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GEOMETER'S ANGLE

Marriages of Incommensurables: ϕ-Related Ratios Joined with $\sqrt{2}$ and $\sqrt{3}$

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Abstract In his "Marriages of Incommensurables: Phi Related Ratios Joined with the Square Roots of Two and Three", artist and geometer Mark A. Reynolds has found two ratios from the golden section family that generate relationships with the square roots of two and three. He includes the grids and procedures necessary for producing these ratios. For him, the significance of the constructions is that they join together ratios from two different groups of rectangles: the golden section family and the square root rectangle progression, two systems that are usually incompatible with each other. In these constructions and grids, the square root of the golden section and the golden section squared are related mathematically to the square roots of two and three, respectively, in ways that he believes have not been seen before. Reynolds calls this series of constructions, "marriages of incommensurables", and the two he presents here are part of a larger group he has been working on for some time.

Keywords Geometry · Geometrical constructions · Incommensurables · Irrationals · Square roots · Phi-related ratios

Introduction

For decades, my daily art practice has focused on the ratios of rectangles (occasionally I also work with triangles, polygons, and circles), and the harmonic compositions they create by generating various grids within their spaces. Figure 1 is an example of my art, part of a series entitled "The Nu Series (Phi Root Two Series)", and shows the results of this research.

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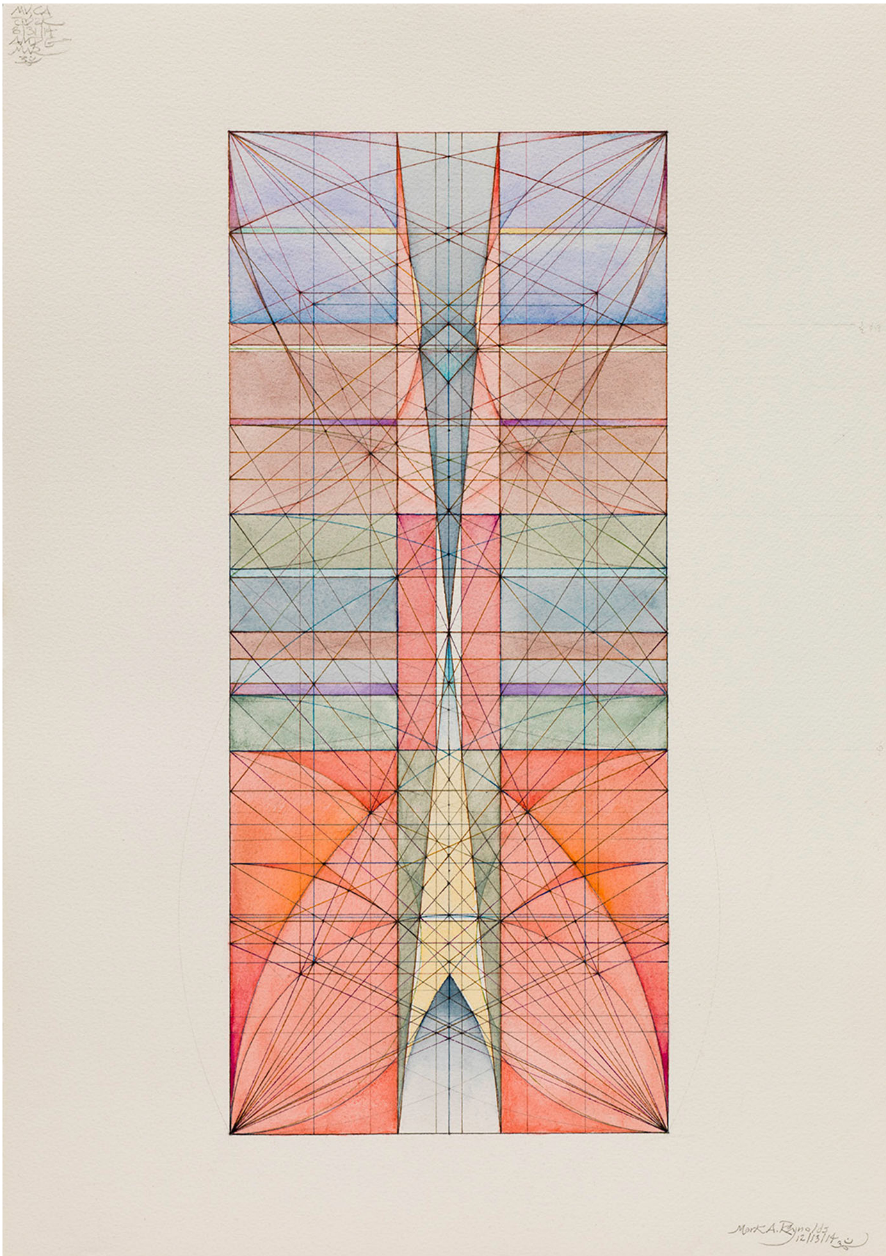


Fig. 1 The Nu Series (Phi Root Two Series), Tulpa XII, 12.13.14

I believe grids can be seen as pictorial representations of the energy fields created by the chosen ratio/shape. These fields are made visible by points, lines, shapes, and intersections through these grids. (The techniques of grid making could be addressed in another paper, as they are too involved and complex to be included here.)

I have worked with integers, Pythagorean musical ratios, the square root rectangle series, the golden section family of ratios, and also what I call "orphan" ratios. Orphan ratios have no true family, or are so well hidden within a family as to be virtually unknown. This paper addresses a couple of these "orphans."

I also reference nature, mathematics, art history, and architecture as well for inspiration and analysis. With that background, many of my compositions now are based on my own inventions, discoveries, and explorations, influenced by all those areas of study.

However, for quite some time, I have been deeply involved in seeing what compositions and visual energies can be produced by mixing ratios that ordinarily are not found together within the same given group. I call these combinations "marriages of incommensurables," where ratios from two incompatible or incommensurable groups may be joined by one element or quality that they share in common. These relationships are "compound ratios" (where more than one ratio or shape is added to another). Not all of the experiments meet with success, but some do, and they're quite amazing in that they demonstrate subtle connections between the various systems that have been utilized over the millennia. They stand alone, being hybrids of relationships found in two families, creating rectangles that have most likely not been seen or discussed before now. Because of their uniqueness and their geometric make-up, I feel it's necessary to name them in order to give them an identity within the great world of geometric systems. Keeping with tradition, I give them names that use Greek letters, like other well-known numbers and geometric relationships that have come down to us through history, such as π (*pi*) and ϕ (*phi*). I will also sometimes name them for their numerical value, for example, "the 2.890" ($\phi + \sqrt{\phi}$). These explorations and investigations are not always without some precedent (as will be seen below). These precedents are mathematical certainties and not approximations. I strive for this mathematical exactitude in my own creations. Sometimes I have to accept that pencil thicknesses may not equate with the proof, but my goals are either mathematical proofs of accuracy or very low percent deviations. The constructions in this paper are mathematically correct as far as I can ascertain. I thank my editor, the architect Kim Williams, and mathematicians Stephen Wassell and Paul Calter for insisting on me being as accurate as one can be, and giving a deviation when one is called for. As an artist, it's made my approach to drawing geometrically, and my ways of thinking about the processes involved in this difficult art form, far better, and much more deeply rewarding.

Before we begin, I'd like to explain that I will be using a convention of indicating irrational numbers whose decimal places go on to infinity. For example, $\sqrt{\phi}$ has a value expressed numerically as 1.2720196. At some point in the decimal expansion, the ellipsis (three dots, …), is added so as to not keep writing the number to infinity. The convention I have adopted here is simply to take the decimal places to the thousandths place and to not add the ellipsis; for example, the value $\sqrt{\phi}$ is given as 1.272.

Also, many of my constructions contain irrational numbers. An irrational number cannot be measured with any reckoning rod, but can be constructed by using the known rational and irrational lengths of compositional elements in geometric shapes, especially the square. There will always be the issues of how line thicknesses and the draughtsman's skills affect the final measurement; basically, it is impossible to be perfectly precise. However, we can accept the construction's "intention" to reference these irrational lengths if we possess some basic knowledge of geometry. Without the lines, we would not be able to see the construction.

Visualizing Marriages of Incommensurables

Before embarking on the new marriages of incommensurables I have found, I want to present several geometric constructions that inspired and motivated me in my journey into these uncharted lands.

Also, because these new discoveries utilize specific members of the ϕ family, I think it's important to say that I am aware of the many pros and cons that revolve around the golden section, from its irrationality and elusiveness regarding precision, to its spiritual and aesthetic qualities or lack thereof. Speaking for myself, I have experienced a wonderful world of possibilities in the ϕ family of ratios, including and beyond those ratios that addressed by Hambidge (1926) and defined by Euclid (1956), while still using the principles of plane geometry in these works. I have found the family to be an endless source for experimentation, discovery, and satisfaction. This present article will discuss some of these qualities.

Drawing I (Fig. 2) is a construction I discovered that relates both to the present article and what I've attempted to say above, especially the concept of finding commonalities in otherwise totally unrelated parts.

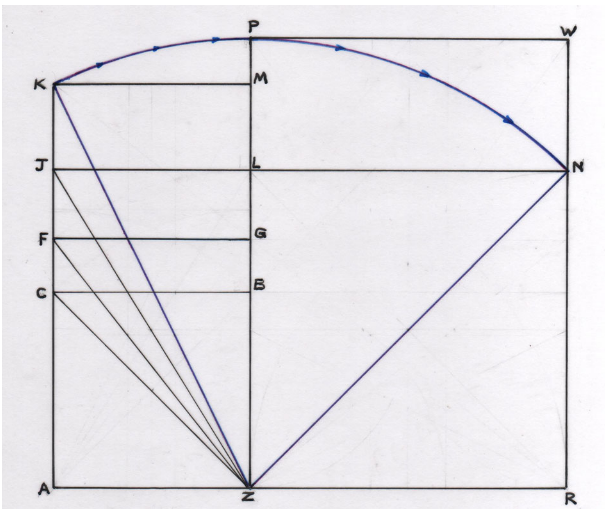


Fig. 2 Drawing I

This first marriage of incommensurables provides a hint and has some relevance for what is to follow. Specifically, Drawing 1 addresses the comment I made about the one shared element that can be found between two different systems for the marriage to take place. Here, it's the ϕ family (seen on the left) and the $\sqrt{2}$ rectangle (on the right), "married" by the diagonal lines ZK and ZN, shared by both families, and being equal.

I have since learned of other wonderful unions of ϕ with the $\sqrt{2}$, demonstrating that there is a relationship between the diagonal of the master square (that generates the square root rectangle progression) and its half-diagonal, which is used to generate a golden section rectangle, from which $\sqrt{\phi}$ and other ϕ family ratios can also be constructed, and even though those two families, ϕ and the square roots, move away from each other compositionally and harmonically to become two increasingly unrelated systems with only rare crossovers, for example, the Double Square and its diagonal, $\sqrt{5}$. They become quite discordant from one another as their numbers/ratios increase numerically.[1]

The marriage of ϕ to the $\sqrt{2}$ was inspirational to me as I continued on to see what other unions between incommensurables I could find that were also valid. The journey has been fruitful.

Drawing I came some time after I was introduced to two mathematically provable right, scalene triangles that many people familiar with geometry know about. We see these two triangles in Drawings II and III (Fig. 3). These drawings are labeled with the lengths of their sides for the reader's convenience.

Both triangles are marriages of incommensurables. Triangle KLM in Drawing II (Fig. 2, left) joins ϕ with the $\sqrt{2}$; triangle PAR in Drawing III joins ϕ with the $\sqrt{3}$, the diagonal of a cube.

Use the Pythagorean Theorem for proof. What additionally impressed me was that there were two ϕ family members—$\sqrt{\phi}$ for the $\sqrt{2}$ hypotenuse, and the reciprocal of ϕ, 0.618, for both, rather than simply ϕ itself—that yielded the crossing over into the family of the square roots. The ϕ family and the square root family became joined by these ϕ family members. If these two members of the ϕ family could yield these marriages, were there others? I was to learn that I had to travel to the far reaches of ϕ to find them, but find them I did.

The μ Ratio

In Drawing I (Fig. 2) rectangle AKMZ is rectangle whose long side measures $\phi\sqrt{\phi}$, and whose short side measures 1. This ratio I have named μ (*mu*), and its numerical equivalent is 2.058:1. To me, this is one of the most beautiful of all ϕ relationships I've encountered, $\phi\sqrt{\phi}$. Contained within this rectangle, we find:

(a) ACBZ is a square, 1:1;
(b) AFGZ is a $\sqrt{\phi}$ rectangle, 1.272:1;
(c) AJLZ is a ϕ rectangle, 1.618:1;

[1] Serlio's *Five Books on Architecture* (1982), Bk I, 23r, mentions the relationship between ϕ and $\sqrt{2}$.

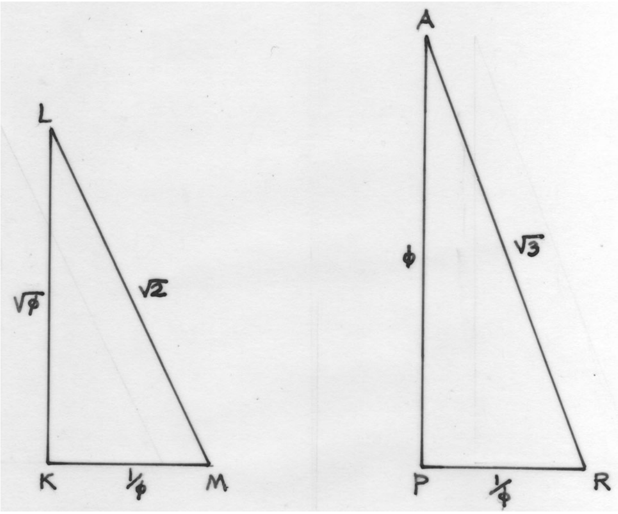


Fig. 3 *Left* drawing II; *right* drawing III

(d) FKMG is the reciprocal rectangle of AFGZ, 1/1.272:1 = 0.786:1 (more about reciprocal rectangles in a minute). It is added to AFGZ at FG to make the μ rectangle AKMZ.

ZK is a diagonal of the μ rectangle. Using Z for my compass pin placement, and opening my compass to K, this diagonal is then rotated up to P and travels to its destination at N.

Therefore, ZK = ZP = ZN. But ZN is also the diagonal of the square, ZLNR (Point Z is the vertex of the 90° angle NZC; because both ACBZ and ZLNR are squares, their diagonals will meet at Z at 90°). Therefore ZPWR is a $\sqrt{2}$ rectangle that uses the diagonal of a square whose side, ZL, equals ϕ, knowable because its side equals the long side, ZL, of the ϕ rectangle AJLZ on the left, This $\sqrt{2}$ diagonal, ZN, is equal to diagonal ZK of the μ rectangle AKMZ. The ratio between the two sides of the squares is AZ:ZR :: 1: ϕ.

Years ago in the *Nexus Network Journal* (Reynolds 2003), I demonstrated the μ ratio, and found it within the $\sqrt{\phi}$ rectangle. To my knowledge, no one had ever addressed this rectangular space and its unique and special qualities. Specifically, the μ initially makes use of a special case of ϕ: its square root. The μ was a kind of "orphan" ratio, for it had never before been described, discussed, or constructed to my knowledge. Even though it is not a true orphan because it is constructed by a $\sqrt{\phi}$ rectangle and its reciprocal (as will be seen by combining Constructions 2 and 3), it can be regarded as a "compound rectangle". It doesn't show up in a simple constructional procedure or superficial examination of the ϕ family. It must be probed deeply for. But it does appear as a unique rectangular space within a $\sqrt{\phi}$ rectangle as a partner to the reciprocal $\sqrt{\phi}$ internally, and exists on its own without any further constructing. It has a singular presence, and its two component $\sqrt{\phi}$ ratios only appear when the golden section is performed on the long side of the μ. It is the result of the remaining space when a reciprocal $\sqrt{\phi}$ is applied to its internal space on either end of the short sides of the $\sqrt{\phi}$ rectangle. In this way, the short side of the master $\sqrt{\phi}$ rectangle becomes the long side of the $\sqrt{\phi}$ internally, that is, its reciprocal. The μ rectangle is formed from this internal ratio's remaining space. As we will see, the μ is made from two $\sqrt{\phi}$ rectangles, one vertical and one horizontal, and joined at the golden section where they are connected. In what follows I show the μ construction, and also address the rectangle's diagonal, which yields the ratio $\phi\sqrt{2} = 2.288{:}1$, which I have named v (*nu*).

Generating v Ratios from μ Ratios

We'll now begin our discussion of the relationship that generates the v from the μ, as will be seen in Constructions 1 through 10 below, replete with the beautiful grid work and construction details that make it exquisite to me. It is the ϕ family of ratios that is responsible for "growing" this new diagonal relationship. My love for this construction is the golden section's essence in it, demonstrating how alive ϕ is. Also, as mentioned above, I depend on the square regularly for the beginning of my constructions; for example, the half-diagonal of the square, $\sqrt{5}/2 = 1.118$, is needed to generate the ϕ rectangle, and build on the fundamentals as the construction develops. I think of the square as the "mother" because almost all of the valued ratios used throughout history for rectangles, both rational (can be measured) and irrational (cannot be measured) may be found, and can be generated from this mother square. My recent work acknowledges this square, the beginning, the unity, or one. Much of what I do has originated from the square in some way, and even though the work I'm doing now is fairly far down the road from its elementary beginnings within the square.

Let's start with the constructions:

Construction 1 (Fig. 4)

(a) Begin with the master square, AKMZ, whose side equals 1;
(b) Use the half-diagonal of the square, CM, to generate the ϕ rectangle APRZ, 1.618:1.

Construction 2 (Fig. 5)

(a) Rotate the long side AP of ϕ rectangle APRZ to the opposite long side RZ to locate point G. AP = AG
(b) Repeat this procedure with the other long side, ZR, to get J.
(c) The resulting rectangle AJGZ is a $\sqrt{\phi}$ rectangle, that is, AJ:AZ::$\sqrt{\phi}$:1 (=1.272:1).
(d) Draw the $\sqrt{\phi}$ rectangle, AJGZ.

This rectangle comes close to framing the chambered nautilus shell, side view. This spiral is sometimes confused with the spiral in the ϕ rectangle. These two spirals are not the same geometrically.

The triangle AGZ is known as the "Triangle of Price," named by and for Professor W.A. Price (see Ghyka 1977: 22). It is the only right triangle whose three

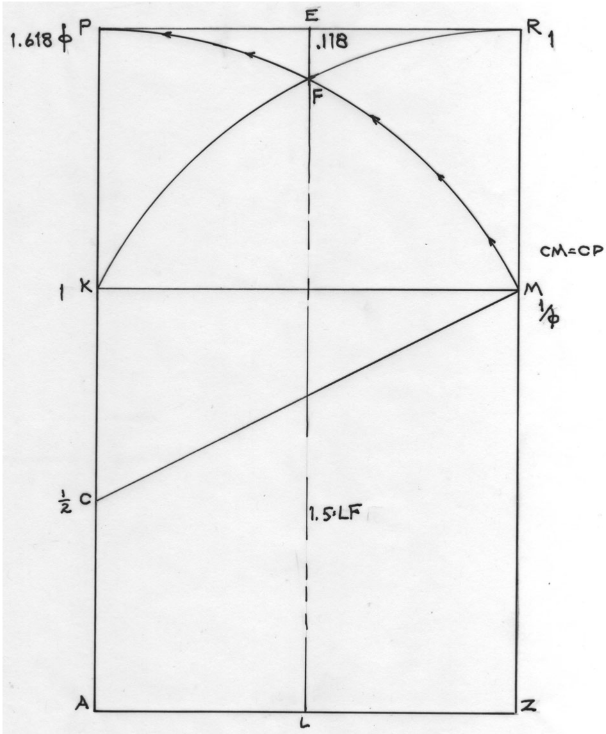


Fig. 4 Construction 1

sides are in a geometric progression. It is based on $\sqrt{\phi}$: AZ = 1, ZG = $\sqrt{\phi}$, and AG = ϕ. Triangle AGZ is also a vertical half of the elevation of the Great Pyramid when opposite faces of the pyramid are vertically cut in half (Reynolds 1999; Herz-Fischler 2000).

Construction 3 (Fig. 6)

(a) Rotate the long side ZG to the opposite side AJ, to point N.
(b) Do the same procedure with the other long side, AJ over to Q.

N is the golden section of AJ, and Q is the golden section of ZG. Points N and Q are the "ϕ caesuras" of sides AJ and ZG, that is, they are the points that define the golden section cuts or breaks of those sides.

(c) Draw the reciprocal $\sqrt{\phi}$ rectangle ANQZ.

Rectangle ANQZ is the reciprocal of AJGZ because the ratios of their sides are reciprocal ratios. AZ (=1, or unity) is both the short side of AJGZ and the long side of ANQZ. The sides of rectangle AJGZ are in the ratio $\sqrt{\phi}$:1, while the sides of rectangle ANQZ are in the ratio 1:1/$\sqrt{\phi}$. When the long side ZG of $\sqrt{\phi}$ rectangle AJGZ is dropped to its opposite side at point N, the $\sqrt{\phi}$ reproduces itself as its own reciprocal from its long side. This is the only rectangle in all the families of rectangles whose long side ZG is its own reciprocal's diagonal ZN and yields its

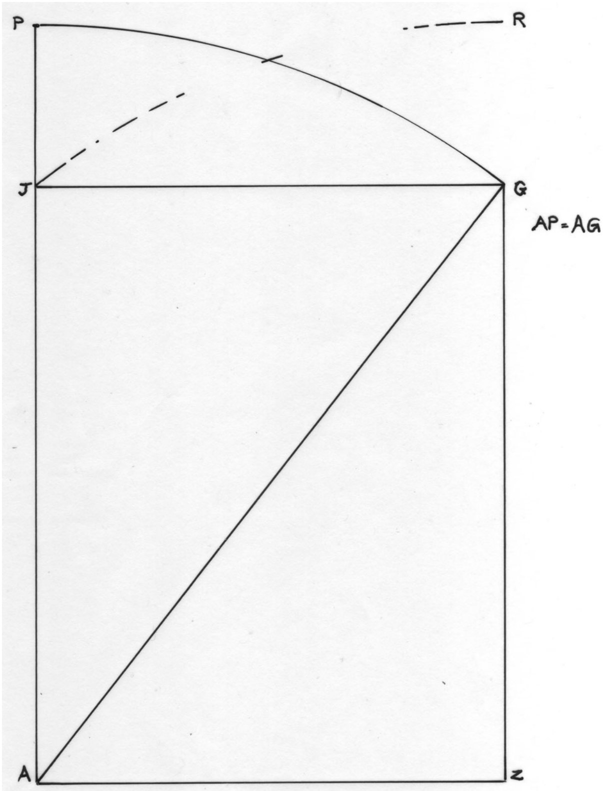


Fig. 5 Construction 2

own reciprocal rectangle ANQZ. All reciprocals only differ in vertical/horizontal orientation. All reciprocals in rectangles do this.[2]

There is another significant rectangle within the area of rectangle AJGZ. The remainder when the reciprocal rectangle ANQZ is taken away is the μ rectangle NJGQ, whose sides are in the ratio $\phi\ \sqrt{\phi}$:1, or 2.058:1. It is almost a double square, but not exactly.

Also, point OC is an "occult center",[3] that is, it is "hidden from the eye"; the eye doesn't see this center as it would see the very still center of the same rectangle. It's a powerful point/focal point. It is the pole around which a spiral winds. The four lines radiating from it (OCN, OCA, OCZ and OCG) are each of a different length, but all are in a geometric progression with each other: OCN is to OCA as OCA is to OCZ, as OCZ is to OCG, and all are equal to $\sqrt{\phi}$:1 (Hambidge 1926: 3–14). The occult center is the intersection of the diagonals of two reciprocal $\sqrt{\phi}$ rectangles,

[2] For more on the reciprocal, see (Calter 2008: 99), and (Hambidge 1926: 30–32, Ch. IV, "The Reciprocal").

[3] In (Calter 2008: 99) Point G in his drawing is an occult center. The term implies "hidden from the eye", not like the "dead" center, which is usually obvious. I don't know the origin of the term, but have used it (both the term and the point) for decades.

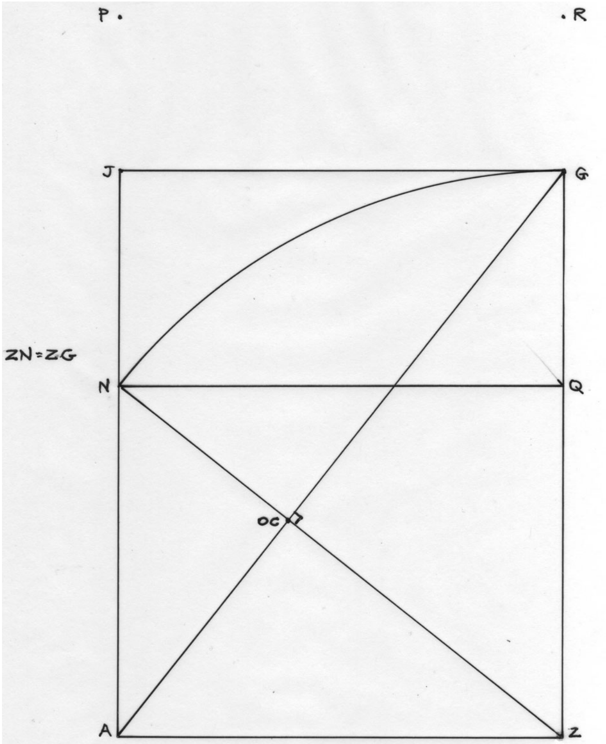


Fig. 6 Construction 3

AG and ZN. Because these diagonals intersect at 90° they form similar rectangular ratios and similar right triangles: AOCN = AOCZ = ZOCG. Also, OC marks the golden section of both the height and the width of the $\sqrt{\phi}$ rectangle AJGX. This will be seen in Construction 4.

Construction 4 (Fig. 7)

(a) Rectangle NJGQ is a μ rectangle. It is a ϕ-related ratio, and a member of the ϕ family of ratios. As stated, the μ rectangle exists a priori, as a specific rectangle within the initial elementary grid constructed in a $\sqrt{\phi}$ rectangle, even though it can also exist as the compound rectangle, $\sqrt{\phi}$ and $1/\sqrt{\phi}$, NJGQ, as in Construction 4.

(b) There is a way to generate this $\sqrt{\phi}$ compound: by the "power of the diagonal" (the diagonal's ability to take a measurement from one side to another side of a rectangle). NQ is the golden section of the long sides, AJ and ZG, of the $\sqrt{\phi}$ rectangle, AJGZ. The diagonal, AG, cuts NQ at X, which is the golden section of NQ. XYGQ is a vertical $\sqrt{\phi}$, and NJYX is a horizontal $\sqrt{\phi}$. Line XY, which is both a short side and a long side of the two $\sqrt{\phi}$ rectangles, NJYX and XYGQ, is the ϕ caesura of the width of the μ rectangle, NJGQ. Further, NJYX and XYGQ are reciprocals of one another. Also note that the rectangle made from these ϕ caesuras, WFYX, is also a μ rectangle.

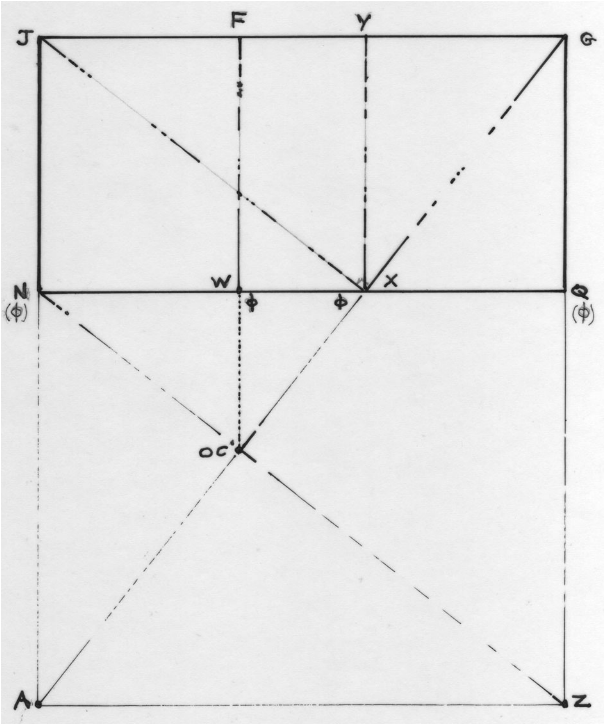


Fig. 7 Construction 4

(c) There is a second way to generate the same twosome: by the occult center, OC. The division of the original μ rectangle NJGQ into its two component $\sqrt{\phi}$ rectangles can be generated from the position of the occult center OC. WF is the golden section of NQ, at W, and JG mirrors this, at F. WF is a ϕ caesura.

Construction 5 (Fig. 8)

Rectangle AKMZ is a μ rectangle. Contained within it are two $\sqrt{\phi}$ rectangles, APRZ and PKMR, reciprocating one another.

PR marks a ϕ caesura on the vertical sides. Point P is the vertex of a 90° angle, where diagonals ZP and PM intersect, thus confirming that the two rectangles are similar and reciprocal.

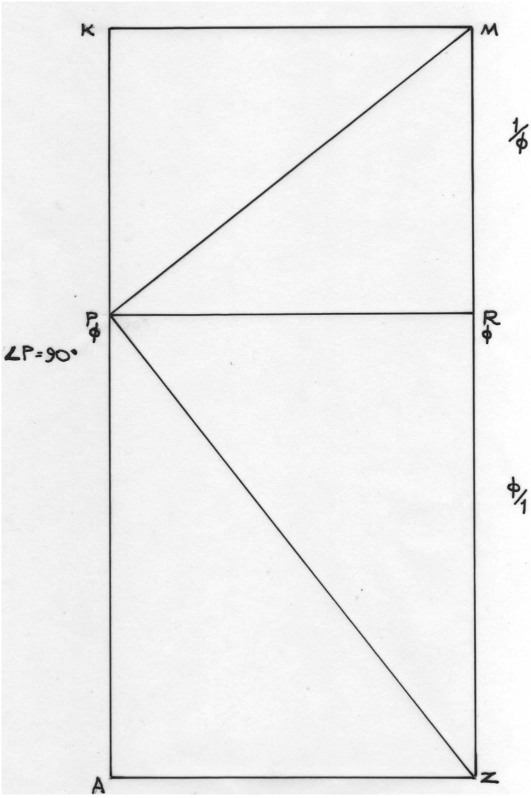
Construction 6 (Fig. 9)

AKMZ is the μ rectangle with its diagonals, AM and ZK added.

(a) With the compass pin in A, open to M, rotate AM to point L, and also, pin in Z, open to K, rotate ZK to N to make the new rectangle, ALNZ, 2.288:1.

This new rectangle is the v rectangle mentioned above, whose sides are in the ratio $\phi\sqrt{2}$:1. The diagonal of a μ rectangle, AM, is 2.288 by the Pythagorean Theorem:

Fig. 8 Construction 5



$$AZ^2 + ZM^2 = AM^2$$

$$1^2 + \phi\sqrt{\phi}^2 = AM^2$$

$$1^2 + (2.058)^2 = AM^2$$

$$1 + 4.236 = AM^2$$

$$2.288 = AM$$

Length AM = AL = 2.288 = $\phi\sqrt{2}$. Here is a naturally occurring marriage of incommensurables that is mathematically verified, proving a link between the ϕ family of ratios with a member of the square root rectangle series. Together, they create a new family of mixed ratios, the off-spring of mixed marriages so to speak. This is another example of the little known and hidden relationship between ϕ and the $\sqrt{2}$ joined by the orphan μ. The Great Pyramid of Khufu also has the ratio $\sqrt{\phi}$ and the $\sqrt{2}$ within it (its height to the diagonal of the pyramid's square base) (Reynolds 1999).

Notice that in calculating the length of AM the first six decimal places 2.28824 are even, that is divisible by 2. As we shall see, this number 2 (also its square root,

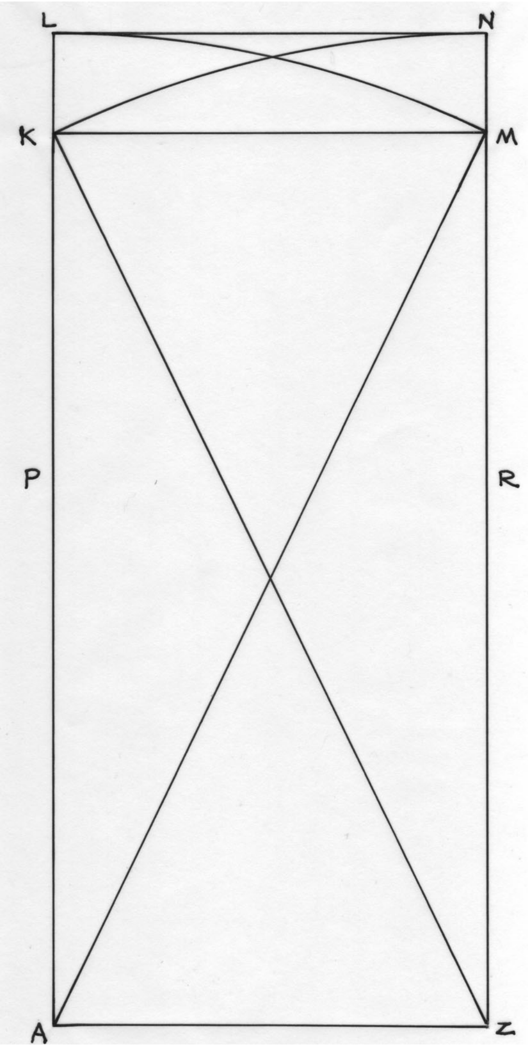


Fig. 9 Construction 6

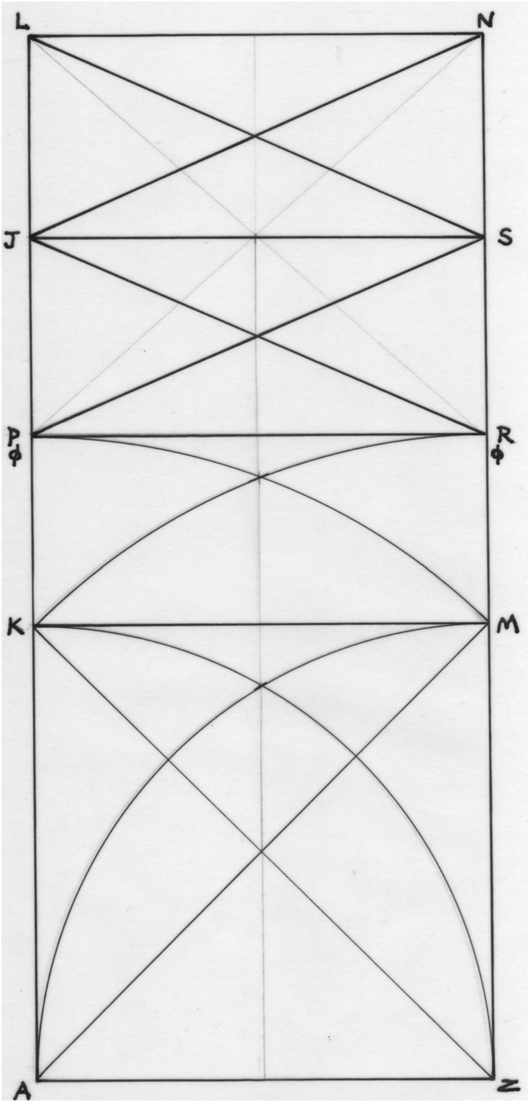
double, or half), is highly significant and will play a key role in the geometry that unfolds in this *v* rectangle. We remember that there is no relationship between ϕ and the number 2 as they exist individually. But here, 2 works naturally with ϕ within the compositional grid, and vice versa. There are other rectangles that perform similar marriages. Later in this article, I will show what I have named the ρ (*rho*) rectangle, $\phi\sqrt{3}:1 = 2.802:1$. It's almost a triple square.

Construction 7 (Fig. 10)

In Construction 7, ALNZ is a *v* rectangle ($\phi\sqrt{2}:1 = 2.288: 1$) and the application of a specific interior grid which is generated as follows:

(b) By rebatment, the short side, AZ, is rotated onto the long sides of the rectangle to form square AKMZ in the base.

Fig. 10 Construction 7



(c) From square AKMZ is generated the $\sqrt{2}$ rectangle APRZ. The top of this $\sqrt{2}$ rectangle, PR, marks a golden section of the height of the rectangle ALNZ.
(d) This leaves rectangle PLNR at the top. PLNR is formed by two reciprocal *v* rectangles to ALNZ, PJSR and JLNS.

Construction 8 (Fig. 11)

(a) The *v* rectangle ALNZ now has a different grid in the top half, CLNE. Diagonals AN and ZL have been drawn, and the two bisectors UF and CE have

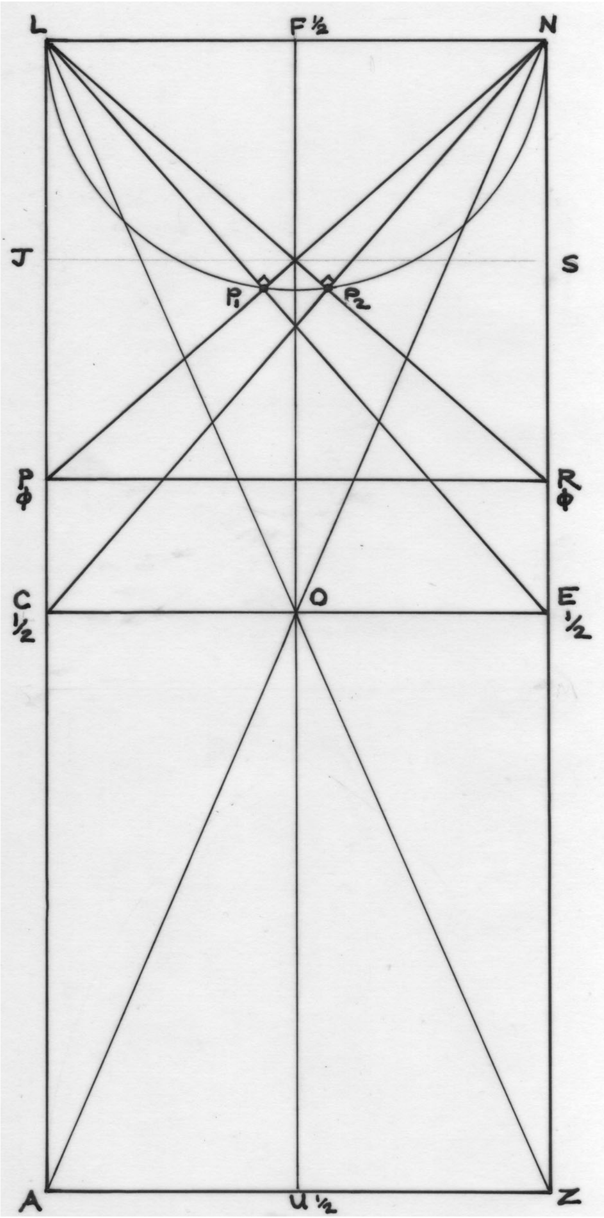


Fig. 11 Construction 8

been added to find halves of the master *v* rectangle. Point O marks the center of the master *v* rectangle.

(b) CN and EL are half-diagonals of *v* Rectangle, ALNZ.
(c) Line PR divides rectangle ALNZ in the golden ratio.
(d) PN and RL are diagonals of the double *v* Rectangle, PLNR.

(e) All these diagonals intersect at points p_1 and p_2, which also intersect at 90°.[4] Therefore, rectangle CLNE is itself a double *v* rectangle, CLFO = OFNE, and so is rectangle PLNR (as indicated in Construction 8) having PJSR and JLNS as a second double *v*.

Construction 9 (Fig. 12)

(a) From the center O, draw a straight line through point p_2 to locate the golden section of LN at point $c\phi4$.
(b) Point $c\phi4$ is the golden section of LN. This is proven by the right triangle in the base, AqZ. AqZ is the 1-2-$\sqrt{5}$ triangle, which is used to generate the golden section. The vertical line $\phi c\phi4$ is applied to locate the golden section of LN at point $c\phi4$ at the top to demonstrate this fact.

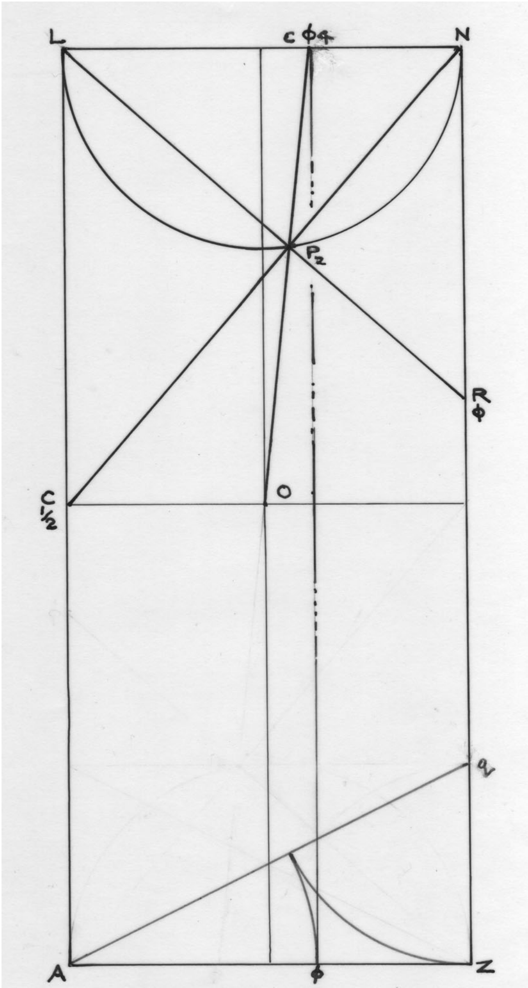
Construction 10 (Fig. 13)

Construction 10 is a *v* rectangle, ALNZ, with a grid that demonstrates the use of the four points, p_1, p_2, p_3, and p_4, that yield the golden sections of AZ and LN. We can generate the other four points, t_1 through t_4, by the "power of the diagonal", on the diagonals, and we now have all eight golden sections of the *v* rectangle's sides, labeled ϕ_1 through ϕ_8.

Now we can see how the *v* finds all eight of its golden sections on its sides by using $\sqrt{2}$ and themes of 2 and its multiples. This wonderful and rational application allows us to find the eight golden sections, the irrational aspects of the rectangle. The theme of 2 running through the constructions comes, I believe, from the evocation of the number 2 and its square root form, in the original *v* number (=$\phi\sqrt{2}$). The golden sections function in the same way; that is, they are found because the golden section was originally invoked at the outset of the *v* ratio. Once called upon, and applied, the numbers in the given ratio stay to work with the geometer in the multitude of grid generating techniques in the grid making, and in a kind of Gestalt-type way. When a specific number or ratio is called for at the outset of the rectangular space, this number will then operate within the structure, at the geometer's commands, through the grids that are generated. The selected numbers form the basis for the inner harmonies. This is how proportioning systems can function by construction and how the influences the ratio selected has on the space within the frame of the rectangle's "personality". The geometry works rationally and instinctually with the mind of the operator. The more skilled the geometer is in making grids, the greater the possibilities for grid harmonies involving the numbers selected for the ratio become. As a result, an enormous variety of compositional grids can be explored and selected.

[4] The Rule of Thales states that the periphery angle over a half circle is a right angle, so half-circle LN proves that p_1 and p_2 are vertices of 90° angles.

Fig. 12 Construction 9



The ϕ^2 and the $\phi\sqrt{3}$ Ratios

There is a second discovery that I would like to present. It is similar to the relationships between μ and v presented already, and shares some qualities with the v rectangle. Most people familiar with the golden section know that $\phi^2 = \phi + 1 = 2.618$. In its rectangular format, the ratio 2.618 expresses itself as a golden section rectangle with a square added to its short side as seen in Construction 11.

Construction 11 (Fig. 14)

(a) AKMZ is the mother square.

(b) CM is the required half-diagonal of the square that generates the ϕ rectangle APRZ, 1.618:1. Rectangle KPRM is a reciprocal ϕ rectangle.

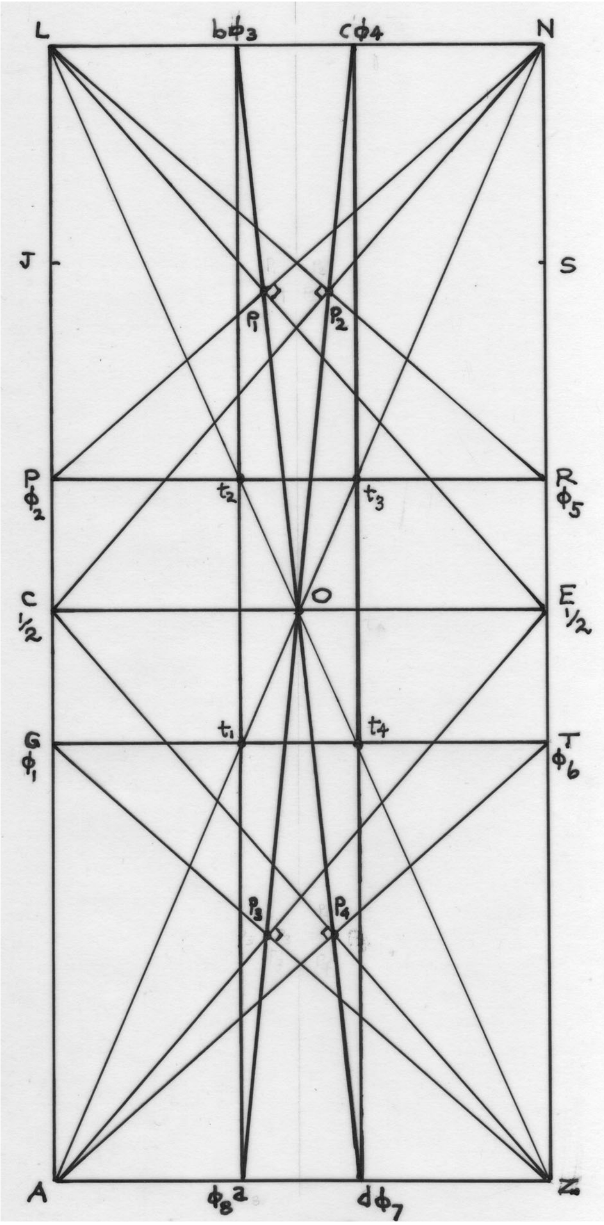
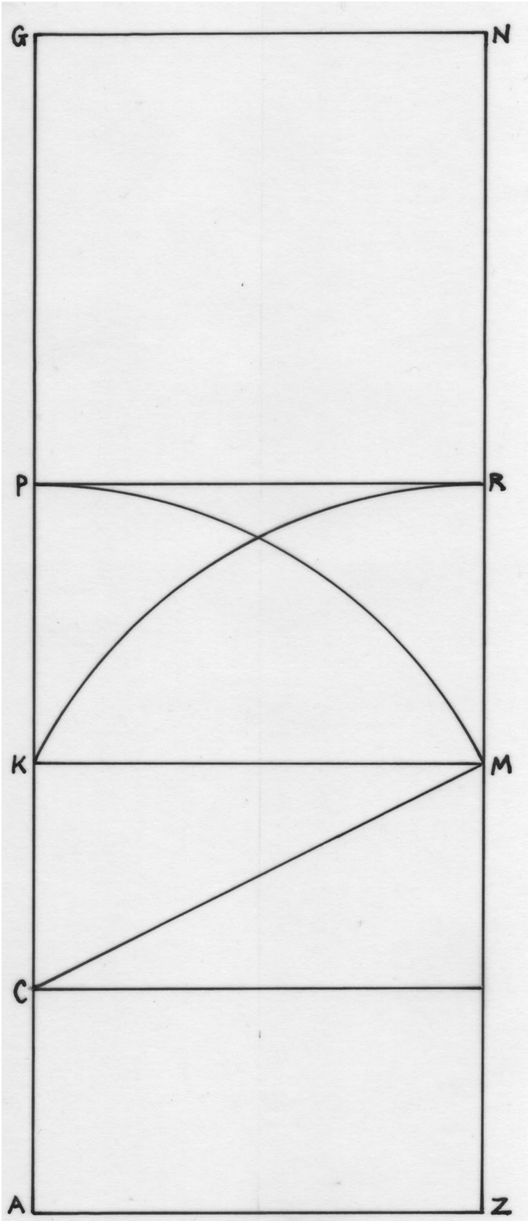


Fig. 13 Construction 10

(c) A second square PGNR is added to the top of the ϕ rectangle APRZ on side PR. Rectangle AGNZ forms the ratio $\phi + 1{:}1$ ($=\phi^2{:}1$) $= 2.618{:}1$.

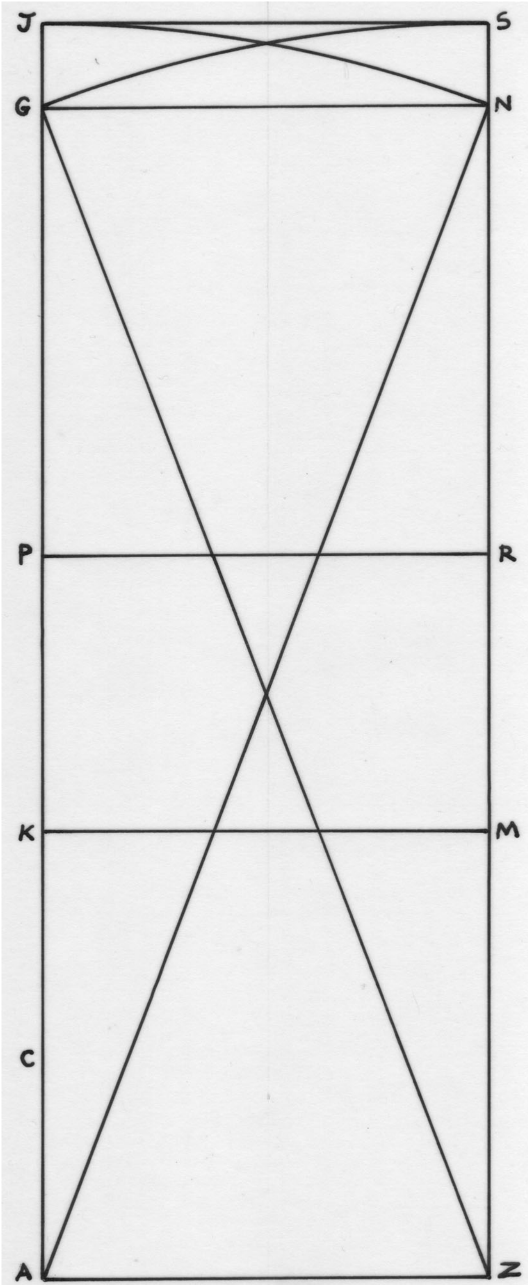
Construction 12 (Fig. 15)

Fig. 14 Construction 11



(a) Rectangle AGNZ is the ϕ^2 rectangle with its diagonals AN and ZG.
(b) Each diagonal, by the Pythagorean theorem, equals $2.618^2 + 1^2 = \sqrt{7.854} = 2.802$.
(c) $2.802 = \phi\sqrt{3}$, another marriage of incommensurables.
(d) These diagonals have been rotated to the vertical, as can be seen in arcs GS and NJ, forming the ρ (*rho*) rectangle AJSZ, “the 2.802”.

Fig. 15 Construction 12



Construction 13 (Fig. 16)

(a) Rectangle AJSZ is a ρ Rectangle.

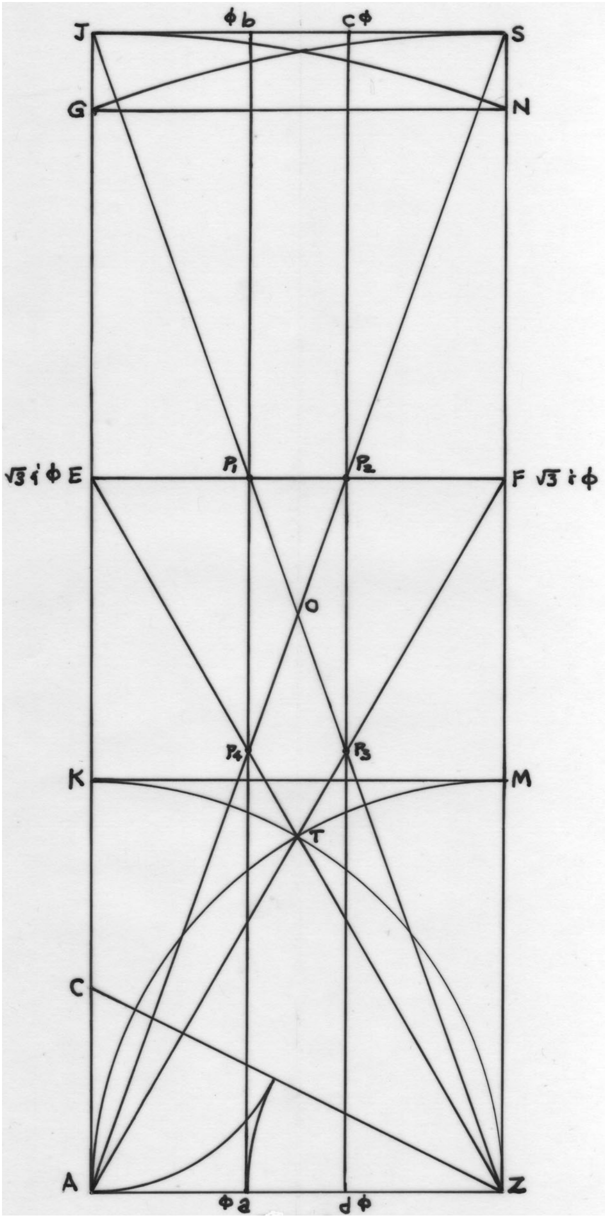


Fig. 16 Construction 13

(b) The mother square, AKMZ, shows the rebatment of AZ to AK and ZA to M, crossing at T, and yielding *ad triangulum*, the equilateral triangle ATZ. Therefore, when AT and ZT are extended to points E on side AJ, and F on side ZS, the generation of the $\sqrt{3}$ rectangle AEFZ ($\sqrt{3}$:1 = 1.732:1) is made.

(c) Also, while working in the master square, AKMZ, the opportunity was taken to generate the golden section. It was applied at point a on the side AZ by the method of the 1-2-$\sqrt{5}$ triangle, AZC. Therefore, the vertical caesura, ab, is a ϕ caesura, and so is dc. This procedure was taken for the purposes of establishing the role of ϕ in the overall construction, as we shall see in the next step.

(d) At points p_1 and p_2, golden section caesuras, ab and dc, intersect with the diagonals of the ρ, AS and ZJ, and the top of the $\sqrt{3}$ rectangle, EF. This demonstrates that the golden section of the height of the ρ rectangle is a $\sqrt{3}$ rectangle, a beautiful marriage of incommensurables at work, ϕ and $\sqrt{3}$!

(e) Also to be seen is how the diagonals of the ρ rectangle, AS and ZJ, intersect with the diagonals of the $\sqrt{3}$ rectangle at points p_3 and p_4. These two points will also yield the ϕ caesuras ab and dc (Points p_3 and p_4 would yield the bottom of a second $\sqrt{3}$ rectangle, another ϕ caesura, placed in the top of the rectangle.) This is the aforementioned "power of the diagonal". The diagonal can move halves, thirds, etc., in fact, any part, from one side to another, in the same way.

This brings us to the last part of the rectangle, the remaining rectangular space EJSF on top of the$\sqrt{3}$ rectangle AEFZ. This rectangle epitomizes, perhaps more so than any other part of the ρ Rectangle, the use of three.

Construction 14 (Fig. 17)

(a) The ratio of this rectangle, EJSF, is 1.070:1. It is nothing less than a rectangle whose ratio is $\sqrt{3}$ by ϕ, another wonderful marriage of incommensurables! And it consists of three ρ Rectangles, EPRF, PGNR, and GJSN. For now, its name is "the 1.070".

(b) There is a second triple ρ at the base, AXYZ. If one triple ρ is placed in the bottom and one in the top, this leaves the central rectangle, XEFY. This ratio is 1.511 (or 1/1.511 = 0.661) to 1. This ratio contains two $\sqrt{3}$ rectangles, one on each side, leaving a ρ rectangle, p_1, p_2, p_3, and p_4, vertically in the center, framed by the ϕ caesuras all around it.

Conclusions

There is most likely more to be found both in the v and the ρ rectangles presented here, and in other rectangles that are at the extreme limits of our known families of ratios and their "orphan" counterparts, especially within the ϕ family. I will certainly continue my explorations.

There are other compounds involving ϕ; see Jay Hambidge's glossary of them (Hambidge 1926: 127ff). I'm certain that there are as yet undiscovered jewels within the ϕ family of ratios. I believe geometry is as much an organic substance as it is a mathematical system. Geometry has a certain intelligence suited to, and compatible with, the human mind's creative abilities, much the way sympathetic magic works; paper and pencil have worked so well together for so very long, it's a kind of sympathetic magic that they do. Geometry is not a dead subject. There will always be another marriage of incommensurables.

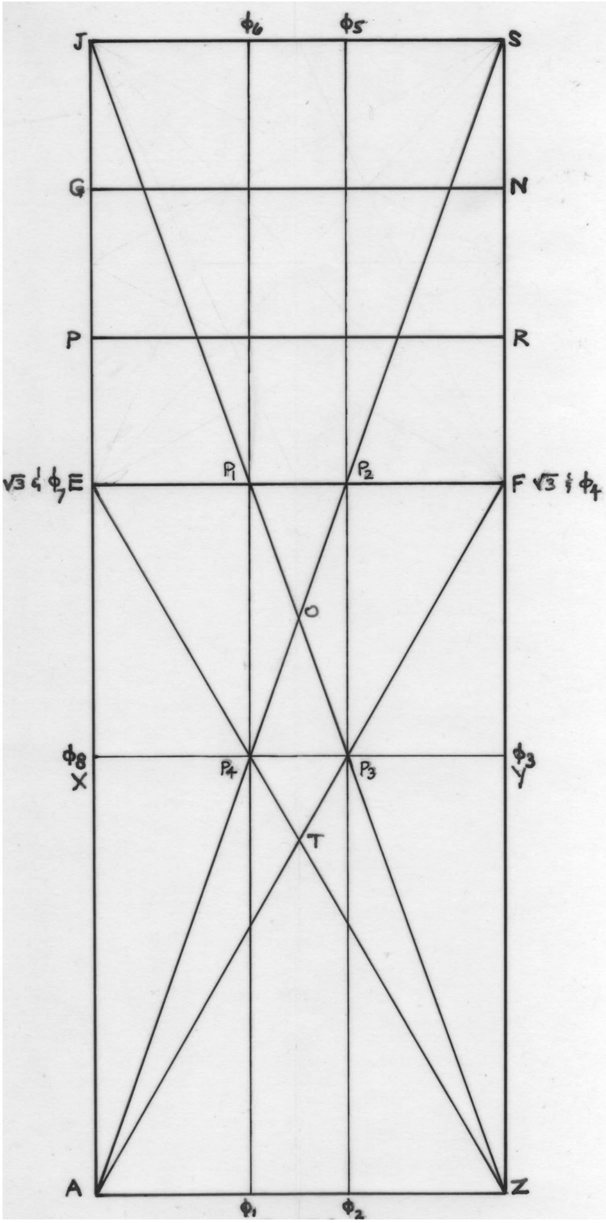


Fig. 17 Construction 14

I believe designers and theorists would do well to look beyond the surface or name of a geometric shape in and of itself, and rather, do the harmonic (de)constructions that you are able to do, break up the space beyond the standard "British Flag" grid—the diagonals and the vertical/horizontal bisectors—and generate intersections of grid lines for development to explore like I have here. See what connections you might find, by pencil or calculator, or preferably, both together. Manifest new areas, connections, intersections, all of it, because there is so much more in the possibilities within that shape than you would think if you just let it pass by.

Always study grids, as many as possible. Learn and understand what the geometer has done, for there are an extraordinary number of things produced with grids. They give you the potential to create spatial relationships inherent within the boundary of the selected shape that are living parts of that shape. If done well, grids will create harmony of parts to whole. They present a world where compositional possibilities are almost boundless, and present a seemingly endless variety of elements for whatever your needs are, be they architectural, design layouts, or artistic compositions. They also provide a way to experience the connections and relationships inherent in all geometry and to see that geometry is a complete language.

This raises questions surrounding the origins of geometry within our universe: how did geometry first come about? If only we knew. But the more you explore, the closer you come to possible answers. If the designer or theorist knows for example that the golden section manifests itself in abundant ways all around us; after asking why, then why not use it to see what you might come up with?

I leave you for now by asking you to remember where these wonderful constructions I've shown you came from: they are found in just some of the possibilities within the anatomical parts of the ϕ family of ratios. The golden ratio's completeness in art, architecture, nature, and mathematics may never be found, especially in terms of its interrelationships with other systems, but the journeys count, and discoveries await.

In a future article, I would like to present another discovery that I believe is most extraordinary. It too was originally a discovery found with the help of ϕ and the $\sqrt{2}$. The golden section is not the only magnificent ratio, but it's one of the best of them.

References

Calter, Paul. 2008. *Squaring the Circle*. Emeryville: Key College Publishing.

Euclid. 1956. *The Thirteen Books of the Elements*, Thomas L. Heath, ed. 3 vols. New York: Dover Publications.

Ghyka, Matila. 1977. *The Geometry of Art and Life*. New York: Dover Publications.

Hambidge, Jay. 1926. *The Elements of Dynamic Symmetry*. New York: Dover Publications.

Herz-Fischler, Roger. 2000. *The Shape of the Great Pyramid*. Waterloo, Ontario: Wilfrid Laurier University Press.

Reynolds, Mark. 1999. A Comparative Geometrical Analysis of the Heights and Bases of the Great Pyramid of Khufu and the Pyramid of the Sun at Teotihuacán. *Nexus Network Journal* **1**: 87–92.

Reynolds, Mark. 2003. The Unknown Modulor: The "2.058". *Nexus Network Journal* **5**(2): 119–130.

Serlio, Sebastiano. 1982. *The Five Books of Architecture*. New York: Dover Publications.

Mark A. Reynolds has worked with graphite, watercolor, pastel, and other drawing media for over thirty years, developing interpretations of traditional principles found in Euclidean and philosophical geometry. He has discovered systems and harmonic constructions with mixed ratios and new geometric relationships he calls "marriages of incommensurables" and "unions of opposites" that unite chaos and order in complex, intriguing, and beautiful geometric drawings and make evident geometry's interdisciplinary relationship with music, nature, mathematics, architecture, and personal discovery. Mark's work in this field includes geometric analyses of architecture, paintings, and design. Some of his studies, writings, and geometric discoveries can be found in the *Nexus Network Journal*. He is also an educator who has taught geometry, linear perspective, drawing, and printmaking to students of art, design, and architecture at the Academy of Art University in San Francisco. He has lectured on his work at international conferences on art, architecture, and mathematics. Mark's drawings are in the permanent collections of the Biblioteca Comunale Leonardiana in Vinci, Italy, the Fine Arts Museums of San Francisco, and the Crocker Museum in Sacramento. He is represented by Pierogi Gallery in New York City.