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RESEARCH

The Geometry of Cuboctahedra in Medieval Art in Anatolia

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Abstract Numerous examples of cuboctahedra found in medieval-era buildings whose dates range from the early twelfth to the early fifteenth century across in Turkey indicate the significant use of such geometrical entities. Here we focus particularly on cuboctahedra with carved-out surfaces. The results show that although the unit cell, which is a combination of cubes and tetrahedra, sufficiently explains all examples, the octahemioctahedron and stella octangula strengthen the possibility of tetrahedral packing with its dual network and indicate a "vector matrix", as suggested by R. Buckminster Fuller. Therefore, their prevalent use as a "geometric solid" in a hollow cube frame and their appearance as an envelope of either tetrahedral packing or highly complex surfaces reveal almost 800-hundred-year-old examples of cuboctahedra as a Vector Equilibrium (VE) producing Isotropic Vector Matrix (IVM).

Keywords Geometry · Architectural ornament · Archimedean solids (cuboctahedron · Rhombic dodecahedron) · Polyhedron/polyhedral · Medieval art · Anatolia

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Birkhäuser

Introduction

The visual elements of medieval-era Islamic Art can generally be grouped into calligraphy, vegetal motifs, and abstract geometrical patterns. Among these, the abstract geometrical patterns are the most prevalent and have been of great interest in Anatolia. These complex geometrical patterns, which were apparently formulated by skilled designers, have generally been regarded as serving only decorative purposes with symbolic content, like other types of art. Recently, however, some scholars have used several examples to show how the artisans of the era constructed an ancient mathematical problem free of symbolic, linguistic, or visual associations imposed by historians (Chorbachi 1989; Makovicky 1992; Bonner 2000; Lu and Steinhardt 2007; Cromwell 2009). Similarly, in Turkey, Alpay Özdural has presented extensive works on the scientific content of medieval art in Anatolia (Özdural 1995, 1996, 1998, 2000, 2015). One of his best-known studies (Özdural 2000) reveals the systematic meetings between mathematicians and artisans in the tenth-seventeenth centuries. According to Özdural (2000: 171–172), the manuscripts written by Abu al-Wafā’ al-Būzjānī in the tenth century in Baghdad, Omar Khayyam in the eleventh century in Isfahan, al-Kashi in the fifteenth century in Samarkand, and Cafer Efendi in the late sixteenth and early seventeenth centuries in Istanbul demonstrate the effects of this collaboration between mathematicians and artisans on the art and architecture of the era. Among these scholars, Abu’l-Wafā’ and al-Kashi studied polyhedral geometry in general, and the cuboctahedron in particular. Furthermore, Thabit ibn Qurra, one of the first translators of Euclid’s *Elements* into Arabic, was recorded as being the first to specificially study and illustrate cuboctahedron in the ninth century.

In this context, numerous examples of cuboctahedra indicate the theoretical interest of medieval scholars in this form as well as the practical skill and knowledge in geometry of medieval builders. Following our earlier discovery of cuboctahedra at the capitals of aiwan in the hospital part of Gevher Nesibe Complex (Hisarlıgil and Bolak-Hisarlıgil 2009), which was built between 1204 and 1206 in Kayseri, we began to study Anatolian Seljuk structures from the literature to find whether there are more examples. Since 2012, we have visited most of the towns to record examples of polyhedra, which can be dated to between the early twelfth to early fifteenth centuries. All examples found in fifty-nine buildings in twenty towns of Turkey (see Figs. 4 and 5 below) are cuboctahedra, but dodecahedra and octahedra were found in two buildings. Moreover, further examples of cuboctahedra found in four other buildings were dated from the first half of the sixteenth century to late eighteenth century. Among the extensive number of medieval era examples, the examples that provide complex information for further geometrical content of cuboctahedra in Ağzıkara Caravanserai, Sarı Caravanserai, Tomb of İzzeddin Keykavus I, and the tombstone in Bursa are particularly analyzed in this present study.

In what follows, first the cuboctahedron that appeared as an individual figure is evaluated as a semi-regular Archimedean solid, which was applied as an engaged column capital in architecture. Then, the examples of cuboctahedra that envelope complex carved-out surfaces as “Polyhedral Clusters” are extensively analyzed. Next, using the data in the third part, we explore the relations between the edges and the complex surface profiles by removing the surfaces of examples as vectors that are associated with Fuller’s “vector equilibrium” (VE) and “isotropic vector matrix” (IVM), and we use Thabit ibn Qurra’s studies on the cuboctahedron as intersecting hexagonal faces. The results demonstrate the accuracy of the craftsmanship behind the complexity of form and reveal the mathematical content of 800-year-old artworks. Although this study suggests that the examples of cuboctahedra progressed linearly from solid to surface and from surface to lines, the results show that they were evidently designed as vectors.

The Cuboctahedron as a “Geometric Solid” in Medieval Art in Anatolia

The origin of polyhedra has been dated back to the Babylonians; they appeared long before the times of the Ancient Egyptians and Ancient Greeks (Friberg 2007: 351). Furthermore, recent findings in Scotland of stones carved with lines that correspond to the edges of regular polyhedra have introduced a new dimension to this topic (Hann 2013: 123). Although the regular polyhedra—tetrahedron, cube, octahedron, icosahedron, and dodecahedron—are discussed in Plato’s *Timaeus* and carry the name “Platonic solids”, Plato should not be credited with their discovery (Cromwell 1997: 71). The Byzantine writer Suidas recorded the history of the regular polyhedra up to the eleventh century in an encyclopedia known as *Suda Lexicon* and contributed that Theaetetus of Athens, an astronomer, philosopher, and disciple of Socrates, first wrote about these five solids (Cromwell 1997:74). Later, Euclid gave a complete mathematical description of them in Book XIII of the *Elements* from 300 BC. The other thirteen semi-regular solids were attributed to Archimedes by Pappus in the fifth book of his *Mathematical Collection* (Cromwell 1997: 79), but according to Heron, Archimedes ascribed one, the cuboctahedron, to Plato (Coxeter 1973: 30).

Numerous manuscripts suggest that medieval-era Muslim mathematicians began to translate Euclid starting in the eighth century, long before Adelard of Bath (1080–1152) and Gerard of Cremona (1114–1187) translated them (Freely 2012: 84). For example, al-Kindi (801–873), the Banu Musa brothers (ninth c.), Thabit ibn Qurra (826–901), al-Khazin (900–971), Abu al-Wafā’ al-Būzjānī (940–998), Abu Nasr Mansur Ibn Iraq (960–1036), Ibn al-Haytam (965–1040), and al-Biruni (973–1048) discussed polyhedra extensively. The mathematicians who are known to have particularly studied and geometrically illustrated cuboctahedra are Thabit ibn Qurra and al-Būzjānī. In his book *On the Construction of a Solid Figure with Fourteen Faces Inscribed into a Given Sphere* (*Kitāb fi al-’amal shakl mujassam dhī arba “ashara qā’ida tuhīt bihi kura ma’luma*),[1] Thabit ibn Qurra considered a spatial construction of a polyhedron bounded by six squares and eight equilateral triangles (Rosenfeld and Youschkevitch 1996: 123) in the ninth century. Studying it as a fourteen-faced polyhedron (Cromwell 1997: 104, Bijli 2004: 33, Rashed 2009: 9), Thabit ibn Qurra illustrated the cuboctahedron as a three-dimensional figure with

[1] We studied this manuscript from the French version translated (Asselah 2009: 324-331).

fourteen faces of equal edges and equal angles, where eight of these faces are equilateral triangles and the other six are squares (Asselah 2009: 320).

In his treatise, *On the Geometric Constructions Necessary for the Artisan (Kitab Fī mā yaḥtaj ilayhi al-ṣāni' min al-a'māl al-handasiyya)*, the tenth-century mathematician-astronomer Abu al-Wafā' al-Būzjānī presented several methods to tessellate a sphere using the properties of the five regular Platonic solids and some of the semi-regular polyhedra (Hogendijk 2012: 38). The geometric constructions of the three-dimensional tessellations recorded in this treatise may have served as a basis for architectural monuments (Sarhangi 2008: 511). Al-Būzjānī's treatise includes constructions of spherical Archimedean solids such as the cuboctahedron (tiling with eight equilateral triangles and six squares) (Sarhangi 2008: 522). Ghiyath al-Din Jamshid al-Kashi also studied cuboctahedra in the late medieval era, shortly before the shape appeared in textbooks during the Renaissance. In 1427, in the seventh section of *Key to Arithmetic (Miftah Al-Hisab*) al-Kashi wrote about the volumes of seven polyhedra including the cuboctahedron. He described them according to the shape and number of faces; their common property is that all of them are composed of right cones whose vertices intersect in the center of the polyhedron (Ta'ani 2011: 47). He also provided instructions for calculating the volumes and surface areas of vaults, rooms, domes, and other structures. Regarding surface areas and volumes of structures, al-Kashi claimed that his methods were more complete and more efficient than those of his predecessors (Azarian 2000: 84–85).

Examples of numerous polyhedra found in medieval art in Anatolia would appear to represent the geometrical studies of the era. These examples, which differ in form, size, and material, are present in different parts of various types of buildings. Although numerous examples are known as individual objects such as weights (Fig. 1a), polyhedral forms that find application in construction are rare, such as the nail top of iron doors (Fig. 1b) and connection nods of the iron window grilles rods (Fig. 1c), which are mostly cuboctahedron. However, further design applications of such polyhedral geometry as an integral part of the art portray the mathematical content of these figures and the skill of the artists. For example, the icosahedral caps on the interior vault surface of Karatay Madrassa in Konya (Fig. 1d) are two-dimensional, whereas the octahedral caps on the surface of the minaret of İnce Minareli Madrassa in Konya (Fig. 1e) connect the brick gap, which shows another geometrical tiling that is only recognizable from a distance. Furthermore, in Korkut Mosque in Antalya (Fig. 1f), the octahedral packing in the circular form transforms the caps into a supporting cantilever for the minaret balcony.

In this context, many medieval polyhedra in Anatolia are found in engaged column capitals and pedestals, which indicate such geometrical use in architecture. Most examples are cuboctahedra (Fig. 2a), while the exceptions are of the octahedra at the portal of Hacıkılıç Mosque in Kayseri (Fig. 2b), the truncated cubes in Murat Hudavendigar Mosque in Behramkale/Çanakkale (Fig. 2c), and the dodecahedron in the prayer niche of Mısri Mosque in Afyon (Fig. 2d). One other example of a dodecahedron is in the Tomb of Gömeç Hatun in Konya, which is dated to 1270. This figure, studied by Özgan and Özkar (2017) for possible surface tiling, is not

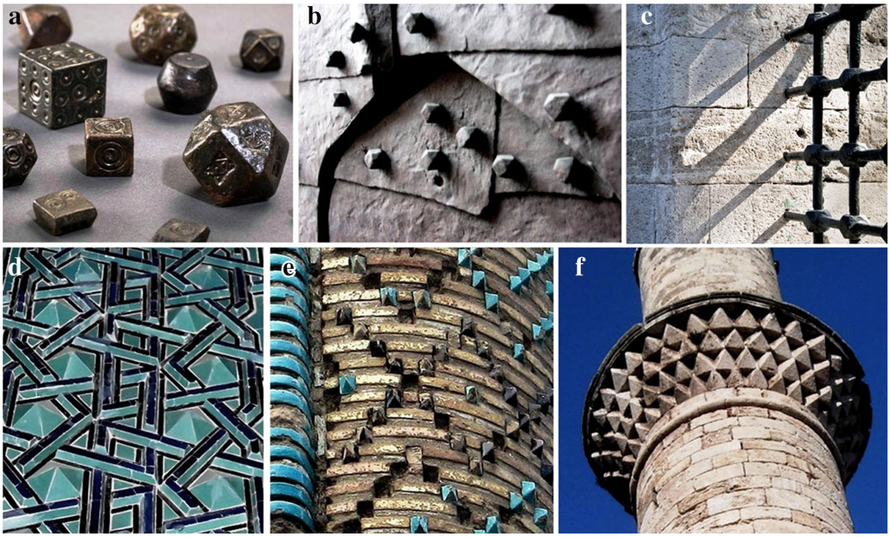


Fig. 1 **a** Islamic Period weights, Bronze, Suna and İnan Kıraç Foundation Anatolian Weights and Measures Collection; **b** cuboctahedral nail tops of iron doors; **c** cuboctahedra as the connection nods of the iron window grilles rods; **d** vertex of the ceramic icosahedral caps on the interior vault surface of Karatay Madrassa; **e** ceramic octahedra on the surface of the minaret of İnce Minareli Madrassa; **f** conical packing of stone octahedra on the surface of minaret balcony of Korkut Mosque

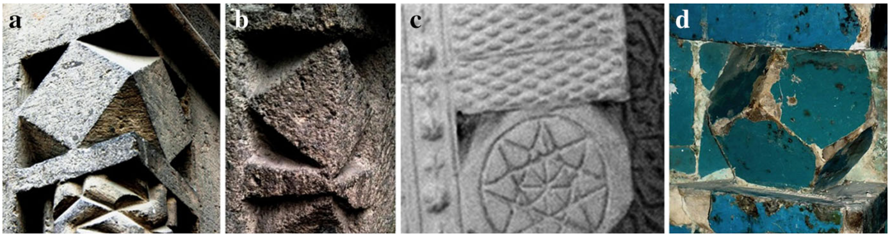


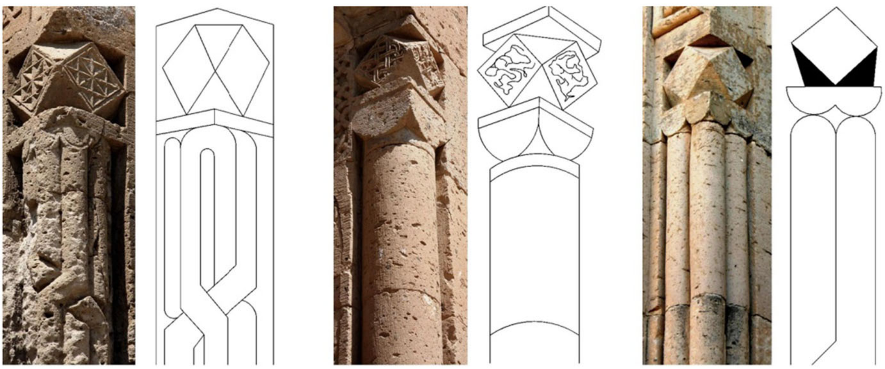
Fig. 2 **a** Cuboctahedron at the portal of Hunat Hatun Mosque; **b** octahedron at the portal of Hacıkılıç Mosque; **c** truncated cube in Murat Hudavendigar Mosque; **d** dodecahedron at the prayer niche of Mısri Mosque

used as an example of dodecahedron because it is not framed by the cubic enclosure and the restoration interventions are unclear.

Described as "a small prismatic capital type placed on the engaged column in Ottoman building terminology" in *Dictionary of Art Concepts and Terms* (Sözen and Tanyeli 1986: 259), these examples are called by different names such as "dice capital", "cubiform" (Aslanapa 1971: 154), or "diamond shape molding" (Önge 2004: 24). Among these names, "dice capital" is the most common one and is used widely in the literature (Diez 1947; Ögel 1966; Akok 1968; Öney 1971; Bakırer 1976; Önkal 1996; Özer 2004; Yıldırım and Üzüm 2010). For example, Ernst Diez, an art historian of the famous Vienna School, who also established the Chair of History of Turkish-Islamic Art in Istanbul University in 1943, described such capitals in his 1947 Turkish article *Endosmos'lar* as "zar başlık" (Diez 1947: 222), and in the German translation they were called "das Würfelcapitäl" (Diez 1947: 231), which means "dice capital". Diez described this dice capital in parentheses as a "cube form" (Diez 1947: 222). In 1966, Semra Ögel (1966: 27) described the capitals of Ağzıkara Caravanserai as "the dice placed upside down". She also made drawings of the capitals of corner engaged columns of the main entrance portal (Fig. 3a) and inner court portal (Fig. 3b) of Ağzıkara Caravanserai, and inner court portal of Kayseri-Sivas Sultan Caravanserai (Fig. 3c) to compare the positions of the dice.

Similarly, Ömür Bakırer (1976: 82), in her book on the mihrabs of thirteenth and fourteenth-century Anatolia, called the figure a "dice capital upside down" and added that "in this type of capital that is most commonly used, the two pointed ends of the dice were placed up and down". Furthermore, Hakkı Önkal (1996: 51) described the portal engaged columns of Ağzıkara Caravanserai and portrayed the "dice capital" as having been placed to sit on its corners. According to Ayla Ödekan (1987: 473), these common types of capitals with considerable variations are encountered in numerous buildings, and the "dice capital" is mostly used by being placed on its vertex.

In our previous study, which focused on 800-year-old cuboctahedra in medieval buildings such as caravanserais, mosques, tombs, and madrassas in Kayseri (Hisarlıgil and Bolak-Hisarlıgil 2009), we state that most of all discovered examples of polyhedra are cuboctahedra, most of which can be dated to between the early thirteenth and fourteenth centuries. As mentioned earlier, examples of cuboctahedra dating from the early twelfth century to the early fifteenth centuries were found in fifty-nine buildings in twenty towns of Anatolia. Although examples are geographically distributed in various locations throughout Anatolia, the majority are found in Konya, Karaman, and Kayseri, which constitute the southern part of central Anatolia and were important centers in medieval-era Anatolia: thirty-five buildings



a b c

Fig. 3 **a** Drawing of the capital of Ağzıkara Caravanserai's corner engaged column of the main entrance portal (after Ögel 1966: 28); **b** drawing of the capital of Ağzıkara Caravanserai's corner engaged column of the inner court portal (after Ögel 1966: 30); **c** drawing of the capital of Kayseri-Sivas Sultan Caravanserai's corner engaged column of the inner court portal (after Ögel 1966: 30) out of the fifty-nine demonstrate the density of examples in this part of Anatolia. Figure 4a shows the distribution of the examples of cuboctahedra in Anatolia according to the number of buildings per town. The cuboctahedra of similar sizes appear in various locations in these buildings. They mostly appear as the capitals of engaged columns at portals (Fig. 4b) and prayer niches (*mihrab*) (Fig. 4c), all of which are considered niches. The lower part of the niche is connected to the upper part via a stalactite vault (*muqarnas*) and the arch in an architectural context. However, they can also be found at other parts of the buildings, such as the pedestals of corner engaged columns (Fig. 4d) or capitals of corner engaged columns (Fig. 4e).

The examples of cuboctahedra in our study are catalogued in chronological order as in Fig. 5. Each example is representatively selected so that more than one can appear in each building. As can be seen, most examples of cuboctahedra can be dated to the thirteenth century, and they appear in forty of the fifty-nine buildings from this century. Among 256 examples of cuboctahedra, most of the catalogued buildings have at least two examples of cuboctahedra, with the greatest number of examples (twenty-four) found in Döner Kumbet in Kayseri. Although some examples appear with adjacent square faces as twin-cuboctahedra, such as in Ebul Fetih Mosque in Karaman and Döner Kumbet in Kayseri, the remainder are

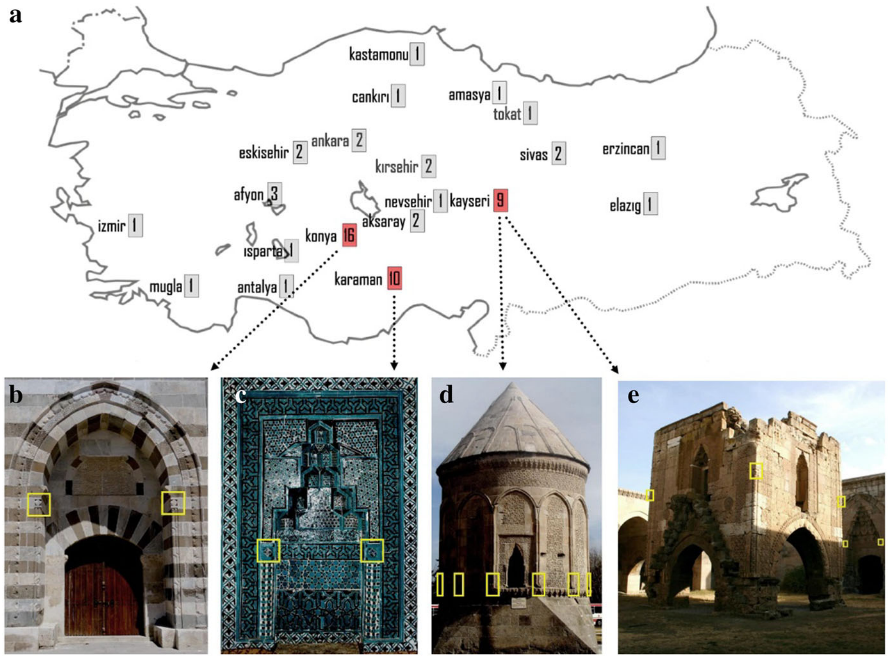


Fig. 4 **a** Map of the example distribution of cuboctahedron in Anatolia, which is related to the number of buildings per towns; **b** engaged column capitals at the portal of Zazadin Caravanserai in Konya; **c** engaged column capitals at the prayer niche of Kazım Karabekir Mosque in Karaman; **d** pedestals of corner engaged columns at Döner Kumbet in Kayseri; **e** capitals of corner engaged columns at Sultan Caravanserai in Kayseri

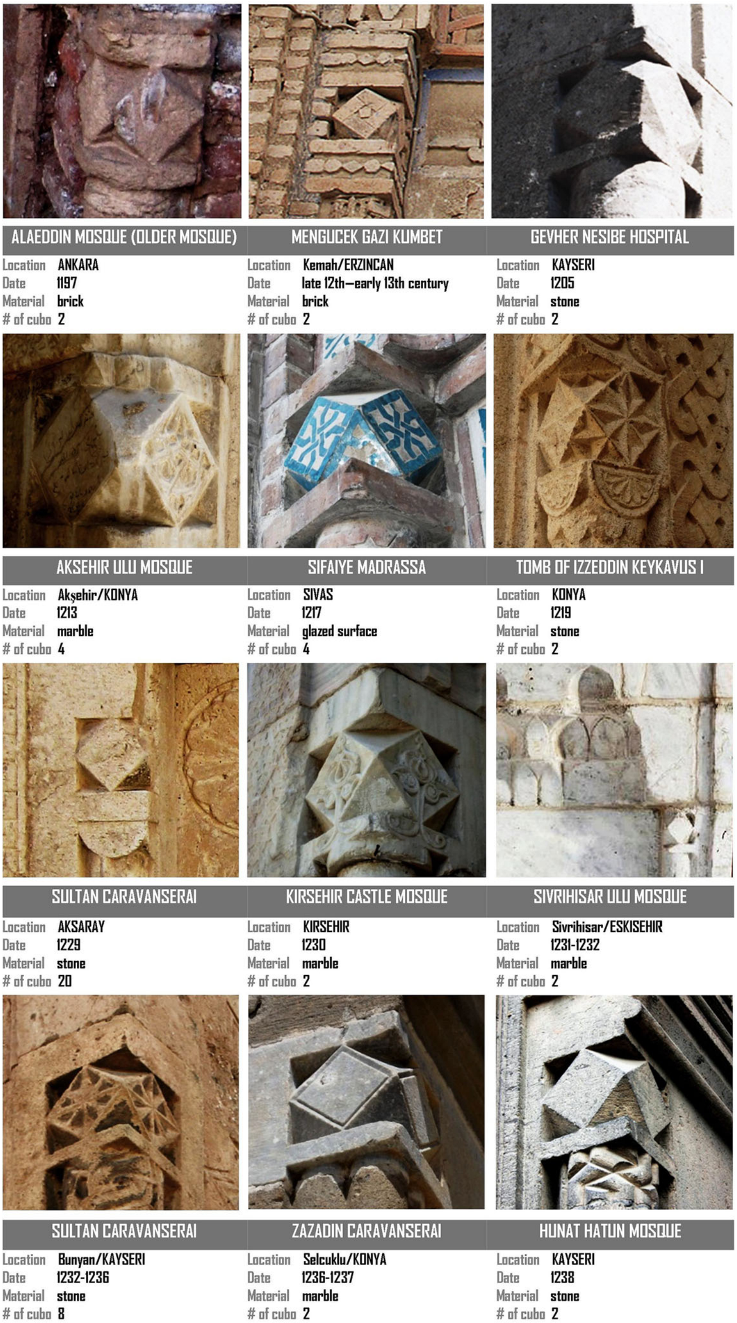


Fig. 5 Cuboctahedra in the Medieval era in Anatolia in chronological order (Drawings of the damaged examples in Kızılören Caravanserai in Beyşehir (After Bakırer 1976: 268), Sırçalı Madrassa in Konya (After Bakırer 1976: 289), and Arap Baba Masjid and Tomb in Elazığ (After Bakırer 1976: 309) are combined with their current photos)

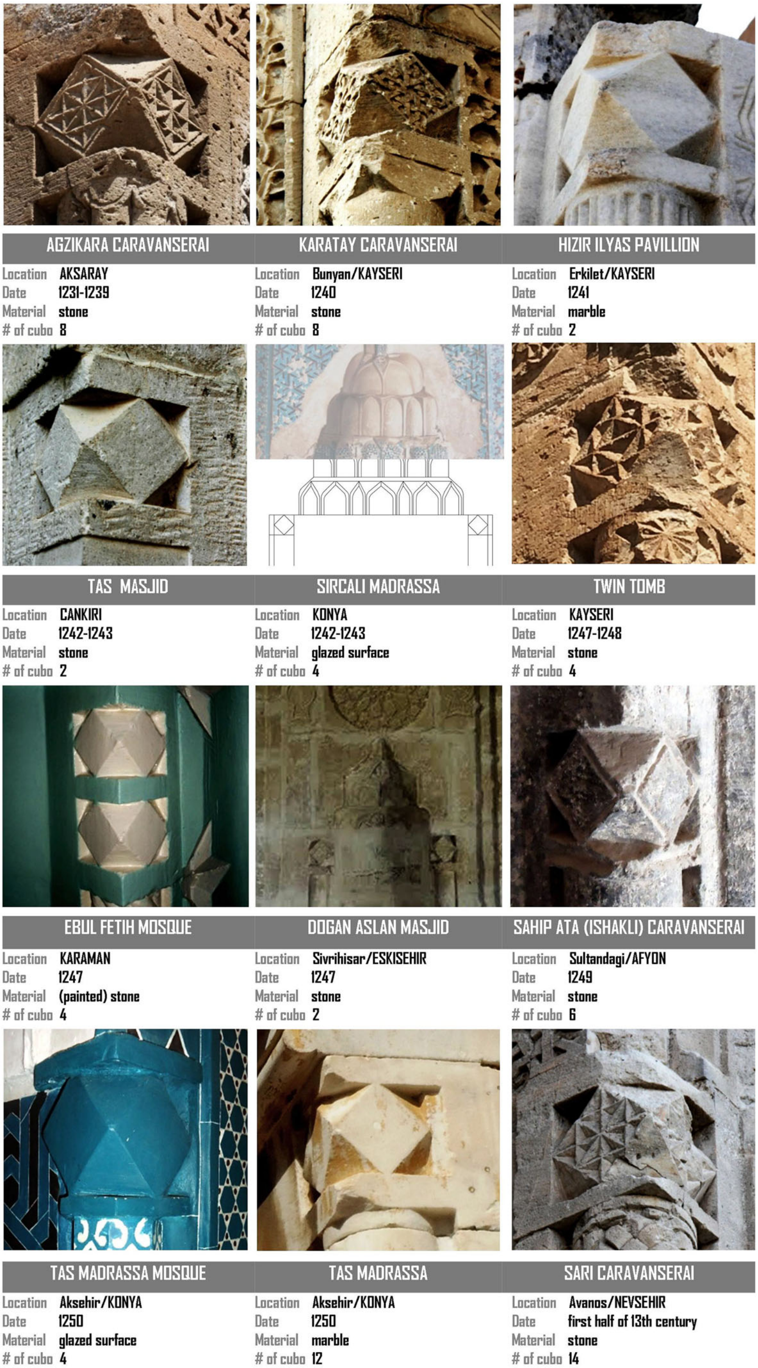


Fig. 5 continued

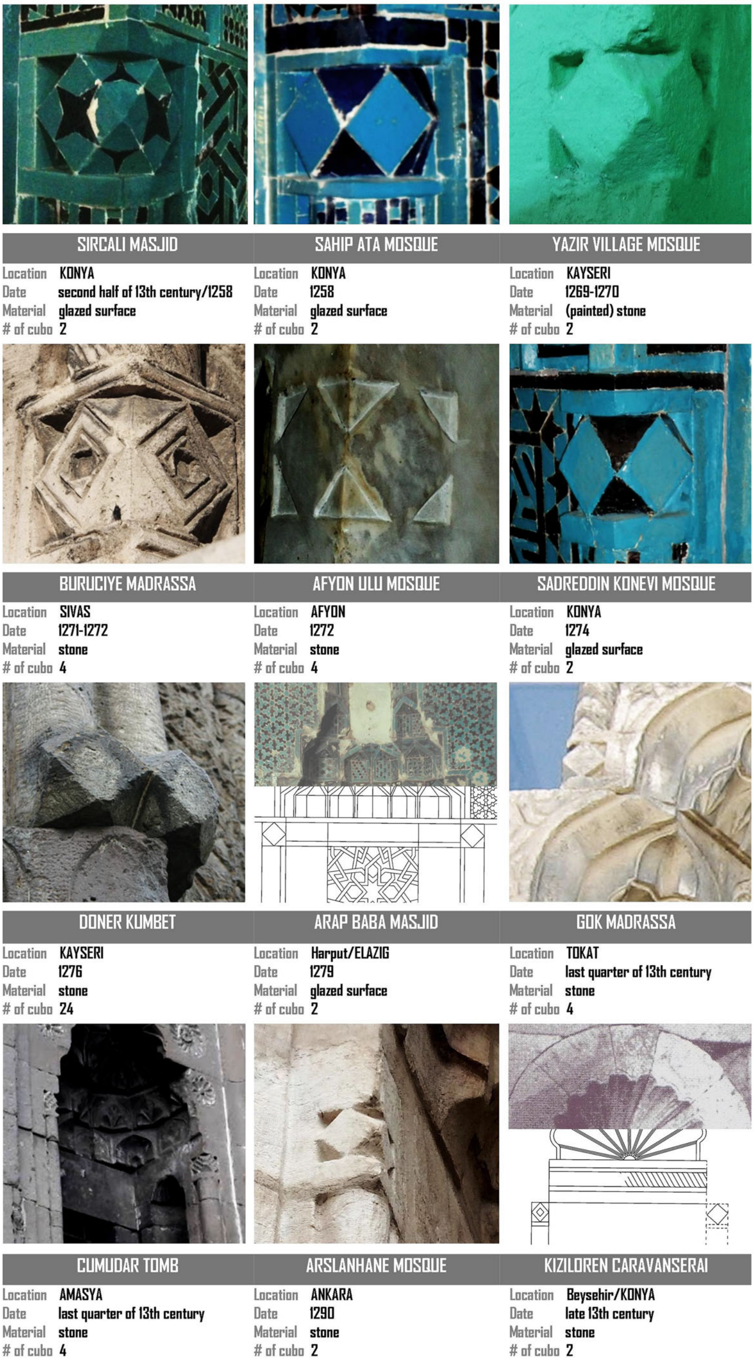


Fig. 5 continued

Fig. 5 continued

Fig. 5 continued

thoroughly individual at capitals or pedestals. Although the materials differ (ceramic, marble, stone, and brick), most examples were made of stone with carved out surfaces and dated earlier. Regarding the size of the figures (which requires an extensive study and is beyond the scope of this study), almost all are hand-size. Furthermore, the inventory of the figures is still expanding because it is always possible to find new ones in structures in small settlements or in the countryside.

All examples are clearly framed by a hollow cube, except the ones in Döner Kumbet in Kayseri. At first glance, two squares and six equilateral triangles are visible and define the visible part of the polyhedra. The faces of the hollow cube frame appear at the corner of the capital, where the corner of the square faces of the cuboctahedron meets with the edges of the cube in the middle (Fig. 6a). Thereafter, the six equilateral triangles in the frame connect with the square faces, and the remaining square faces and two triangles are invisible. Therefore, the geometrical relation between the framing cube and the visible faces, vertices and edges of cuboctahedron makes the remainder imaginable. Moreover, the voids of the vertices of the cubic enclosure intersection of two bodies indicate that another polyhedral part is an irregular tetrahedron, which is 1/8 of an octahedra with three right triangles and an equilateral triangular base, which the cuboctahedron and octahedron share (Fig. 6b). Thus, the cuboctahedron is either the bisection of twelve edges of either cube or octahedron or the truncation of the eight corners of the cube or six corners of the octahedron, which result in the intersection of both figures (Fig. 6c).

Considering this relation, we developed a symbolic approach to the cuboctahedron in an ancient fashion (Hisarlıgil and Bolak-Hisarlıgil 2009). Since Antiquity, Platonic solids represent elements with polyhedra: fire with the tetrahedron, earth with the cube, air with the octahedron, water with the icosahedron, and the cosmos with the dodecahedron (Lindberg 2008: 40). When analyzed, the cuboctahedron at the engaged column, arch and vault satisfies almost all examples of this representation. Therefore, in Platonic terms, the cuboctahedron, generated from the octahedron (representing air) and the cube (representing earth), metaphorically represents a phase of a transformational process between earth and sky (Fig. 7). This commentary dates to as far back as the ninth century A.D. by Qustā ibn Lūqā, a mathematician, astronomer and physician. In his translation and commentary on Qustā ibn Lūqā's *The Introduction to Geometry*, Hogendijk informs us that in Chapter 3 of his book, Lūqā discusses the five regular polyhedra, stating that the ancients compared these polyhedra to the four elements (Hogendijk 2008: 167). In this chapter, Lūqā particularly describes a solid with fourteen faces that comprises eight equilateral triangles and six equilateral and equiangular quadrilaterals, which can represent "air and earth" (Hogendijk 2008: 168).

Cuboctahedron as a "Polyhedral Cluster" in Medieval Art in Anatolia

Such geometrical relations between solid and void parts cannot be limited to the intersection of two solid bodies. The example at the capitals of the mihrab of Ebul Fetih Mosque (Fig. 8a) packs two cuboctahedra, which indicates another solid-and-

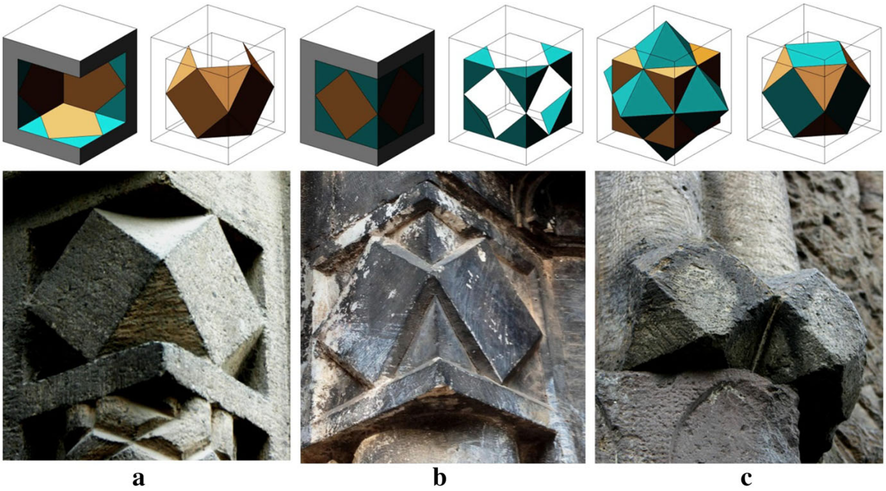


Fig. 6 **a** Visible parts of the cuboctahedron in the cube in Hunat Hatun Mosque; **b** visible parts of the octahedron in the cube in Yivli Minaret Madrassa; **c** cuboctahedron as the intersection of the cube and octahedron in Döner Kumbet

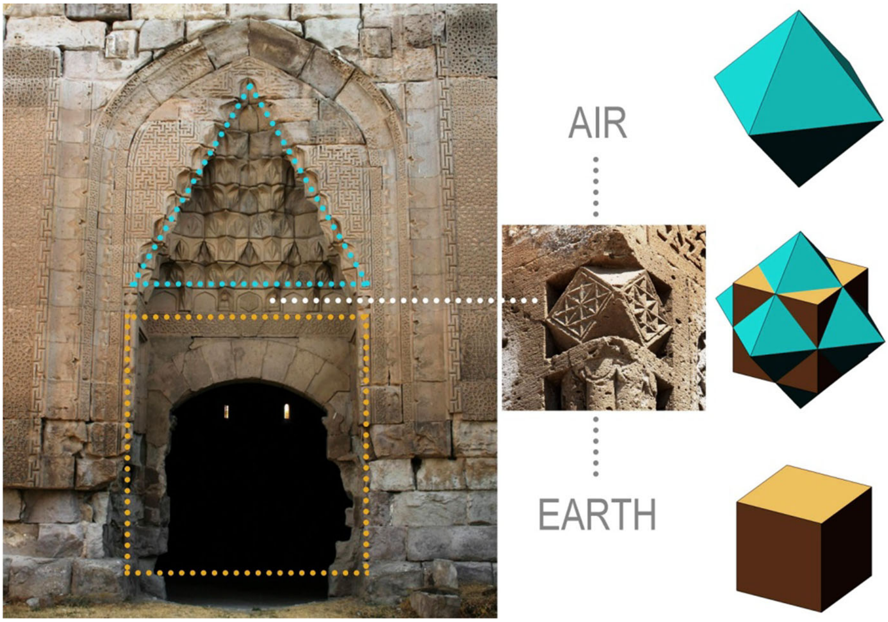


Fig. 7 Cuboctahedral capitals where the engaged columns rise from the ground (earth) meet the sky (stalactite vault)

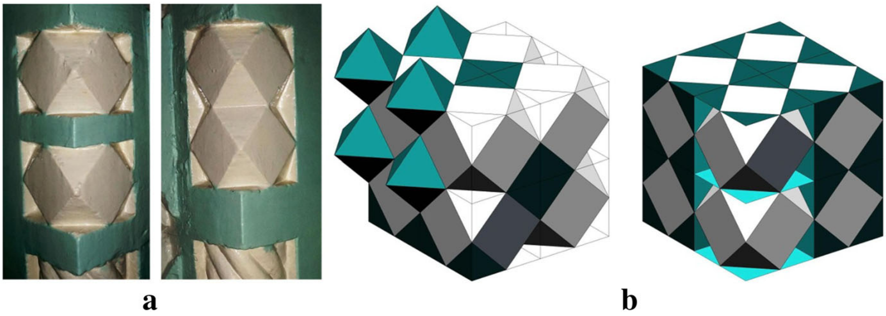


Fig. 8 **a** Cuboctahedra in Ebul Fetih Mosque in Karaman; **b** remaining octahedral voids of the packing of a cuboctahedron

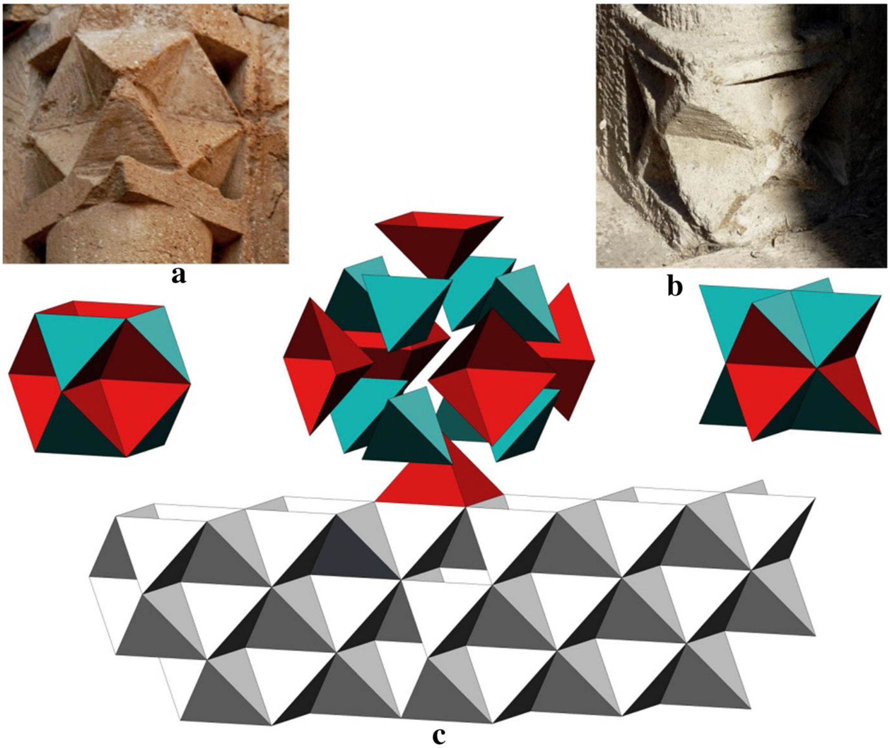


Fig. 9 **a** Stone octahemioctahedron in Sarı Caravanserai; **b** stone stella octangula in Sahip Ata (İshaklı) Caravanserai; **c** octahemioctahedron and stellated octahedron as the result of an alternatingly tetrahedral–octahedral combination

void relation between cuboctahedron and octahedron. In this example, the octahedral voids become more apparent than the single examples, where the cuboctahedron is more frontal. Here, the leftover voids, each of which is an octant (1/8 sectors) of an octahedron, are not limited to the leftover space of solid cuboctahedra; instead, the corners of the framing cube now become the volume centers of the octahedra (Fig. 8b). Similarly, supposedly, when the hollow cube frame is filled with the maximum number of cuboctahedra, the square surfaces of the cuboctahedron that represents one of three square planes that cross one another at the center become octahedral cavities. Hence, these polyhedral relations in the examples of cuboctahedra inspire us to search for other examples of polyhedral clusters.

Strikingly, our 2012 discovery of figures in Sarı Caravanserai in Nevşehir (Fig. 9a) and Sahip Ata (İshaklı) Caravanserai in Afyon (Fig. 9b), which are clusters of tetrahedra, implies that the cuboctahedron cannot be imagined simply as the result of the designer's intersecting an octahedron and a cube. Each of these figures, which were constructed in 1249, shares tetrahedral units that connect alternatingly in half. The example in Sarı Caravanserai was generated from eight tetrahedra with octahedral cavities, and is today referred to as an octahemioctahedron or octatetrahedron in geometry. Its generation can be illustrated as arising from switching the stella octangula at halfway points through the edges of the octahedron at the intersection of two tetrahedra or more simply as the three-dimensional packing of either, since the intersection of one usually generates the other (Fig. 9c). The figure in Sahip Ata Caravanserai is known as the stella octangula in mathematics and is made up of a compound polyhedron of eight tetrahedra. Hence, being the only stellation of the octahedron, the stella octangula is often incorrectly referred to as the "star tetrahedron". The cube is the convex hull of this form, which is also the faceting of a cube with twelve diagonals on six faces because the edges of two intersecting tetrahedra generate an octahedron. Historically, the stella octangula was first depicted in Pacioli's *Divina Proportione* in 1498, where it was referred to as an "octahedron elevatum". It was named "the stella octangula", which means "eight-pointed star", by Johannes Kepler in 1611 (Cromwell 1997: 152).

An example of a twin octahemioctahedron was found in the tombstone in Bursa (Fig. 10a) and clearly illustrates the transferrable relation between the stella octangula and octahemioctahedron as compounds of tetrahedra. However, in addition to this combination, there is a three-dimensional cross cap inside the square faces of two octahemioctahedra, which was either carved from the cuboctahedron or

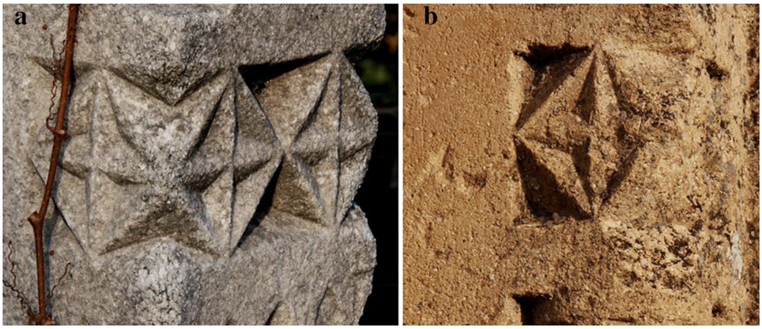


Fig. 10 **a** Twin octahemioctahedron with cross caps in the tombstone in Bursa; **b** octahemioctahedron with a cross cap in Sarı Caravanserai

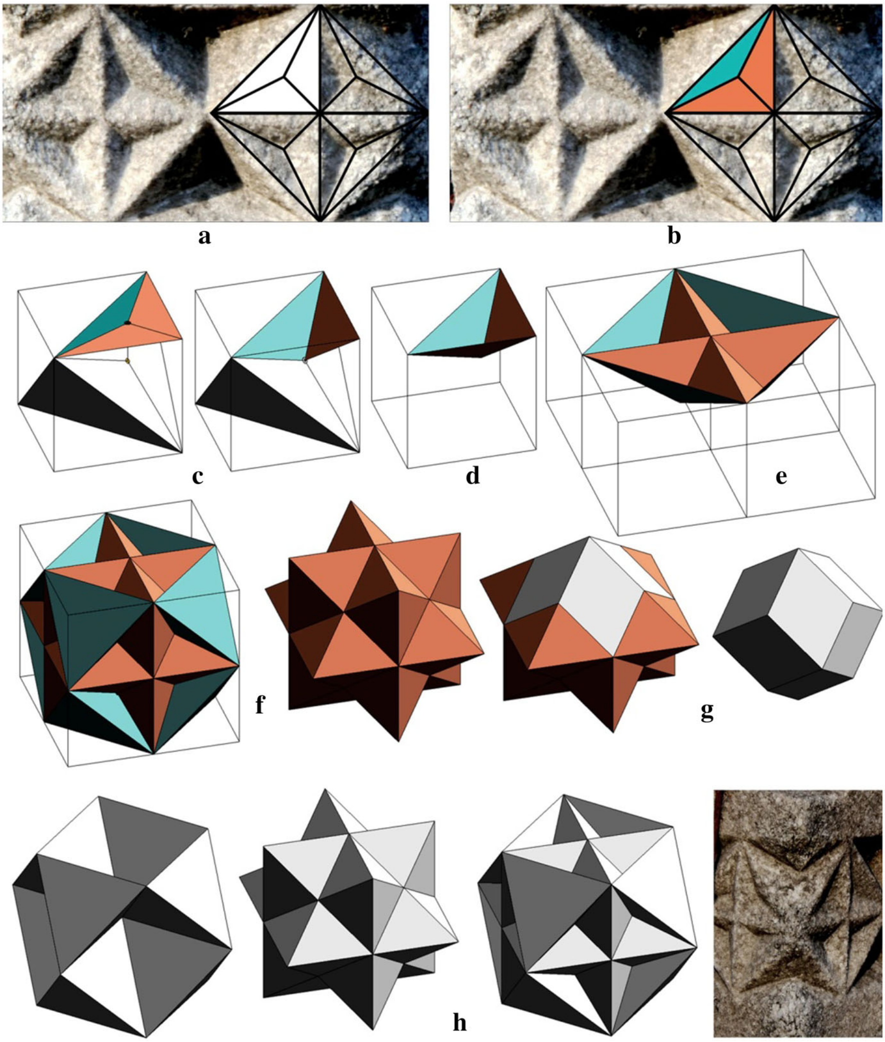


Fig. 11 **a** Vectors of the ridges of the cross intersecting with that of the square suggest that it was levelled in three dimensions, the tombstone in Bursa; **b** the intersection of either bisectors or medians of the three triangles, where one is different from the other identical two, the tombstone in Bursa; **c** the bisectors or medians intersecting at the right triangle with the equilateral triangle of the tetrahedron as the geometrical centre; **d** a three-dimensional profile emerges when the corners of the right triangle connect to the geometrical centre; **e** three-dimensional profile of the whole cross on one surface of the octahemioctahedron; **f** three-dimensional contours of stellated rhombicdodecahedron with nesting rhombicdodecahedron; **g** The edges of three squares intersected at a 90° angle and the body diagonals of eight cubes; **h** The stellated rhombicdodecahedron inside the octahemioctahedron as in the tombstone in Bursa

inserted into the octahemioctahedron. Another example from Sarı Caravanserai in Nevşehir explicitly suggests a similar individual relation (Fig. 10b). In either case, we first observe the edges of the square and the vectors that intersect at different angles inside the square. The edges of the square are the ridges of the octahemioctahedron or cuboctahedron.

From the above example (Fig. 10a), the vectors of the ridges of the cross that intersect with the square also appear to have been levelled in three dimensions in the tombstone in Bursa (Fig. 11a). The other vectors that intersect inside the four right triangles that are generated by the square and the cross appear to form the intersection of either bisectors or medians of the triangles, which results in three triangles, one of which is different from the other two, which are identical (Fig. 11b). To identify the dihedral angles of the three-dimensional profiles on the square surface of the polyhedra such as an octahemioctahedron, the right triangle in the square is taken as the orthographic projection of the equilateral triangle of the tetrahedron. Thus, bisectors or medians that intersect the right triangle are also those of the equilateral triangle of the tetrahedron as the geometrical center (Fig. 11c). If the corners of the right triangle are connected to this center, we obtain the three-dimensional profile of this triangle (Fig. 11d). Being one-fourth of the square, four of these pieces generate the three-dimensional profile of the entire cross on one surface of the octahemioctahedron (Fig. 11e). By applying the identical process to all surfaces of the octahemioctahedron, we generate the three-dimensional contours of a stellated rhombicdodecahedron, inside of which nests an additional rhombicdodecahedron (Fig. 11f). Briefly, a stellated rhombicdodecahedron with twelve slightly flattened or oblate pyramids with 8 isosceles triangular faces is generated by turning the intersection of the body diagonals of a rhombicdodecahedron inside out. This result can also be explained by either packing twelve around a center or intersecting three perpendicularly, which also generates a rhombic dodecahemioctahedron inside. Furthermore, the combination of six bisections of those octahedral units, which are also six sextants (1/6 sectors) of a cube, suggests either the four body diagonals of a cube inside or a rhombic dodecahedron outside. Figure 11f shows that the edges of the stellated rhombicdodecahedron represent both the edges of three squares that intersect at a 90° angle and the body diagonals of eight cubes that are generated by this intersection (Fig. 11g). In summary, placing the stellated rhombicdodecahedron inside the octahemioctahedron generates the figures discovered in Sarı Caravanserai, Nevşehir and the tombstone in Bursa (Fig. 11h).

The two examples in Ağzıkara Caravanserai, Nevşehir (Fig. 12a) and Sarı Caravanserai, Nevşehir (Fig. 12b) give the impression of three-dimensional packing according to the example in the tombstone in Bursa (Fig. 10a) and Sarı Caravanserai (Fig. 10b) at first glance. However, these examples can be simply explained as the intersection of twelve spheres of the same radius around a centre (Fig. 12c). The centers of these spheres define the twelve nodes of the cuboctahedron. The three-dimensional network in these examples simply comprises the intersection of twelve octahemioctahedra and twelve closely packed stellated rhombicdodecahedra (Figs. 12d, e) in a similar way. In our example, the unit polyhedron takes the form of the stellated dual cuboctahedron in the octahemioctahedron (Fig. 12f). In brief, the example of the single octahemioctahedron in Sarı Caravanserai becomes two in the tombstone of Bursa and twelve in Ağzıkara Caravanserai (Fig. 12a) and Sarı Caravanserai (Fig. 12b).

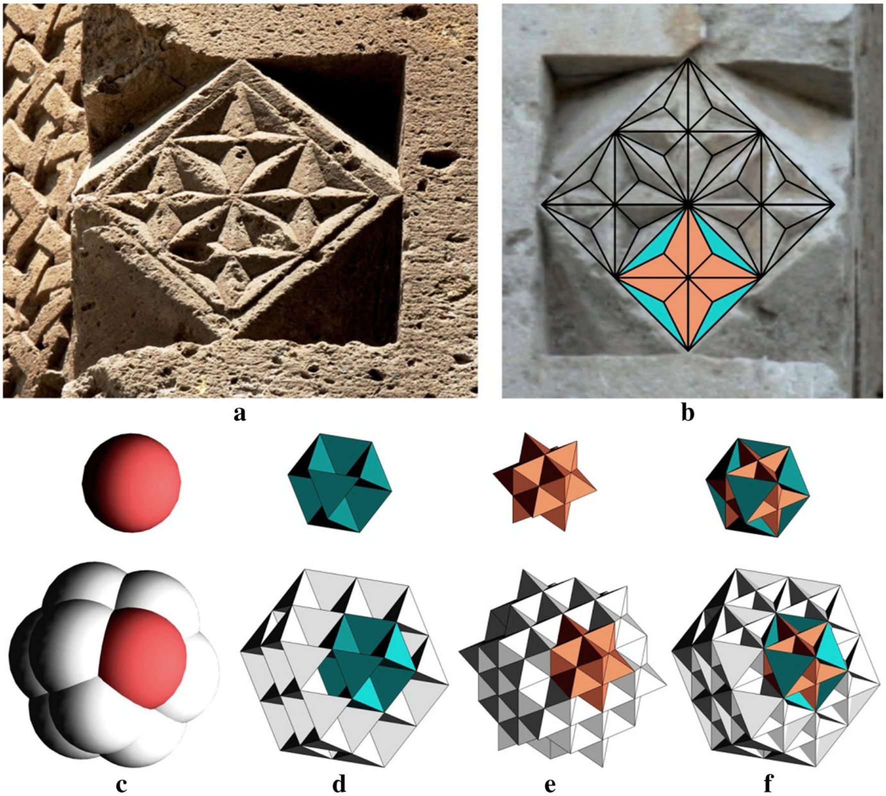


Fig. 12 **a** A stone octahemioctahedron and stellated rhombicdodecahedron cluster in Ağzıkara Caravanserai; **b** a marble octahemioctahedron and stellated rhombicdodecahedron cluster in Sarı Caravanserai; **c** packing twelve spheres intersecting around the nucleus sphere; **d** packing twelve octahemioctahedra intersecting around the nucleus sphere; **e** packing twelve stellated dual cuboctahedra intersecting around the nucleus sphere; **f** packing twelve stellated dual cuboctahedra in the octahemioctahedron intersecting around the nucleus sphere

Two other examples in Sarı Caravanserai, Nevşehir (Fig. 13a) and the Tomb of İzzeddin Keykavus I, Konya (Fig. 13c) are fairly similar to the examples in Sarı Caravanserai (Fig. 12b) and Bursa (Fig. 11a, b). These examples can be generated simply by intersecting the octahemioctahedron, stellated octahedron, and stellated rhombicdodecahedron (Fig. 13b), or one stellated rhombicdodecahedron with twelve closely packed octahemioctahedra (Fig. 13d). For example, if we analyze the right triangle in the tombstone in Bursa (Fig. 11a), we see that it becomes two via bisectors in Sarı Caravanserai (Fig. 13a). By applying the projection process of the former to the latter, we can conclude that the surfaces of the triangles dramatically differ not in terms of the dihedral angle but in the number of surfaces of tetrahedra that generate the octahemioctahedron. Furthermore, the carved-out profiles of triangular faces of the cuboctahedron in the Tomb of İzzeddin Keykavus I indicate the possibility of both scenarios based on the tetrahedral network, which generate either a tetrahedron or an octahedron (Fig. 13c). These faces are composed

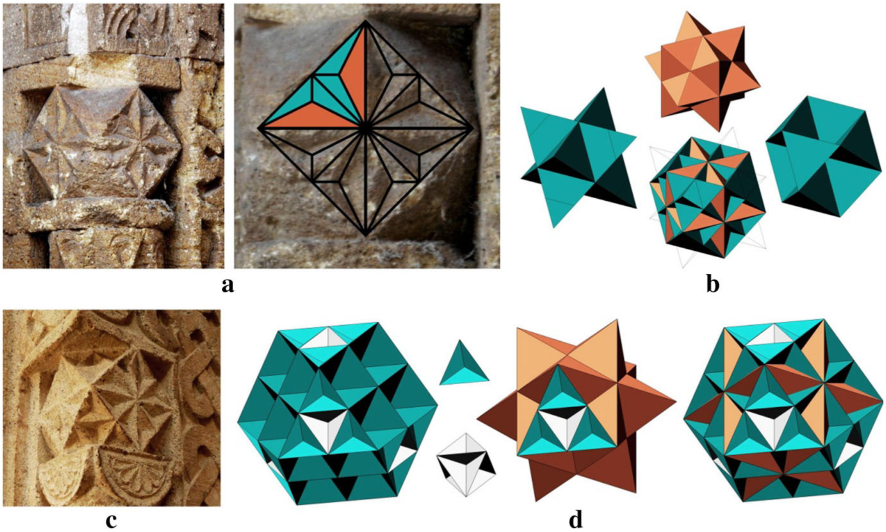


Fig. 13 **a** Two right triangles with bisectors of the octahemioctahedron in Sarı Caravanserai; **b** intersection of the octahemioctahedron, stellated octahedron, or stellated rhombicdodecahedron; **c** cuboctahedron in Tomb of İzzeddin Keykavus I; **d** intersection of twelve octahemioctahedra with one stellated rhombicdodecahedron where the exact centers of three tetrahedra and an octahedron meet at the triangular surfaces of the figure

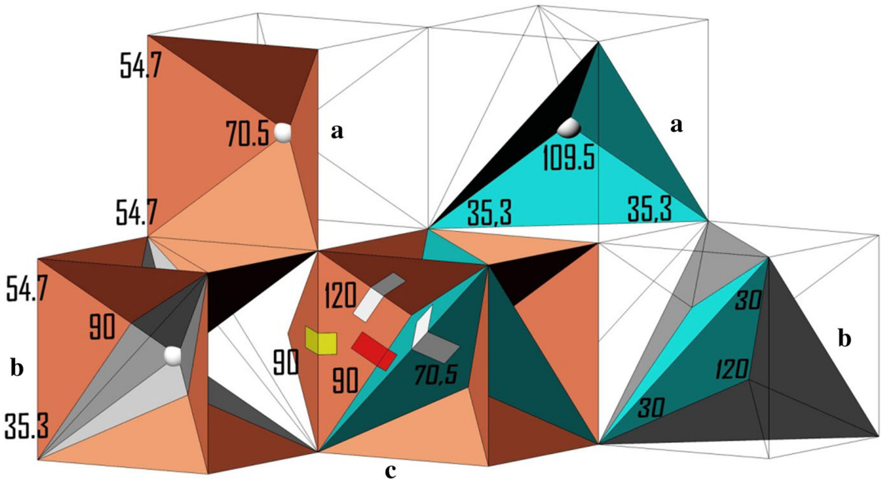


Fig. 14 **a** Central angles with inclination angles in a unit module; **b** dihedral angles in a unit module; **c** surface angles in a unit module

of four equilateral triangles, which are compounds of three isosceles triangles to make the concave caps suggest the volume centers of the exact centers of each tetrahedron and octahedron. In this case, the isosceles triangles join in a pyramidal manner; the single octahedron and four tetrahedral units with central nodes show radial vectors into each original cell. This network relates to the edges of the stellated rhombic dodecahedron inside, where the vertices and edges join (Fig. 13d).

All examples can be explained by a module unit in which a tetrahedron embedded in a cube so that the four threefold axes of the figures coincide where the vertices of the tetrahedron fall on four of the cube's eight vertices (Holden 1971: 29). The other four vertices determine the position of its dual (a second tetrahedron rotated 180°) in the cube. In brief, the combination of six sextants (1/6 sectors) of a cube and four quarter tetrahedra around a center gives not only the vertices, edges and surfaces of the module unit, but the surface diagonals, body diagonals and body centers as well (Fig. 14a). Here, the body centers of either cube or tetrahedron meet with the central angles of 70.5° and 109.5°, where their inclination angles are 54.7° and 35.3° at all vertices. The intersection of the cube and tetrahedra gives two types of triangles on all surfaces: isosceles triangles with angles of 120°, 30°, and 30° and scalene triangles with 54.7°, 35.3°, and 90° (Fig. 14b). Dihedral angles on all surfaces are determined by either an oblate octahedron with 120° and 90° or a tetrahedron with 70.5°. Furthermore, the intersection of the equilateral triangle with the isosceles triangle with 70.5°, 54.75°, and 54.75° gives the fourth dihedral angle, which is also 90° (Fig. 14c).

In general, the intersection of the tetrahedron with the cube explains the entire vocabulary of the entire geometric structure, where any result can be read as polyhedral clusters that results in the assemblage of triangular surfaces at any scale in the manner of fractals. In this structure, all elements can be isometrically combined at any size, where the increase in size of one component compensates with the multiplication of other components (Fig. 15).

Figure 16 shows the complex entity as an intersection of polyhedral enclosure of two types, which shows the cross relations of several elements from triangular surfaces of polyhedral units to polyhedral clusters.

Cuboctahedron as a "Vector Matrix" in Medieval Art in Anatolia

When the surfaces of the unit cell are removed, the remaining edges give the skeleton that is the combination of a tetrahedron and a cube with body diagonals, which suggest that the volume centers of each polyhedron are congruent with one another. Furthermore, the vertices of the unit cube represent the volume centers of the octahedron. Packing this unit cell with eight octahedra generates the complex compound lattice and indicates the edges of the cuboctahedron and the cube frame in which it is placed (Fig. 17a). As analyzed in the previous section, the edges of the example from tombstone in Bursa perfectly match with all vectors of the eight-unit cell (Fig. 17b).

This result is closely related to the findings of R. Buckminster Fuller in 1940 with respect to developing a cuboctahedron using vector equilibrium (VE) and to the "isotropic vector matrix" (IVM) as a coordinated system of alternating tetrahedra and octahedra (Fuller et al. 1975). The VE is also the result of the intersection of four hexagonal planes. The radius and length of the edges of the three-dimensional frame of the cuboctahedron are equal, as these planes suggest (Fig. 18a). Similarly,

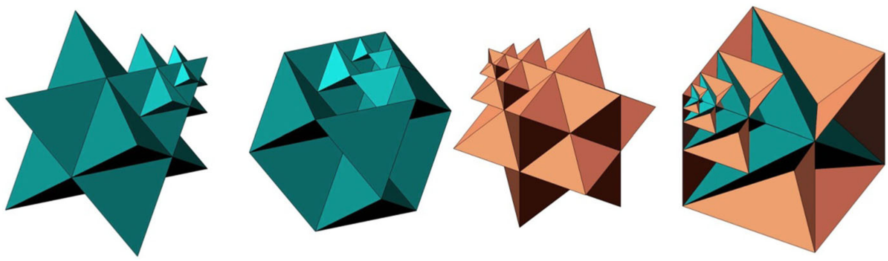


Fig. 15 Isometric scaling of unit elements

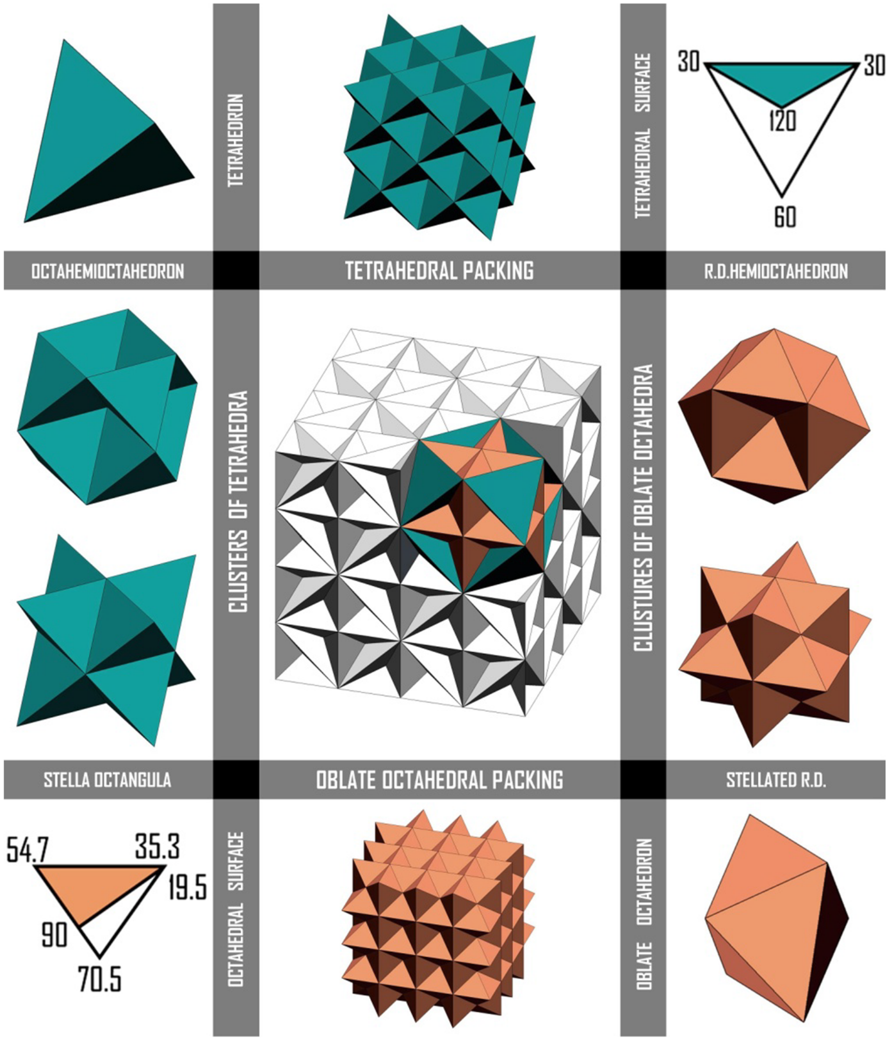


Fig. 16 Relations of elements from triangular surfaces of polyhedral units to polyhedral clusters generating the vocabulary of the geometric structures of the examples

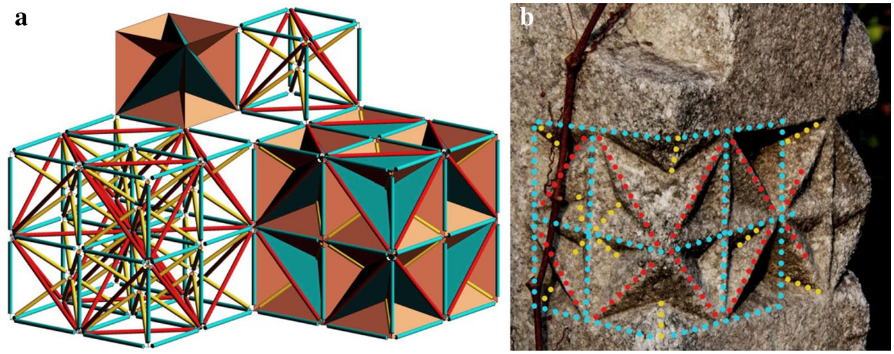


Fig. 17 **a** Unit cell and its packing of eight octahedral with removed surfaces; **b** cuboctahedra in the tombstone in Bursa, which perfectly match with all vectors of the eight-unit cell

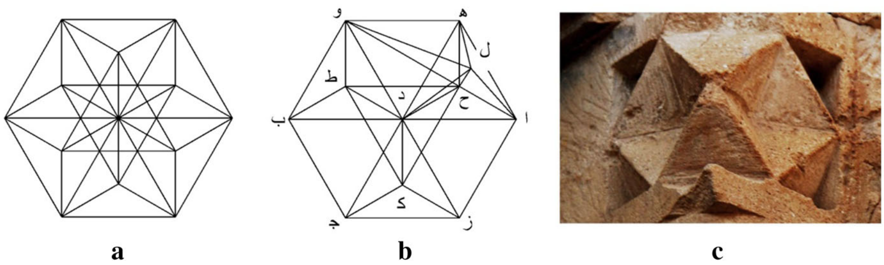


Fig. 18 **a** Illustration of VE (after Fuller 1975: 250); **b** illustration of half an octahemioctahedron with a hexagonal base by Thabit ibn Qurra (after Asselah 2009: 327); **c** octahemioctahedron in Sarı Caravanserai

Thabit ibn Qurra explores the metric relations between the edge of the polyhedron and the diameter of the sphere in the ninth century. In his illustration of half an octahemioctahedron with a hexagonal base (Fig. 18b), which is called a “triangular cupola” in mathematics (Weisstein 2002: 1580), he shows that the figure has fourteen faces of equal edges and equal angles; eight of these faces are equilateral triangles, and the other six are squares (Asselah 2009: 317). His construction begins from a “great circle” of the sphere, from which a regular hexagon is built, because Euclid constructed from the diameter of the sphere a first polygon, on which he built his construction (Asselah 2009: 320). The octahemioctahedron in Sarı Caravanserai, which is composed of six tetrahedra, implies such a hexagonal construction (Fig. 18c).

When all examples are traced in a similar manner, the removal of the surfaces that enclose the six tetrahedra in Sarı Caravanserai in a hollow cube frame turns the solid cuboctahedron into the skeleton of the VE. The combination of edges and body diagonals results in an octahemioctahedron with six tetrahedra and six octahedra parts (Fig. 19a). The edges of four hexagonal planes intersect symmetrically around a center and give the edges of a cuboctahedron, while the diagonals of these hexagonal planes produce six directions of twelve vectors that radiate from or meet on these six axes at central angles of 60° (Fig. 19b). In addition, the volume

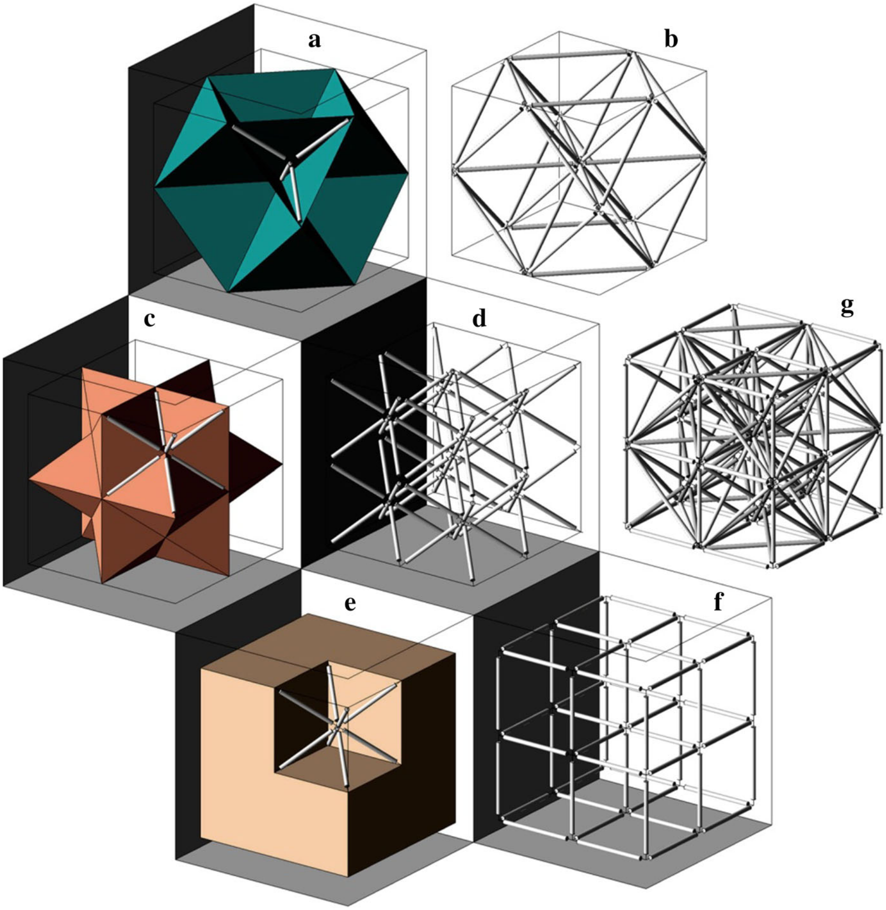


Fig. 19 **a** Intersection of four hexagonal planes; **b** six directions with twelve vectors of octahedra; **c** stellated rhombic dodecahedron defined by four directions of tetrahedra and cube; **d** four directions with eight vectors of tetrahedra and cube; **e** Combination of (**d**–**f**); **f** three directions with six vectors of octahedra; **g** combination of all systems

centers of each unit give other complementary axes, which are four directions with four vectors of tetrahedra and 109.5° central angles (Fig. 19d) and three directions with six vectors of octahedra and 90° central angles (Fig. 19f), each of which suggests that the framework of the hollow cube encloses all examples (Fig. 19e). When these complementary axes are connected, the stellated rhombic dodecahedron appears, implying the rhombic dodecahedron inside (Fig. 19c). Finally, the combination of three separate axis systems around a center illustrates the final picture (Fig. 19g), which is the shape of space according to Fuller (Edmondson 1987: 114).

This system connects all simple elements of space points, lines, surfaces, and solids in a unified manner and provides the resulting complex forms with a uniform change at any scale as demonstrated in the previous section. Fuller explains that this

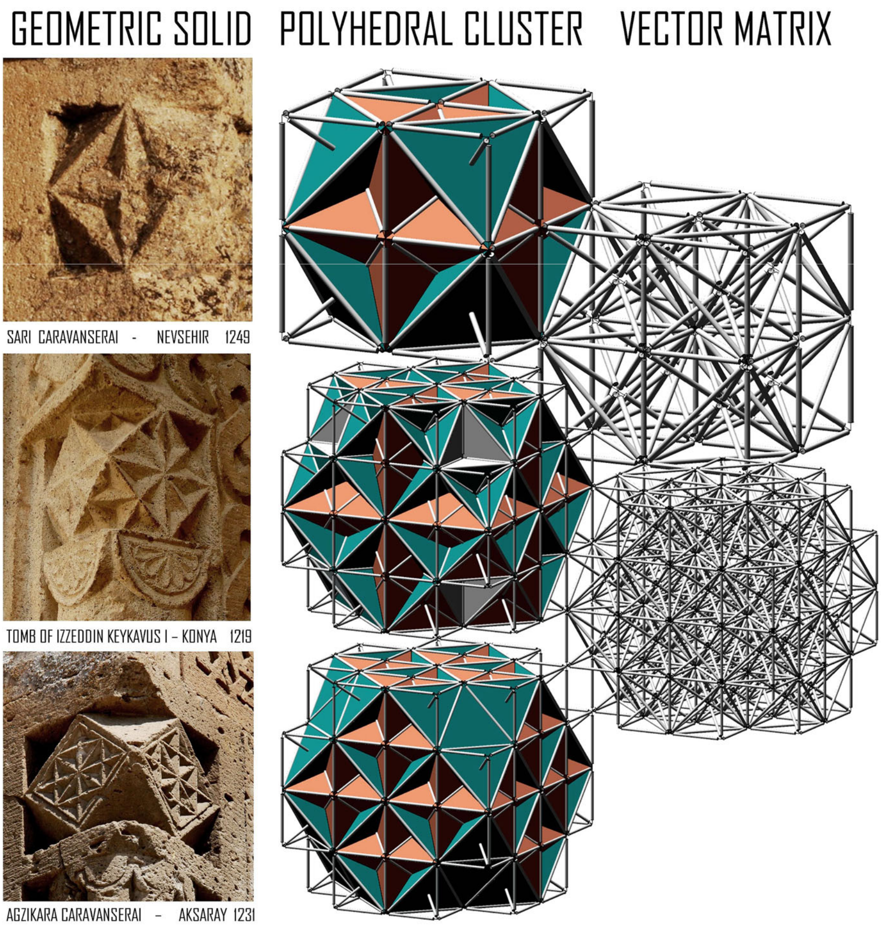


Fig. 20 Three examples of cuboctahedra as geometric solid, polyhedral cluster, and vector matrix

vector system is also a face-centered cubic (closed cubic packing) (Fig. 19b), body-centered cubic along with the diamond point complex (Fig. 19d), and simple cubic lattices as a single unit (Fig. 19f); this was analyzed further as the Vector Equilibrium Principle (VEP) by Arthur Loeb (1970: 237). Although all three systems can be derived from the geometry of a cube because they have the 4.3.2 rotational symmetry (Kappraff 2001: 283), Fuller's preference of the cuboctahedron for this explanation connects with Thabit ibn Qurra's particular interest in the cuboctahedron in the ninth century. In addition, this preference can be traced to the fourteenth century, when al-Kashi studied it as a polyhedron because the length of each edge is equal to the radius of the circumsphere.

Although most examples portray the cuboctahedron as a solitary solid figure at first glance from a distance, the minority of examples with carved-out profiles on their surfaces accentuates the complex content of the remainder. Therefore, when we compare the cubic frames that reveal a relatively limited message, a closer observation of these examples provides the actual information that extends its geometrical content. Hence, these encouraging examples, which were successively analyzed as "geometric solid", "polyhedral cluster" and "vector matrix" in this study, reveal their genuine content in terms of mathematical intelligence, craftsmanship, and logic of design. Figure 20 outlines the polyhedral content of the examples in a progressive manner and illustrates the possibility of collaboration between mathematicians and artisans in medieval art in Anatolia. In this context, the results also show the preference for cuboctahedra from the ninth century to the fourteenth century and illustrate the profound geometrical content of medieval art in Anatolia.

Conclusion

These findings discussed here deepen our knowledge of the mathematical content of medieval art works that have been simply considered decorative elements until now, and provide the future scholars with a new vocabulary for a meaningful discussion of the craftsmanship and mathematical intelligence of artisans to rigorously master polyhedral geometry. Hence, the results are important for the history of art, proper restoration or reconstruction of damaged examples, and further comprehension of the logic of the design of the cultural heritage, which will be more sustainable for future generations in addition to simply preserving the existing examples.

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