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Research

Method of Modulation and Sizing of Historic Architecture

Abstract. Both the base duodecimal arithmetic and geometrical procedures derived from the diagonal of a square were recurrent resources in the design and construction of past architecture. The hypothesis of a double metric scale justifies the modulation size in buildings throughout a long historical period, using a simple and practical procedure whose fundamentals and characteristics are presented here. Validation by other researchers of the method proposed, would be a milestone in the history of the proportion of architecture, a step towards gaining knowledge of a common metric system used since ancient times in the construction of important buildings.

Introduction

What I have proposed is a method for modulation and dimensioning buildings of the past. It is based on a dual scale, a duodecimal arithmetic base and an octagonal graphic base. The modules are determined on two orthogonal axes of reference.

Duodecimal arithmetic base

The classical system of anthropometric units, which is in base 12, has a remote origin, as does the sexagesimal – base 60 – system, used to measure angles and time. Historically it has been used extensively in determining distances, areas and volumes until it was adopted by of the current international metric system.[1] It survives today in the metric imperial system and in isolated local areas.

The base magnitude of length L – and therefore the magnitudes derived from area L^2 and volume L^3 – is defined from a anthropometric base unit value corresponding to the height of man. Derived units are determined by fractions of the duodecimal arithmetic base (prime factors of the base 2, 3).

$$\mathbf{L(2,3) = 1, 1/2, 1/3, 1/4, 1/6, 1/8, 1/9, 1/12,\ldots}$$

The different civilizations that have used this system never established their base unit as full anthropometric module (the fathom) – perhaps because it was too big and impractical – but rather as a fraction of it (the foot, the cubit or the yard). The palm is a common sub-multiple (1/24), which is divided into 4 digits, or 3 inches in the uncial system (fig. 1).

The dimensional value of the base unit was fixed for each power or government – arbitrarily, it appears – in the same way that the value of the units of the weight or currency system was established. For this reason, the dimensions of the units of different historical and local areas (fig. 2) do not coincide, although in each area the proportional system is similar.[2]

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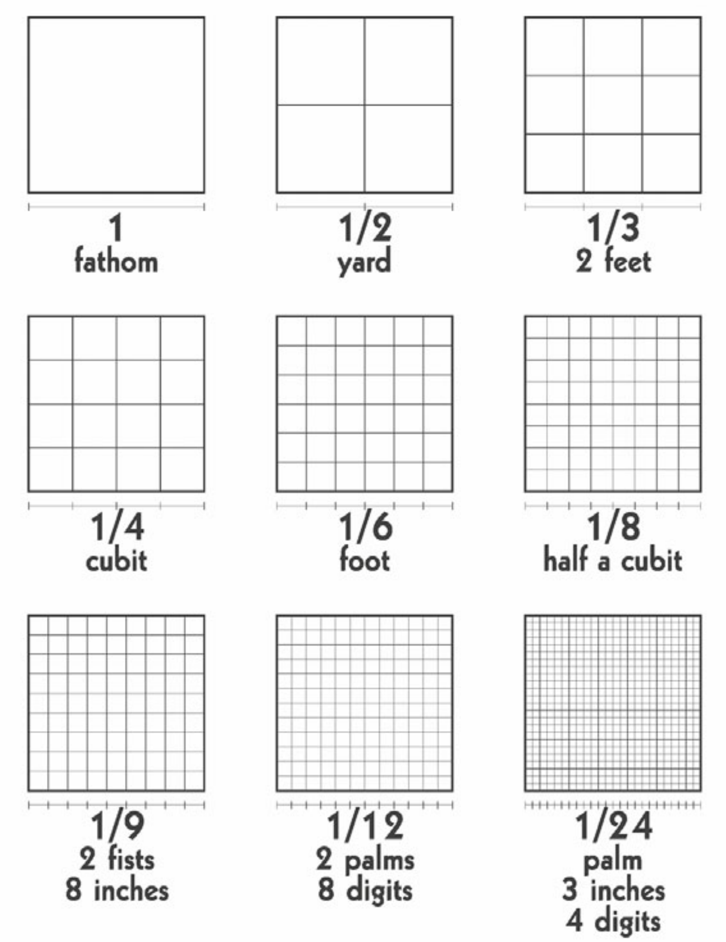


Fig. 1. Duodecimal frames

Unit	*Unidad*	*Fraction*	*SI*	*Roma*	*Imperial*	*Castilla*
fathom	**Braza**	1	200.00	177.42	182.88	167.181
yard	**Vara**	1/2	100.00	88.71	91.44	83.59
cubit	**Codo**	1/4	50.00	44.36	45.72	41.80
foot	**Pie**	1/6	33.33	29.57	30.48	27.86
	Curata	1/8	25.00	22.18	22.86	20.90
	Sesma	1/12	16.67	14.79	15.24	13.93
	Ochava	1/16	12.50	11.09	11.43	10.45
fist	**Puño**	1/18	11.11	9.86	10.16	9.29
palm	**Palmo**	1/24	8.33	7.39	7.62	6.97
inch	**Pulgada**	1/72	2.78	2.46	2.54	2.32
digit	**Dedo**	1/96	2.08	1.85	1.91	1.74

Fig. 2. Table of values

Linking units of length with human body parts – as a mnemonic code – facilitates simple arithmetic operations between fractions. Nevertheless, this base system is not operative when divided by other prime factors (1/5, 1/7, 1/11 ...).

Octagonal graphics base

The double square rotated 45° is one of the graphic resources most used throughout the history of architecture. It is known as a measurement tool called Double Egyptian Remen, whose value corresponded to the $\sqrt{2}$ – square root of two – of a Cubit.

In all medieval and Renaissance treatises[3] the procedure called *ad quadratum* or "in quadrature" is used. This consists in dividing each side of the square in the middle. Connecting these points results in another square that is inscribed and rotated 45°; this new square has an area that is half of that of the first square. If the procedure is repeated with the second square, a third horizontal square is produced, and its sides measure half of those of the first square. Its area is half of that of the second and one fourth of the first, and so on (fig. 3).

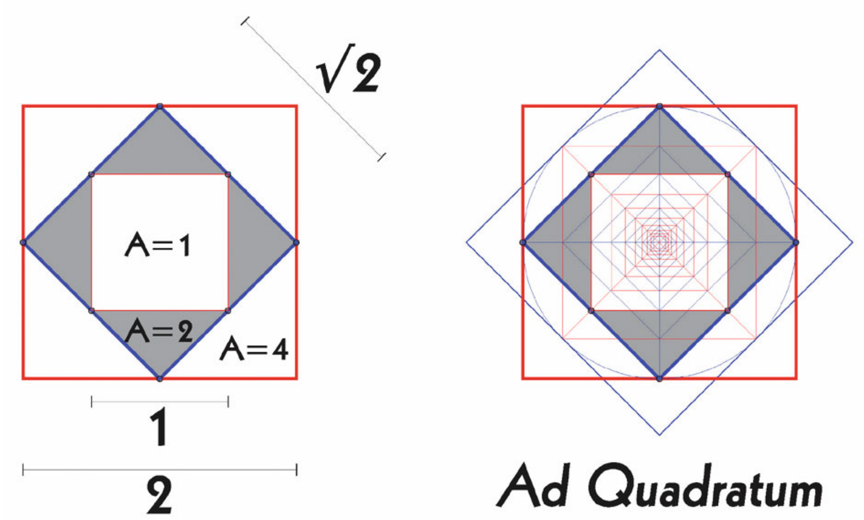


Fig. 3. Ad quadratum

These and other properties, such as those that are used in the DIN-A format, are caused by irrational value $\sqrt{2}$ – which is obtained from the diagonal of the square –, so the application was often used to duplicate and divide areas as well as to make dimensioned templates to lay out different architectural elements.[4]

$$\mathbf{L\ (\sqrt{2}) = (1,\ 1/\sqrt{2},\ 1/2,\ 1/2\sqrt{2},\ 1/4,\ \ldots)}$$

Modulation procedure

This involves using the duodecimal arithmetic base and octagonal graphic base together. The dimensions of the modules are established by one of the following procedures (fig. 4):

a) Corresponding duodecimal system units;

b) $\sqrt{2}$ of these values;

c) Both a) and b).

Option c) can be $1+\sqrt{2} = \theta$ – the silver mean – or other combinations such as $\sqrt{2}-1$, $2+\sqrt{2}$, $1+2\sqrt{2}$…

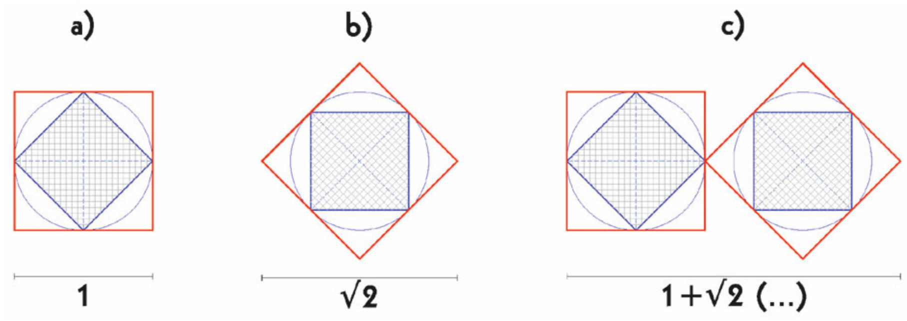


Fig. 4. Procedures

The modulation is static if you use only one of these procedures, and dynamic if you use more than one. Frames originating from orthogonal modulations are static if they use the same procedure on the two axes. These modular frames are dynamic if you use different procedures in each axis. If they are octagonal dynamic frames you can also delimit the two axes rotated 45° by just exchanging the scale of the $1+\sqrt{2}=\theta$ modules. This frame generates the silver rectangles and the Cordovan triangles[5] within the octagon (fig. 5).

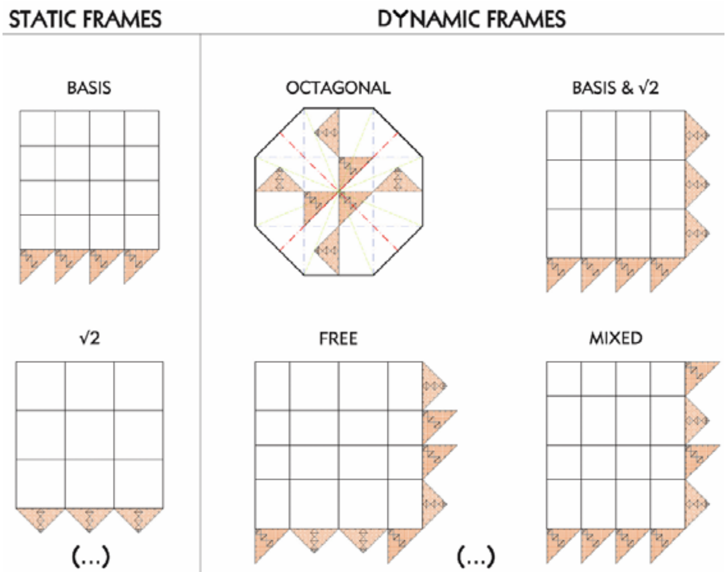


Fig. 5. Typology of frames

Total and partial linear dimensions dynamically modulated are expressed by the positive value of:

$$M = (a \pm b\sqrt{2}) / (2, 3)$$

where a and b are positive integer numbers, and / (2, 3) is a fraction of the duodecimal base. These dimensions cannot be divided into integer values because they present irrational values.

In statistic modulation, a or b is zero.

Adjustment of limits: Residues

Modulations that approximate overall dimensions determined by another, different modular method are allowed. The differences or irrational remainders accumulate symmetrically at the ends, and are called Residues (fig. 6).

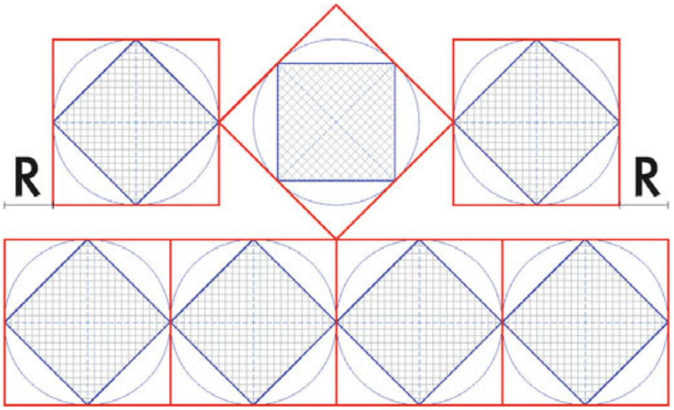


Fig. 6. Residues

Dynamic approaches

Dynamic modulations offer a practical and operational approach – less than 1% error – both fractions of prime factors not present in the duodecimal system (1/5, 1/7, 1/11, etc.) as in other irrational values used in architecture (Phi, Φ: golden number or divine proportion; Pi, π: ratio of length and radius of the circle; Cordovan proportion, c: ratio between the radio and the side of an octagon, or between two unequal sides of a Cordovan triangle).

Fig. 7 shows correspondences of the combination 1+√2 with the approaches to 1/5, Cordovan proportion and Φ, and 2+√2 to 1/7 and π.

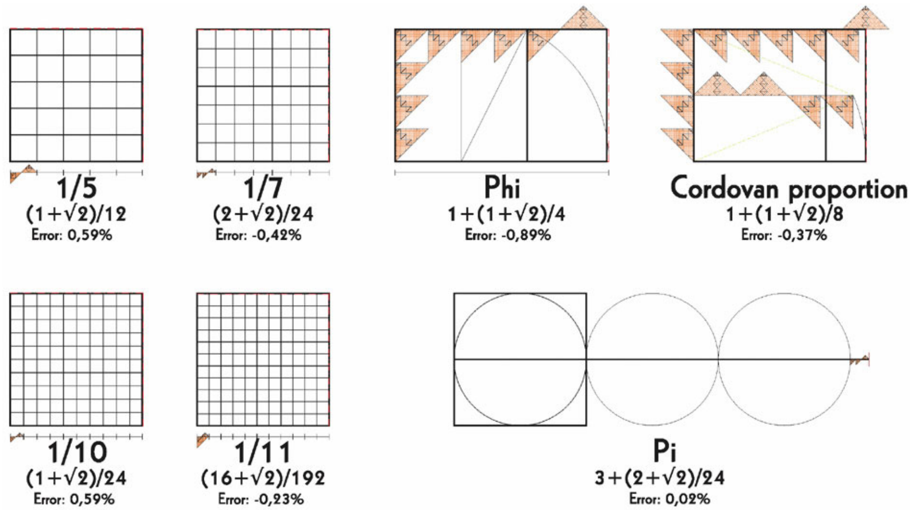


Fig. 7. Approximation

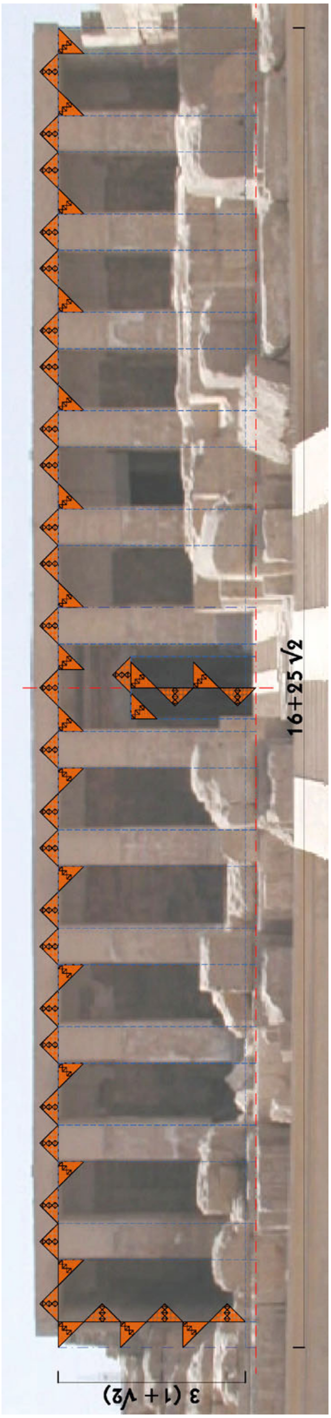
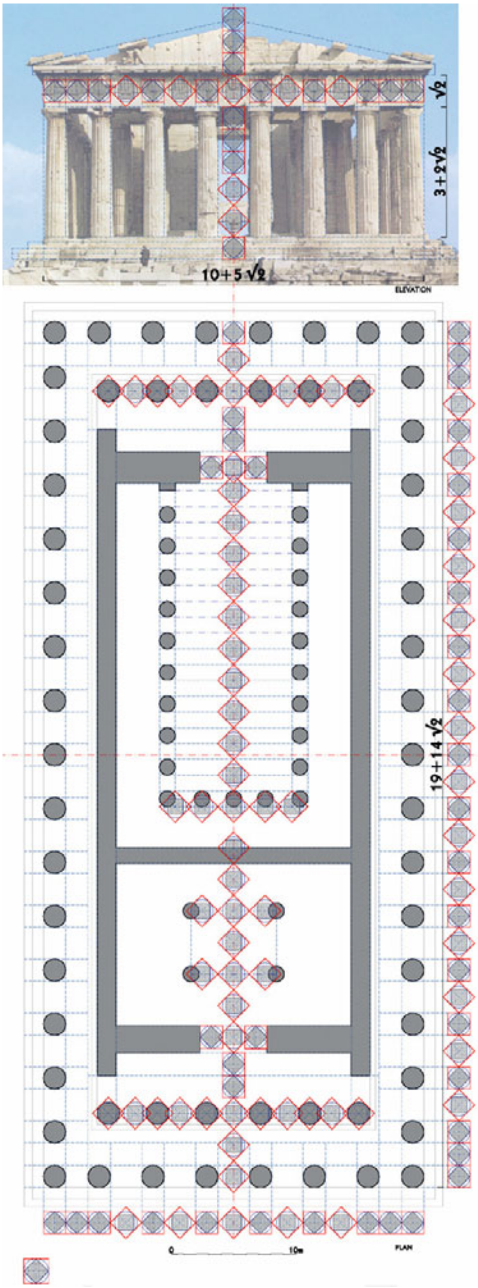


Fig. 8. Temple of Sethy I



UNIT	fathom	yard	cubit	foot	palm	inch	digit
Unidad	BRAZA	VARA	CODO	PIE	PALMO	PULGADA	DEDO
FRACTION	1	1/2	1/4	1/6	1/24	1/72	1/96
CENTIMETERS	178,97	89,49	44,74	29,83	7,46	2,49	1,86

Fig. 9. The Parthenon

Application Examples

Photogrammetric surveys of buildings are preferably used as a metric reference base, over which the modulations are produced by CAD software (Autocad). Use is also made of ortho-projected photographs, and overall documented dimensions. If dimensions are not known we cannot determine the metric values of the modules; however, we can establish the proportions between them.

Temple of Sethy I (twelfth century B.C), Adibos, Egypt (fig. 8).

The pilaster portico design of this temple corresponds to dynamic modulation. The width of each pillar is √2, and is spaced 1+√2, except at the widest central space between columns, which is 2+√2. The extremes of the pilasters or sidewalls have a thickness of 1. Because there are thirteen bays the total width of this facade is 16+25√2.

The height of the pillars from the base is determined by 3(1+√2). In addition, the main door measures 1+√2 in width, by twice its height.

The Parthenon (fifth century B.C), Athens, Greece (fig. 9).

Debated proportions in this temple[6] conform to the dynamic modulations both in floor plans and height. Its anthropometric module measures 178.98 cm, which corresponds to a foot which measures 29.83 cm. Its exterior columns are one module in diameter and 3+2√2 high (1043.17 cm). They are spaced by √2 of the module, except at the far ends where they are only separated by one module. In this way the main facades, with eight columns each, measure 10+5√2, while the seventeen side columns are 19+14√2. Above them rests the entablature of √2 units high.

Through the restored floor plans, one can appreciate the √2 rate arrangement of the interior columns that shape the paths and central spaces.

Arch of Medinaceli (first century A.D.), Soria, Spain (fig. 10).

The regular grid of the static modulation dominates the composition of this Roman civil engineering work, which is dimensioned as follows:

Width: 3 +3 +3 +11 +3 +3 +3 = 29

Height: 8 +9 +3 = 20

Bottom: 4

Furthermore, on both sides of the central arch there are two separate elements formed of two Corinthian pilasters topped by a triangular pediment. The total width of each element is 3√2 and the interior space is set to 2√2.

According to the general dimensions published,[7] these modules correspond to cubits value 45.29 cm (181.16 cm anthropometric module).

Xcalumkin (eighth century A.D.), Campache, Mexico (fig. 11).

Dynamic modulation is detected in various elements through the analysis of a photograph of the southern building of the Initial Series of this Mayan city. If the diameter of each of the columns is 1 module, then their total height equals 3, their axes are spaced in 2+√2, the width of the abacus measures √2 and its height is one quarter of √2.

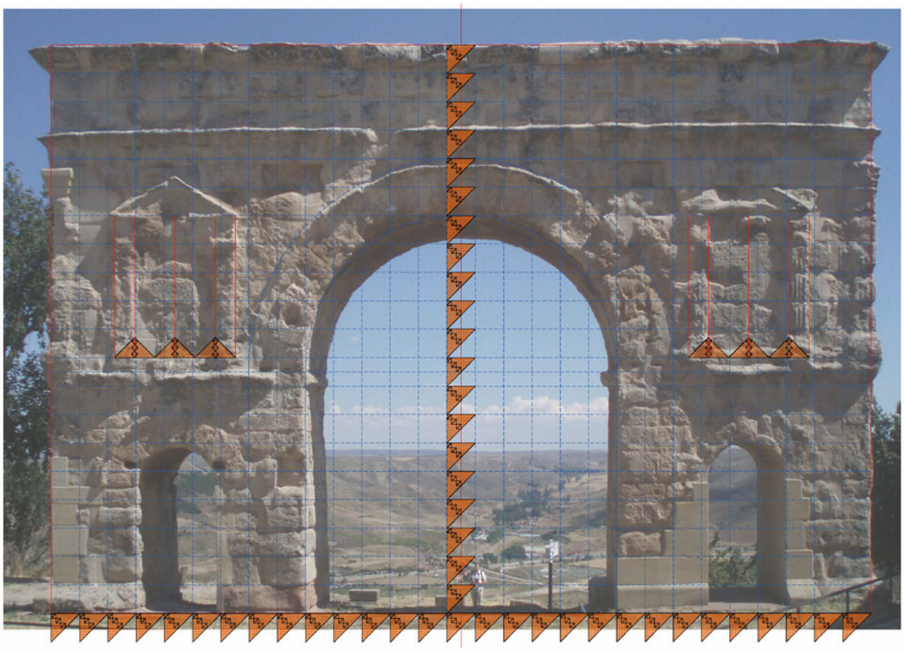


Fig. 10. Arch of Medinaceli

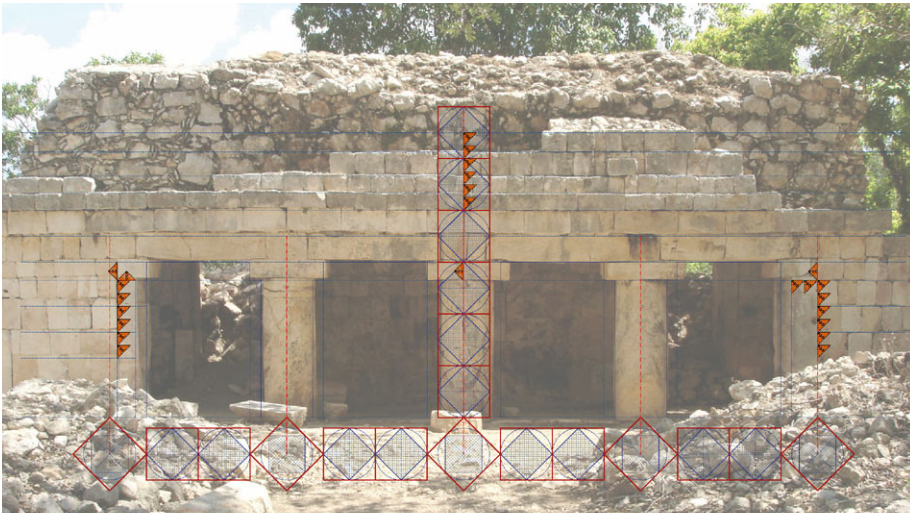
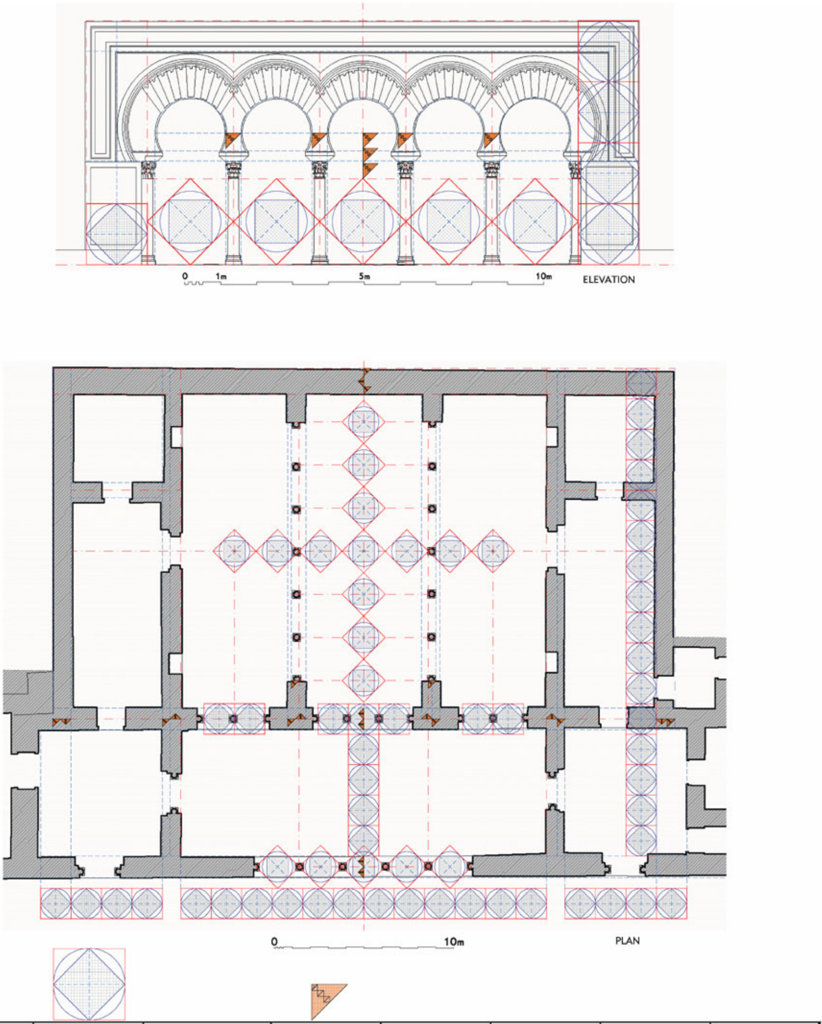


Fig. 11. Xcalumkin

Madinat Al-Zahra (tenth century A.D.), Cordoba, Spain (fig. 12).

The facade of the Salón Rico[8] of Caliph Abd al-Rahman III in this palatial city is composed of an arcade of five arches separated by the √2 anthropometric module, corresponding to a 42.48 cm cubit. It is framed by a decorative module, the total width being 2 +5√2 and the height 4.

On the floor the interior spaces are sized in straight modules, except through the arcade of the Salón Rico, which matches the scale √2. The thickness of the longitudinal walls is dimensioned dynamically (1+√2). The two transverse walls have a width of 2√2 and the measurement of the bottom is 2+√2.

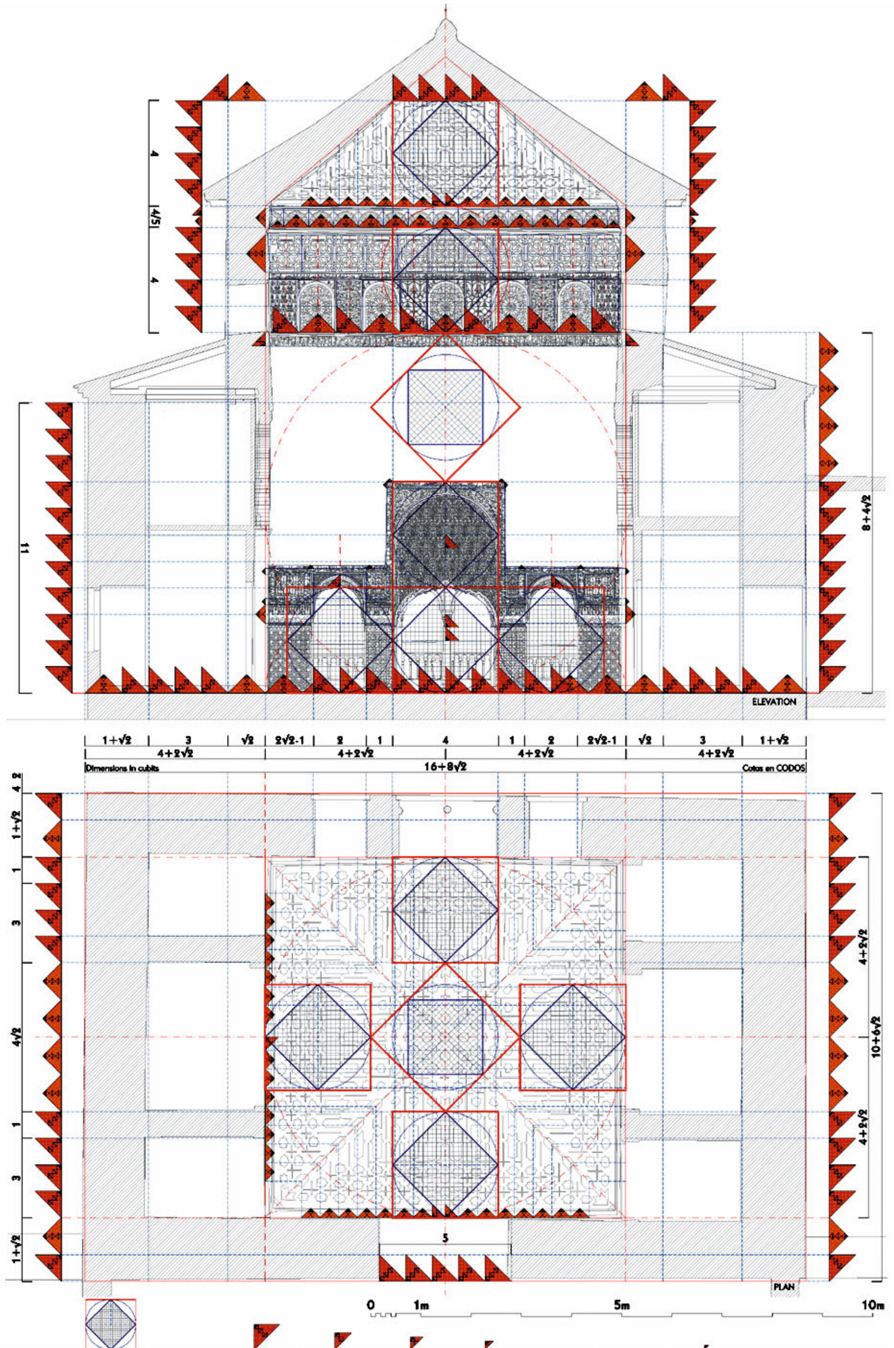


UNIT	fathom	yard	cubit	foot	palm	inch	digit
Unidad	*BRAZA*	*VARA*	*CODO*	*PIE*	*PALMO*	*PULGADA*	*DEDO*
FRACTION	1	1/2	1/4	1/6	1/24	1/72	1/96
CENTIMETERS	169,92	84,96	42,48	28,32	7,08	2,36	1,77

Fig. 12. Salón Rico in Madinat Al-Zahra

Cuarto Real De Santo Domingo (thirteenth century A.D.), Granada, Spain (fig. 13).

This is an emblematic Nasrid palace earlier than and very similar to those of the Alhambra. The results of the study of their modulation[9] – of which only the general dimensions are presented here – have enabled the author to establish the characteristics of the metrology of classical architecture described in the present work.



UNIT	fathom	yard	cubit	foot				fist	palm	inch	digit
Unidad	BRAZA	VARA	CODO	PIE	CUARTA	SESMA	OCHAVA	PUÑO	PALMO	PULGADA	DEDO
FRACTION	1	1/2	1/4	1/6	1/8	1/12	1/16	1/18	1/24	1/72	1/96
CENTIMETERS	210,37	105,19	52,59	35,06	26,30	17,53	13,15	11,69	8,77	2,92	2,19

Fig. 13. Cuarto Real de Santo Domingo in Granada

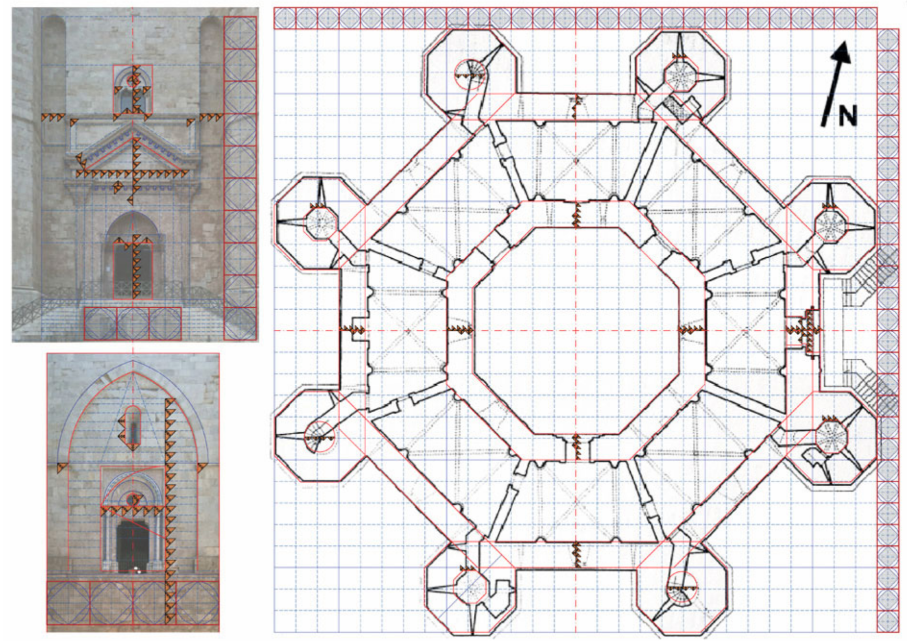


Fig. 14. Castel del Monte on Andria, Apulia

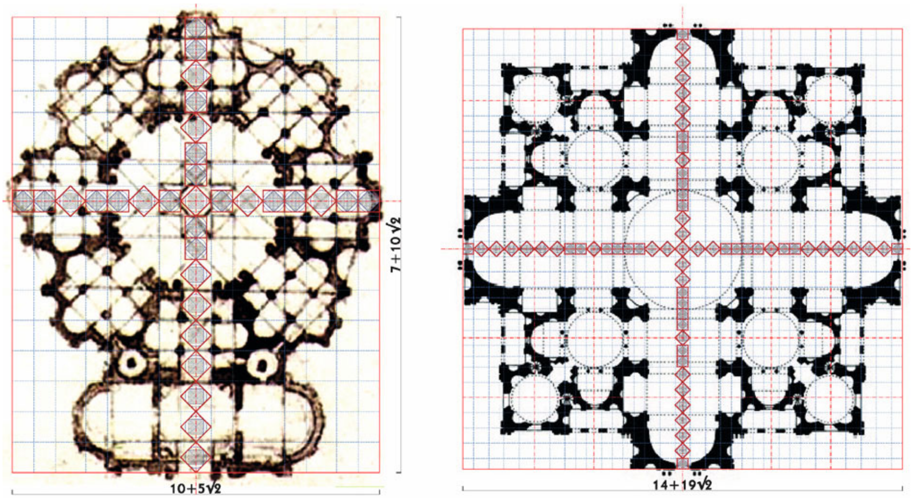


Fig. 15. The Study of a Central Church by Leonardo da Vinci and Bramante's floor plans for St. Peter's in Rome

Castel Del Monte on Andria (thirteenth century A.D.), Apulia, Italy (fig. 14)

The overall dimensions of this building, constructed by the Emperor Frederick II, are justified by a static modulation. The inner octagon has a width of 12 modules and the outside has 22. The thickness of the walls is such that the side of the octagon of the patio closely approximates 4, and determines the width of each of the eight octagonal towers totaling 28 as the total width of the building. The thickness of the wall can be justified statically by 29/24, or dynamically by $(1+\sqrt{2})/2$. Other dynamic modulations are detected in the widths of the holes of the main facade, the skylight over the balcony of the courtyard, and other elements. The framed part of this balcony is an exact golden rectangle.

The study of a centrally planned church by Leonardo da Vinci (ca. 1488 A.D.) and Bramante's floor plans for St. Peter's in Rome (1505-1506 A.D.) (fig. 15)

Both central compositions are substantiated by dynamic modular frames. The scale $\sqrt{2}$ can be appreciated in determining the axes and routes of each sacred space.

Conclusions

The deduced metric system coincides with the units derived from anthropometric duodecimal canon, but includes one other scale in proportion to $\sqrt{2}$. It can be used separately or together. Since the number 2 is contained in the $\sqrt{2}$ series, we can express it like this:

$$\mathbf{L(\sqrt{2},3) = (1, 1/\sqrt{2}, 1/2, \ldots)\ (1, 1/3, 1/9, \ldots)}$$

In this way, it is compatible with the static scale in regular grid, and in addition, with all the dynamic *ad quadratum* designs generated from octagonal symmetry, which are traced with total metric control on the four axes of symmetry. One example of this is the elaborate works of Hispano-Muslim decoration proper to the "Mudéjar style".

Using a modulation method and a very simple and practical layout, this method also provides another possible variety of combinations, good approximations to other architectural constants, and facilitates prefabrication in the workshop.

This explains the principle of proportionality between the parts, in works from antiquity.

There is more use of the $\sqrt{2}$ scale in palaces of royalty, and especially in temples and sacred motives.

The double scale system described here represents an effective tool for documenting and analyzing the architectural heritage, and its application seems advisable for in all historical studies.

Notes

1. Throughout history, people have used other mathematical, numerical systems: the binary numeral system, which is in base 2; the ternary system, which is in base 3; the quinary system, which is in base 5; the octal system, which is in base 8; the decimal system, the most common system in use today, which is in base 10; the hexadecimal system, which is in base 16; the vigesimal system, which is in base 20. However, in arithmetical terms, the duodecimal system and the sexagesimal system admits a greater number of fractions, while in geometrical terms, they are compatible with square and triangular-hexagonal frames.
2. The observed variation in the size of the different units was a great disadvantage of the classical system of proportions, but it was not the only one. For example, Filarete [1990: 52-53], in his treatise on architecture, describes a system of different modules defined by Vitruvius [2007]. He highlights the language problem that occurred when using similar names for units, which corresponded to different fractions of the anthropometric module. To this, we must add that even in the same area, one unit would have different size depending on the type of merchandise to be measured. The variety of measures and units has always favoured a host of local names and misunderstandings in the original texts, translations and interpretations. According to Vitruvius the parts of a temple should be in proportion to each other, as they are in a well-formed man. The modules could be many sizes and denominations, however in the

construction of a building there should be metric unity in order to carry this out. Each work can have its own and unique module, as each man has a different height.

3. Among the most ancient treatises we can cite those of Matthäus Roritzer, Hans Schmuttermayer and Lorenz Lechler, by using the procedure thoroughly. The Renaissance treatises prefer to use measurable fractions, but continue to use it for certain paths [Ruiz de la Rosa 1987].
4. The ratio between different parts of buildings according to the $\sqrt{2}$ has been studied by Violet-le-Duc, Jay Hambidge, Tons Brunes [1967], Carol M. Watts [1987], Rafael Vila [1997] and Jay Kappraff [2002], among others.
5. The Cordovan triangles are present in the octagonal dynamic frames, as Tomás Gil López [2012] verifies. The Cordovan rectangle or Cordovan proportion is proposed by Rafael de la Hoz [1973] as ratio of several rectangular architectural bodies of the Great Mosque and other buildings in Córdoba, Spain.
6. Luis Moya, in his *Relación de diversas hipótesis sobre las proporciones del Partenón* [1981], examines the proposals of Barnacles Nicolas, Jay Hambidge, Viollet-le-Duc, Georges Tubeuf, Lesueur, Charles Chipiez, C. J. Moe, Maruis Clayet-Michaud, Henri Trezzini, D. R. Hay, August Thiersch, Alexander Speltz, Zeysing, Mossel, Matila C. Ghyka, Ernst Neufert, Wedelphol, Hans Plessner, Funck-Heller, Otto Hertwing, Odilo Wolff, Karl F. Wieninger, Victor D'Ors, Uhde and other theorists. Moya begins his conclusions as follows: "A close look at the diverse hypothesis with respect to the proportions of the Parthenon has demonstrated that no system can explain, simultaneously, the two aspects of the problem: First of all, the real dimensions of the measurements at present, and secondly, how they were obtained in the construction process" (my translation). Other studies have appeared, usually emphasizing Matila C. Ghyka's theory of the Divine Proportion, spearheaded by Le Corbusier, as a counterpoint to highlight those proposed by Tons Brunes [1967] and Anne Bulckens [2002].
7. Overall dimensions published by Lorenzo Abad Casal [2002].
8. Photogrammetry performed by Antonio Almagro [2011].
9. Francisco Roldán [2011] with photogrammetry performed by Antonio Almagro and Antonio Orihuela [1997].

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