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Remarks on the Surface Area and Equality Conditions in Regular Forms Part III: Multi-sided Prisms

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Abstract Applying the methodology and rules that were previously established in Part I of this work, this part presents the remarks on the mathematical analysis for the regular multi-sided right prisms. According to the shape of their bases, these include shapes from pentagon to circle. The first remark examines the effect of θ on S. The second remark calculates the minimum total surface area (S_{Min}) in two cases, the case of constant θ and the case of variable θ. The third remark calculates walls ratio R_W and the critical walls ratio R_{Wo}. The last remark studies the required conditions for the numerical equality in two cases, the case of Per = Ar, and the case of S = V. Finally, the findings of the first group (right regular prisms) are generalized and discussed.

Keywords Trigonometry · Algebra · Differential equations · Volume · Area · Total surface area · Perimeter · Regular polygons · Multi-sided prisms · Minimum total surface area · Walls ratio · Numerical equality

Introduction

Regular multi-sided rooms have been used repeatedly in many historical and contemporary buildings either in simple residential buildings or multi-function complex projects. The Holy Dome of the Rock in Jerusalem (Elkhateeb 2012) (built between 687 and 691, Fig. 1a) is one of those famous regular octagonal buildings. Regular multi-sided shapes were also utilized on the level of city planning, as there are many examples of cities that have regular multi-sided shapes. The plan of the city of Palmanova (near Venice, Italy, constructed during the renaissance, Fig. 1b) is a good example of such a city (Wikipedia 2013).

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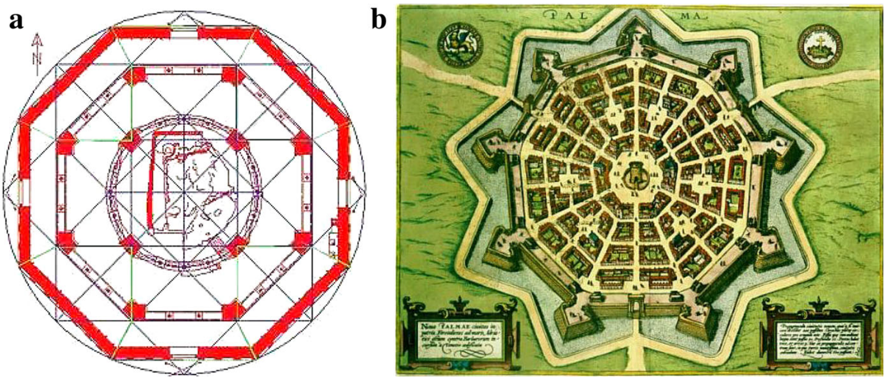

Fig. 1 Regular multi-sided shapes have many architectural applications. **a**, *left* Plan for the Holy Dome of the Rock. **b**, *right* Plan of the city of Palmanova near Venice

For the purposes of this part, the regular multi-sided right prisms are those prisms that have more than four sides. Thus, and according to the shape of their bases, they include a wide range of shapes, from pentagonal to circular. In the first part of this work (Elkhateeb 2014) assumptions and methodology were set out to mathematically analyze isosceles triangular right prisms in order to answer five questions:

- How the angle θ (or θ and β) affects S?
- When S becomes minimum (S_{Min})?
- What is the ratio between walls surface area S_W and S ($S_W/S = R_W$)?
- When Ar numerically equals Per? and,
- When S numerically equals V?

Following the same methodology and assumptions that were previously applied, this part investigates the case of the regular multi-sided right prisms.

Notations

In this part, the following terms mean:

Ar	: Room floor area (m^2)
h	: The altitude of the triangle, see Fig. 2 (m)
H_R	: Room height, the height of the prism (m)
$\boldsymbol{H_{Ro}}$	: The critical room height, the height that fulfills (S–V) equality (m)
n	: Number of sides
Per	: Perimeter (m)
S	: Room total surface area (m^2)
S_{Min}	: The minimum total surface area (m^2)
S_W	: Walls total surface area (m^2)
r	: Radius (m)
$\boldsymbol{r_o}$	: The critical radius, the radius that fulfills (Per–Ar) equality (m)
R_W	: Walls ratio, S_W/S (Ratio)

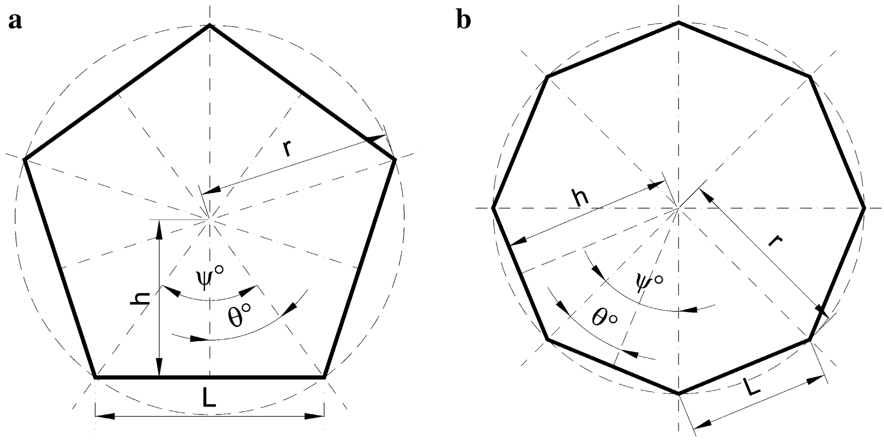

Fig. 2 Multi-sided shapes, the different variables. **a**, *left* Example for a pentagonal shape. **b**, *right* Example for an octagonal shape

R_{Wo} : The critical walls ratio, the ratio between walls total surface area and total surface area when S is minimum (S_{Min}) (Ratio)

V : Room volume (m^3)

ω_o : The critical ratio, the ratio between H_R and r when S is minimum (S_{Min}) (Ratio)

The other terms will be illustrated in figures according to each case as required.

Room Assumptions

The term "regular" as will be used hereafter means that the shape under discussion must fulfill two conditions (see Fig. 2):

- To be contained in a circle.
- Its central angle ψ is constant and equal to 360/n.

Through this part, it is assumed that the angle θ, Ar and V are the independent variables whereas Per and S are the dependent ones. Figure 2 shows the terms: θ, ψ, h, L and r in the regular multi-sided shapes.

The Mathematical Relationships of Multi-sided Shapes

The regular multi-sided shapes can be identified knowing both Ar and n. This section derives the main mathematical functions among the different variables of such shapes. These functions will be utilized later to determine S_{Min} and calculate the equality conditions. From the first principles, it can be proved that:

$$\theta = \frac{180}{n}\ (degrees) \tag{1}$$

$$L = 2r\sin\theta. \tag{2}$$

Also

$$h = rcos\theta. \tag{3}$$

And

$$r = \sqrt{\frac{Ar}{n\sin\theta\cos\theta}}. \tag{4}$$

The perimeter Per of the regular multi-sided shapes can be calculated as:

$$Per = n \times L \tag{5}$$

By substitution for L according to Eq. 2, Eq. 5 can be rewritten as:

$$Per = 2nr\sin\theta \tag{6}$$

The Area of the regular multi-sided shapes can be calculated as:

$$Ar = \frac{n \times L \times h}{2} \tag{7}$$

By substitution for L and h according to Eqs. 2 and 3 respectively, Eq. 7 can be rewritten as:

$$Ar = nr^2\sin\theta\cos\theta. \tag{8}$$

In the third dimension, a regular multi-sided shape can be extruded to form a right prism with a height H_R. In this case, its volume $V = (H_R \times Ar)$ can be calculated from:

$$V = H_R \times nr^2\sin\theta\cos\theta \tag{9}$$

Thus

$$H_R = \frac{V}{nr^2\sin\theta\cos\theta}. \tag{10}$$

In this part, the ratio ω will be defined as:

$$\omega = \frac{H_R}{r}. \tag{11}$$

The total surface area S of a right prism in this case can be calculated as:

$$S = 2Ar + (Per \times H_R) \tag{12}$$

Given the values of Per (Eq. 6), Ar (Eq. 8), and H_R (Eq. 10) as a function of θ, Eq. 12 can be rewritten as:

$$S = 2nr\sin\theta\,(H_R + r\cos\theta) \tag{13}$$

By substitution for H_R according to Eq. 10, Eq. 13 will be:

$$\boldsymbol{S} = 2\boldsymbol{nr}\sin\boldsymbol{\theta}\left(\frac{\boldsymbol{V}}{\boldsymbol{nr}^2\sin\boldsymbol{\theta}\cos\boldsymbol{\theta}} + \boldsymbol{r}\cos\boldsymbol{\theta}\right). \tag{14}$$

Remark 1: Effect of θ on S

In multi-sided shapes, the numerical solution of Eq. 13 shows that S is an increasing function of θ (see Fig. 3), thus it is a decreasing function of n according to Eq. 1. Figure 3 is a graphical representation for Eq. 13. As can be concluded from this figure, the function reaches its minimum value when θ → 0°.

Remark 2: the Minimum Total Surface Area, S_{Min}

Following the same approach that was previously applied in triangular rooms, two cases will be considered here:

- Case of constant θ, where both Ar and H_R are variables. Or;
- Case of variable θ, where both Ar and H_R are constants.

Case I, Constant θ, Variable Ar and H_R

In this case, among the different multi-sided rooms that have the same θ and V, S_{Min} occurs when the first derivative of Eq. 14 equals zero, i.e.

$$\frac{\boldsymbol{dS}}{\boldsymbol{dr}} = 4\boldsymbol{nr}\sin\boldsymbol{\theta}\cos\boldsymbol{\theta} - \frac{2\boldsymbol{V}}{\boldsymbol{r}^2\cos\boldsymbol{\theta}} = 0 \tag{15}$$

By Substitution for V according to Eq. 9, and applying the rules of algebra, Eq. 15 can be rewritten as:

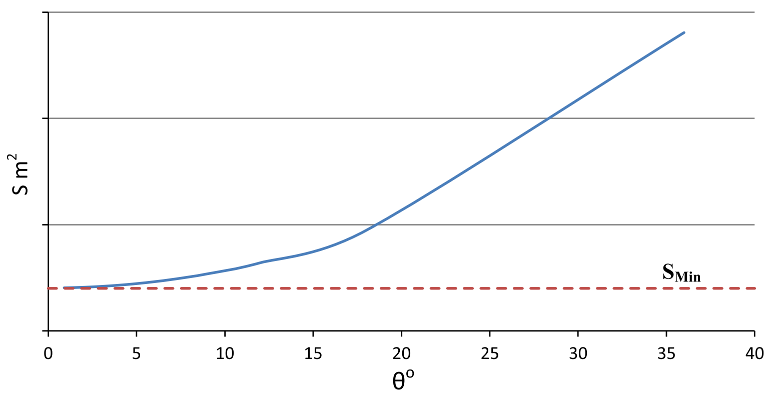

Fig. 3 The relationship between θ and S according to Eq. 13

$$\boldsymbol{H_R} = 2\boldsymbol{r}\cos\boldsymbol{\theta} \tag{16}$$

Thus, the critical ratio $\boldsymbol{\omega_o}$ (see Sect. "Notations") can be calculated from Eq.16 as:

$$\boldsymbol{\omega_o} = 2\cos\boldsymbol{\theta}. \tag{17}$$

As can be concluded from Eq. 17, $\boldsymbol{\omega_o}$ is a function of θ (consequently n). Thus, for every n (i.e. for every regular multi-sided right prism) there is $\boldsymbol{\omega_o}$ that fulfills S_{Min}. Eq. 17 also tells that $\boldsymbol{\omega_o}$ is a decreasing function of θ, therefore, it is an increasing function of n, see Fig. 4. When n reaches ∞, then θ → 0°, consequently, $\boldsymbol{\omega_o} = 2$, this is the case of a circle. To determine room dimensions that fulfill S_{Min} in such prisms:

- Determine both n and V;
- Calculate θ by applying Eq. 1;
- Calculate $\boldsymbol{\omega_o}$ by applying Eq. 17;
- Apply Eq. 10 to get r;
- Apply Eq. 17 again to get H_R;
- Utilize Eqs. 2 and 4, respectively to get L and Ar.

It is worth mentioning in this context that Eq. 17 also applies in the two special cases of equilateral triangle (θ = 60°) and square (θ = 45°). Both shapes can be considered regular multi-sided shapes according to the above definition of this term (see Sect. "Room Assumptions"). If r in Eq. 16 was replaced by its equivalent value of h (***r = 2/3 h*** in equilateral triangle, and ***r = h/2*** in square), then Eq. 17 will yield $\boldsymbol{\omega_o}$ according to Eqs. 14 or 15 (Part I) and Eq. 14 (Part II) (Elkhateeb and Elkhateeb 2014).

Similar to the cases of triangular and quadratic right prisms, the two relationships (H_R–S) and (Ar–S) depend on $\boldsymbol{\omega_o}$ which divides these functions into two zones (see Figs. 5 and 6):

- Zone [a]: where $\boldsymbol{\omega} < \boldsymbol{\omega_o}$. In this zone, S is a decreasing function of H_R (see Fig. 5) and an increasing function of Ar (see Fig. 6), note that the location of the

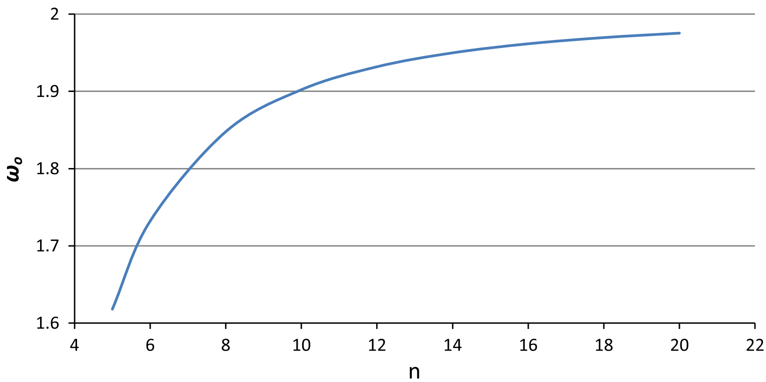

Fig. 4 Values of ω_o in the range $5 \leq n \leq 20$ according to Eq. 17

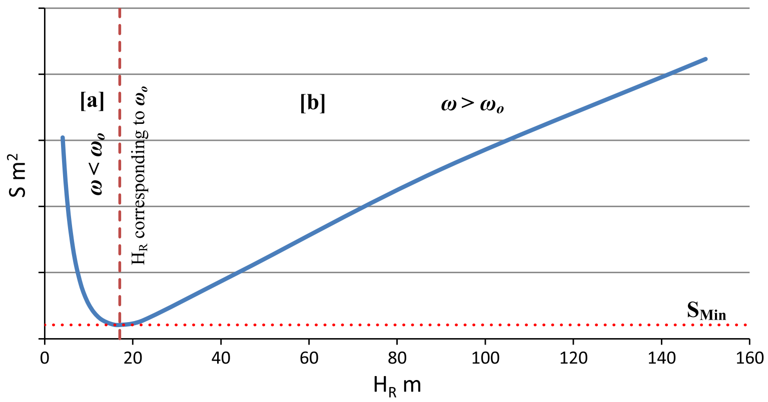

Fig. 5 The relationship of H_R to S (case of pentagon, n = 5)

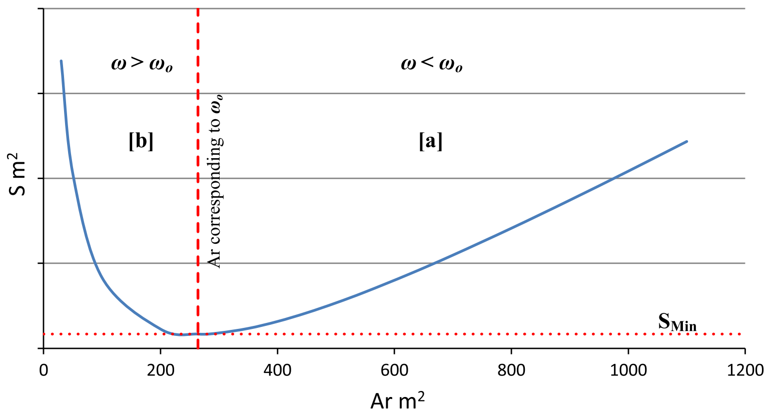

Fig. 6 The relationship of Ar to S (case of pentagon, n = 5)

zones is reversed in Fig. 6. Thus, any increase in room height will decrease its total surface area.

- Zone [b]: where $\boldsymbol{\omega} > \boldsymbol{\omega_o}$. In this zone, S is an increasing function of H_R and a decreasing function of Ar. This means that the increase in H_R will increase S.

Case II, Variable θ, Constant Ar and H_R

In this case, the perimeter will control the value of S according to Eq. 12 as long as the other three parameters V, Ar and H_R are constants in prisms under consideration. In this case, S reaches its minimum value when the first derivative of Eq. 6 equals zero after replacing both n and r by their equivalent values according to Eqs. 1 and 4, respectively, thus:

$$\frac{dPer}{d\theta} = \frac{1}{2}\sqrt{180Ar}\sqrt{\frac{\sin\theta}{\theta\cos\theta}}\left(\frac{\theta cos^2\theta - sin\theta(-\theta\sin\theta + \cos\theta)}{\theta^2\cos^2\theta}\right) = 0 \quad (18)$$

This leads to:

$$\theta = \sin\theta\cos\theta. \quad (19)$$

Eq. 19 fulfills only if θ equals 0. This result completely agrees with the mathematical axiom that among the different regular shapes that have the same area, the circle (n = ∞) possesses the minimum perimeter. As a result, among the different regular multi-sided right prisms, a cylinder has the minimum total surface area. Another proof to this result is that Per is an increasing function of θ according to Eq. 6 (consequently, a decreasing function of n according to Eq. 1), thus Per is a decreasing function of n; accordingly S is a decreasing function of n (see Figs. 3 and 9).

Remark 3: Walls Ratio R_W

In regular multi-sided right prisms, R_W can be mathematically defined as:

$$R_W = \frac{Per \times H_R}{2Ar + Per \times H_R} \quad (20)$$

By substitution for Per and Ar from Eqs. 6 and 8 respectively, Eq. 20 can be rewritten as:

$$R_W = \frac{H_R}{H_R + r\cos\theta}. \quad (21)$$

The relationship between R_W and θ resembles the relationship between S and θ (see Fig. 3), thus it is an increasing function of θ, consequently, a decreasing function of n. R_W reaches its minimum value when θ → 0° (circular shapes). To calculate R_{Wo}, the conditions for ω_o must be applied, thus, Eq. 21 can be rewritten as:

$$R_{Wo} = \frac{2r\cos\theta}{2r\cos\theta + r\cos\theta} \quad (22)$$

This leads to:

$$R_{Wo} = \frac{2}{3} = 0.6667. \quad (23)$$

Consequently, the critical walls ratio R_{Wo} (see Sect. "Notations") in regular multi-sided right prisms is also constant for any θ and equals 2/3.

Remark 4: Case of Equality

This section calculates two cases of numerical equality in regular multi-sided prisms. The first considers the numerical equality between Per and Ar. The last considers the numerical equality between S and V.

Case I, Equality of Per and Ar

In regular multi-sided rooms, and according to Eqs. 6 and 8, the numerical equality between Per and Ar occurs when:

$$2nr_o \sin\theta = nr_o^2 \sin\theta \cos\theta \tag{24}$$

By applying the rules of algebra and trigonometry, the critical radius r_o (see Sect. "Notations") can be calculated from Eq. 24 as:

$$r_o = \frac{2}{\cos\theta}. \tag{25}$$

In the special case where n → ∞ (i.e. circular shape), θ → 0. As $\cos 0 = 1$, thus, $r_o = 2$ m.

Similar to ω_o, the numerical equality between Per and Ar also depends on θ according to Eq. 25. The values of r_o were plotted in Fig. 7. As can be concluded from this figure, the relationship between θ and r_o is similar to the relationship between θ and S (see Fig. 3), where r_o is an increasing function of θ (consequently a decreasing function of n).

Case II, Equality of S and V

According to Eqs. 9 and 13, such numerical equality occurs when:

$$2nr\sin\theta(H_{Ro} + r\cos\theta) = nr^2\sin\theta\cos\theta H_{Ro} \tag{26}$$

By applying the rules of algebra, Eq. 26 will be:

$$H_{Ro} = \frac{2r\cos\theta}{r\cos\theta - 2}. \tag{27}$$

Thus, in regular multi-sided right prisms with given θ and Ar, the numerical equality between S and V occurs only when Eq. 27 fulfills. This can be calculated in the following sequence:

- Determine both θ (or n) and Ar;
- Apply Eq. 4 to get r, then;
- Substitute in Eq. 27 to get the critical room height H_{Ro} (see Sect. "Notations").

In the special case where n → ∞ (i.e. a cylinder), θ → 0. As $\cos 0 = 1$, thus, Eq. 29 will be:

$$H_{Ro} = \frac{2r}{r - 2} \tag{28}$$

Similar to triangular and quadratic rooms (see Parts I and II), the minus sign (−) in the denominator of Eqs. 27 and 28 indicates that for every θ there is a minimum r under which this numerical equality will never exist. This occurs when H_{Ro} tends to ∞, i.e., when Ar equals Per according to Eq. 25. Figure 8 represents the relationship between Ar and H_{Ro} calculated from Eq. 27 for a regular pentagonal right prism (θ = 36°). As can be seen from the figure, in the acceptable range, H_{Ro} is a decreasing function of Ar. In this range, the function can be divided into two

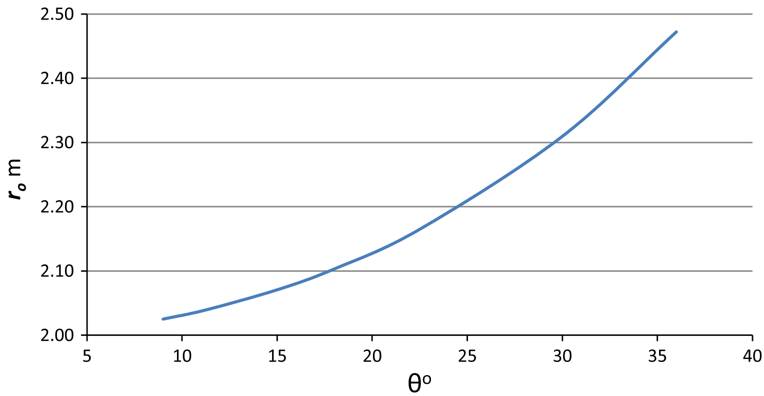

Fig. 7 Values of r_o for the common regular multi-sided shapes according to Eq. 25

main zones, zone of rapid decay (when Ar tends to be equal to Per) and zone of slow decay (when Ar is far from this equality).

Generalization of Results, Group of Regular Traditional Forms

The right prism that has regular bases (from the isosceles triangle to the circle) constitutes the core of the first three parts of this work (see Elkhateeb 2014; Elkhateeb and Elkhateeb 2014). It was assumed that the volume of the prism V is constant. Through this work:

- A complete set of mathematical functions that relates Per, Ar and S to the angle θ (or θ and β) was derived.
- The effect of θ (or θ and β) on S was investigated.
- The minimum total surface area S_{Min} of the prism and walls ratio R_W were calculated.
- The conditions to fulfill two cases of numerical equality (Per–Ar) and (S–V) were calculated.

When θ is variable, in triangular and rectangular rooms that have the same area, S is a decreasing function of θ until a specific θ where this relationship reverses and S becomes an increasing function of θ. This specific θ = 60° in triangular shapes (i.e. equilateral triangle) and 45° in rectangular shapes (i.e. square). The same fact also applies in trapezoidal shapes but the angle at which the function reverses its direction depends on both θ and β. In regular multi-sided rooms, S is an increasing function of θ (accordingly a decreasing function of n).

In all of the examined shapes (from triangle to circle), the mathematical analysis indicates that S is a decreasing function of n. This means that a triangular right prism (n = 3) possesses the maximum S in comparison with a cylinder (n = ∞) that has the minimum S assuming that both have the same Ar and V (see Fig. 9). This can be clarified from the fact that when θ is variable and Ar is constant, the

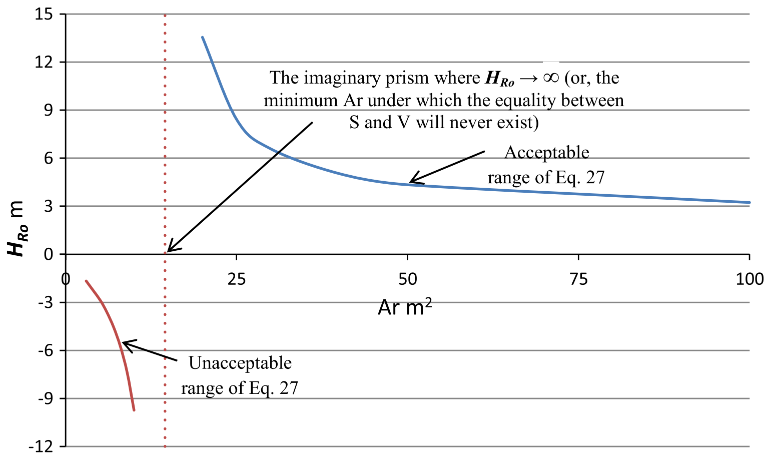

Fig. 8 The relationship of Ar to H_{Ro} when S = V (case of n = 5, θ = 36°)

perimeter becomes the main variable that controls S. As Per is a decreasing function of n (i.e. a triangle possesses the maximum Per, whereas a circle possesses the minimum Per), thus S will be also a decreasing function of n. It should be mentioned here that the variation in S as a function of n (***ds/dn***) becomes limited at the higher values of n (n ≥ 10) as can be seen in Fig. 9.

In trapezoidal shapes, the analysis indicates that Per is a decreasing function of both θ and β. This can be expressed mathematically as:

$$\boldsymbol{Per} \propto \frac{1}{\boldsymbol{\beta}} (\boldsymbol{when\ \theta\ is\ constant}) \tag{29}$$

$$\boldsymbol{Per} \propto \frac{1}{\boldsymbol{\theta}} (\boldsymbol{when\ \beta\ is\ constant}). \tag{30}$$

When θ is constant (so, Ar is variable) and in prisms under consideration, the mathematical analysis proves that there is a critical ratio $\boldsymbol{\omega_o}$ that makes the total surface area of a room reaches its minimum value. This $\boldsymbol{\omega_o}$ depends on the shape of the base and is a function of θ (or θ and β in trapezoidal shapes). When $\boldsymbol{\omega} < \boldsymbol{\omega_o}$, S becomes a decreasing function of H_R and an increasing function of Ar. This means that any increase in room height will decrease its total surface area. On the contrary, when $\boldsymbol{\omega} > \boldsymbol{\omega_o}$, S becomes an increasing function of H_R and a decreasing function of Ar. This means that an increase in H_R will increase S.

When θ is variable, the mathematical analysis indicates that walls ratio R_W of the prisms under discussion is also a function of θ (or θ and β in trapezoid). This relationship resembles the relationship (θ–S). Hence, in triangular and quadratic prisms, R_W is a decreasing function of θ until a specific θ (θ = 60° in triangle and 45° in rectangle) then the function reverses and R_W becomes an increasing function of θ. In regular multi-sided rooms, R_W is an increasing function of θ (accordingly a decreasing function of n).

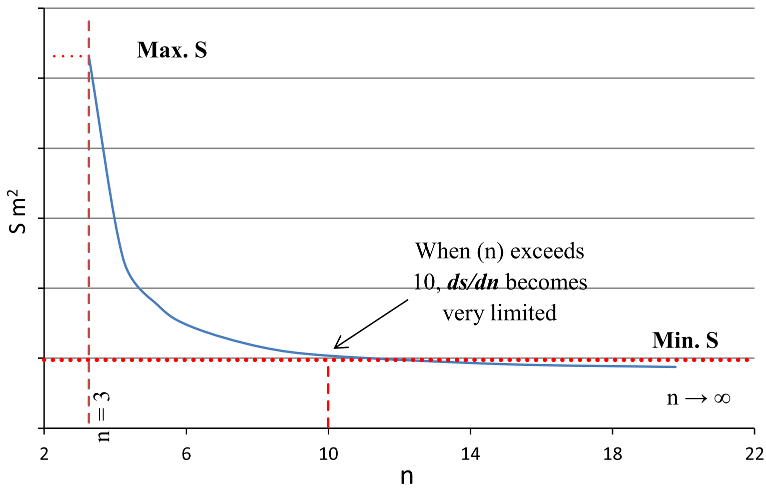

Fig. 9 The relationship of n to S

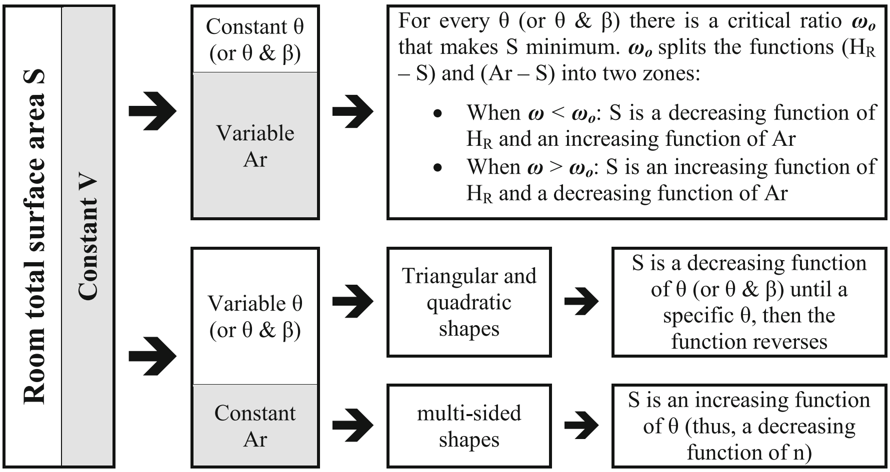

Fig. 10 Cases and mathematical characteristics of S

When S reaches its minimum value, the mathematical analysis proves that $\boldsymbol{R_{Wo}}$ is constant in all prisms under consideration and is equal to 2/3. Figures 10 and 11 summarize the findings of this work (in Parts I, II and III) for both S and R_W.

The numerical equality between Per and Ar fulfills when the critical altitude/diagonal or radius ($\boldsymbol{h_o}$ or $\boldsymbol{r_o}$) fulfills. Similar to $\boldsymbol{\omega_o}$, $\boldsymbol{h_o}$ (or $\boldsymbol{r_o}$) is a function of θ (or θ and β in trapezoid). In the case of circles, $\boldsymbol{r_o} = 2$ m.

The numerical equality between S and V also fulfills when the critical room height $\boldsymbol{H_{Ro}}$ fulfills. In the acceptable range of the derived formulae to calculate this equality, $\boldsymbol{H_{Ro}}$ is a decreasing function of Ar. In the prisms under consideration, the

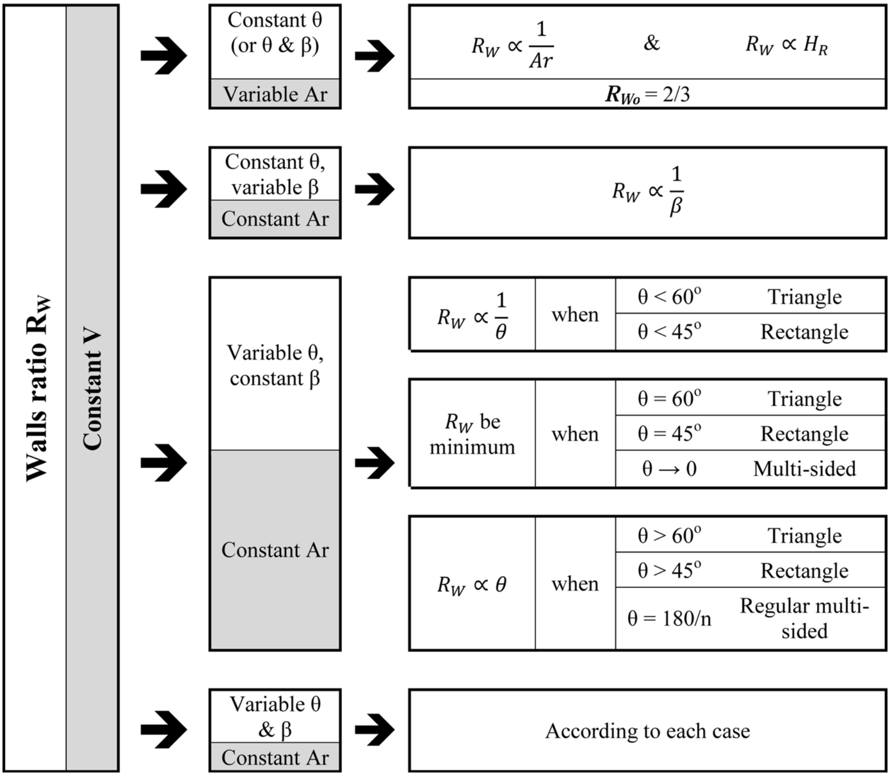

Fig. 11 Cases and mathematical characteristics of R_W

mathematical analysis indicates that for every θ (or θ and β in trapezoid) there is a minimum Ar under which this equality will never exist. This occurs when $\boldsymbol{H_{Ro}}$ tends to ∞, i.e., when Ar equals Per.

Conclusions

Following the same methodology, assumptions and rules that were applied previously in Parts I and II, this part examines the cases of the regular multi-sided right prisms. According to the shape of their bases, such prisms include shapes from pentagon to circle. The first remark examines the effect of θ on S. In the second remark, the minimum total surface area S_{Min} for the prisms under discussion was calculated in two cases, the case of constant θ and the case of variable θ. In the first case, the critical ratio $\boldsymbol{\omega_o}$ was calculated. Results showed that $\boldsymbol{\omega_o}$ depends entirely on θ. The values of $\boldsymbol{\omega_o}$ were calculated and presented. In the second case, where θ is variable, results showed that S_{Min} occurs when θ → 0 (i.e. cylindrical rooms). The third remark calculates the ratio R_W, results showed that R_W reaches its minimum value in circular rooms ($n = \infty$). Results also showed that the critical walls ratio $\boldsymbol{R_{Wo}}$ is constant for any n and is equal to 2/3. The last remark investigates the conditions for the numerical equality either between Per and Ar or S and V. In the first case, the critical radius $\boldsymbol{r_o}$ that fulfills Per–Ar equality was calculated. Results showed that $\boldsymbol{r_o}$ depends entirely on θ. In the second case, the critical room height $\boldsymbol{H_{Ro}}$ that fulfills S–V equality was calculated. Results also indicated that for every θ there is a minimum r under which this equality will never exist; this corresponds to $\boldsymbol{r_o}$ (i.e. Ar = Per). Finally, the results of the first group (regular right prisms) were generalized and discussed.

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