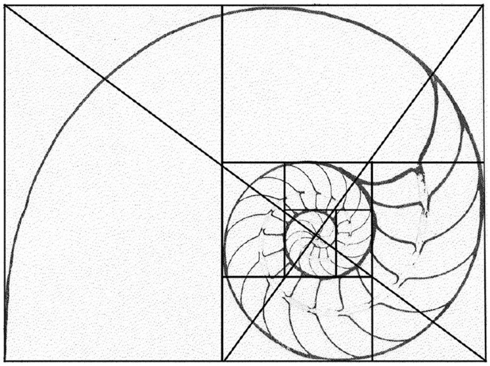

Fig. 10 Crusty Nautilus scan overlaid with a meta-golden ratio Chi and corresponding subdivision

Huylebrouck 2013: 451–452). An example in art was examined by Michael Trott (2015), chief scientist for Wolfram Research. In looking at voluminous datasets (over a million paintings) of the aspect ratio of paintings through the ages, for movements and painters, he found that: "The median of the aspect ratios of all paintings decreased over the last 500 years… The mean also decreased and seems to stabilize slightly above 1.35." He also notes that "… the "meta-golden ratio chi" could be included… and might be a more significant ratio in aspect ratios in painting canvases in recent times… suggesting the use of a 1:1.35 ratio in structuring the geometry of their compositions."

Recent studies have asserted that the Nautilus shell aspect ratio is 4:3 (1:1.333). Undoubtedly, many artists and architects, including Vitruvius, Pacioli, Leonardo, and others, used this classical 'sesquitertia' ratio. Yet, based on the evidence of this study, the *Nautilus pompilius* spiral appears to have an aspect ratio of 1:1.310, not 1:1.333. Accordingly, to supplant contemporary claims, could there be some other mathematical constant that, in light of the results of this study, is a more appropriate contender for the actual ratio of 1.31?

In her papers on 1.3 ratios in art and architecture, Redondo Buitrago (2013: 2014) discusses the Cordovan proportion, $c = (2 - \sqrt{2}) - 1/2 \approx 1.3065$. This number was discovered by the Spanish architect Rafael de la Hoz while attempting to demonstrate the presence of the golden ratio in the architecture of the city of Cordoba. The Cordovan number, approximated to 1.31, could be one ratio to consider for *N. pompilius*. Like Chi, intimately connected to the golden ratio, a more apposite candidate is the solution of $(\varphi + 1)/2$ (or φ^2: 2) ≈ 1.309. This number, 1.309 matches the median aspect ratio of *N. pompilius* and rounded to 1.31, is an excellent fit to the mean aspect ratio of the whole *Nautilus* genus (excluding, of course, the Crusty Nautilus) (Fig. 11).

In architecture, 1.309 features predominantly in Le Corbusier's 'Second Modulor'. The well-known Modulor is a system of anthropometric dimensions that Le Corbusier based on the height of a 6 ft. (183 cm) fictional British policeman. His amended Modulor employed two sequences, a 'Red Series' established with a key measure of 113 cm (the height of the male's navel), and a 'Blue Series' with a key measure of 226 cm (twice the navel height and the

Fig. 11 Crusty Nautilus shell spiral (photo: author)

total height of the man with an upraised arm). Canons of proportions of the human figure have often suggested that the navel 'ideally' sub-divides the height in golden ratio proportions (Bartlett 2014: 307). Corbusier used the same belief to calculate his fundamental number, 113 cm (183 cm/φ). By extrapolating 113 and 226 by 1.618, all the numbers both higher and lower were calculated in each of the red and blue sequences. They are presented side by side but staggered so the numbers are not opposite. This closed otherwise large gaps occurring in successive dimensions in each series and allowed for a greater sophistication of applied aesthetic proportions. As in the Fibonacci sequence, if we add any two numbers in either sequence, we obtain the next term. Across the two series, there are other intrinsic relationships, one of which is revealed if each higher number of the Red series is divided by the opposite lower number of the Blue series (Le Corbusier 1961: 82–83). For example: …113/86.3 ≈ 1.309; 182.9/139.7 ≈ 1.309; 295.9/226 ≈ 1.309…. In this context, and approximated to 1.31, it has a fitting relationship to the Nautilus spiral, especially considering Le Corbusier's apparent personal affinity for the shell. For example, in the 2013 MoMA exhibition, 'Le Corbusier: An Atlas of Modern Landscapes', the Nautilus shell was conspicuously displayed as one of his eclectic inspirational natural objects.[9]

Conclusion

That the Nautilus shell spiral is governed by the golden ratio, has been essentially debunked, although strong vestiges of the popular myth still remain. Nevertheless, a newer myth with increasing traction claims that the Nautilus spiral has an aspect

[9] Unfortunately, photographs of the exhibition were prohibited. Besides a graphic sketch (used as a book cover for *Who was Le Corbusier*), he also painted the shell in an illustration of his poem on proportion in *Le Poeme de l'Angle Droit*.

ratio of 4:3 (1.333). This assertion has been propagated by many authors, either by generalizing from their own specimen's ratio or quoting Falbo's (2005) study that provided no details of species, methods, or sample size.

Therefore, the initial hypothesis for this study was that the putative ratio of 4:3 for the aspect ratio of the Nautilus shell spiral is a misapprehension. By measuring a larger sample of Nautilus shells across species, the aim was to challenge this new myth and to propose that the meta-golden ratio Chi (1.356) might be a better fit. With compelling empirical evidence, two ratios emerged as mathematical standard bearers; 1.310 for the *Nautilus* genus and for the Crusty Nautilus 1.356, the meta-golden ratio Chi.

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