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Research

Interlocking Star Polygons in Persian Architecture: The Special Case of the Decagram in Mosaic Designs

Abstract. This article analyzes a particular series of Persian mosaic designs illustrated in historical scrolls and appearing on the surfaces of historical monuments. The common element in these designs is a special ten-pointed star polygon, called a decagram for convenience; it is the dominant geometric shape in several polyhedral tessellations. This decagram can be created through the rotation of two concentric congruent regular pentagons with a radial distance of 36° from each others' central angles. To create a decagram-based interlocking pattern, however, a craftsman-mathematician would need to take careful steps to locate a fundamental region. A modular approach to pattern-making seems to be more applicable for this design than compass-straightedge constructions. Such designs include patterns that are sometimes called aperiodic or quasi-periodic tilings in the language of modern mathematics.

1 Introduction

From a few documents left from the past, it is evident that the designers of patterns on the surfaces of medieval structures in Persia and surrounding regions, were well-equipped with a significant level of knowledge of applied geometry. Nevertheless, they never exhibited the same level of effort or interest in providing the pure side of the subject by proving theorems and establishing mathematical facts about such designs. The main concern of a designer or craftsman was to present a visual harmony and balance, not only in deep details, but also as a whole. Nevertheless, the steps taken for creating such designs include techniques that are only acquired and understood by mathematicians. An individual with the knowledge of such detailed techniques is a mathematician, artist or not.

The infrastructures of a large number of patterns are the three regular tessellations of equilateral triangles, squares, and regular hexagons. Square shape tiles are the most popular ones because they are produced more conveniently than the others. Between the other two shapes, since each six equilateral triangles can be presented as a single hexagon, the economy for molding, casting, painting and glazing tiles would have made craftsmen more inclined to create tiles of hexagonal shapes than triangular ones. The following are two examples of patterns that are created on the basis of regular tessellations. Even though the two patterns may appear to be the same to an untrained eye, as both include the same repeating twelve-pointed star polygons, they are very different in construction. Fig. 1a shows a pattern made from a square tile; Figure 1b shows a pattern made from a regular hexagon. Fig. 2 demonstrates the geometric constructions for each tiling illustrated in Fig. 1.

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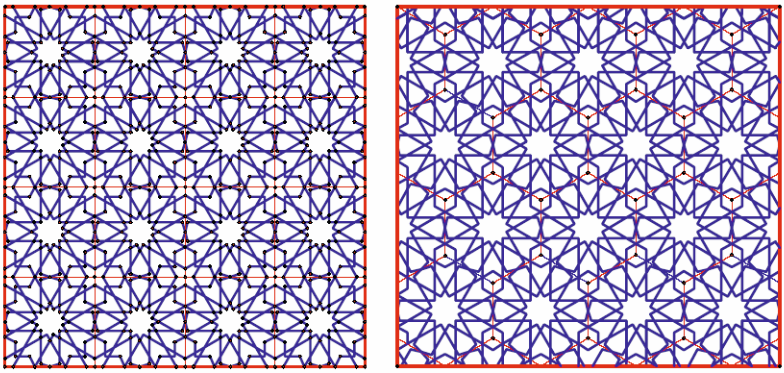

Fig. 1. a, left) Square-based twelve-pointed star tiling; b, right) Hexagonal-based twelve-pointed star tiling

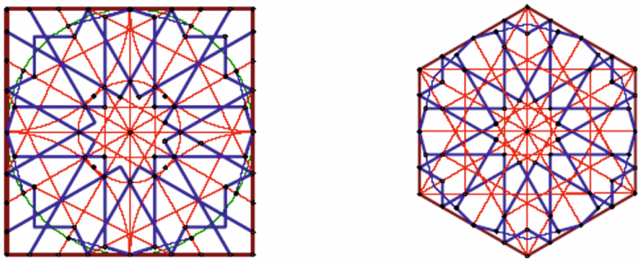

Fig. 2. a, left) Square-based twelve-pointed star tiling; b, right) Hexagonal-based twelve-pointed star tiling

A recent book on this subject illustrates step-by-step compass-straightedge constructions of tile patterns and is dedicated specifically to patterns derived from these two shapes, square and hexagon [Broug 2008].

After studying designs on existing structures of medieval Persia and similar buildings in neighboring regions, one may notice that some patterns cannot be based on either squares or regular hexagons; their underlying structure is based on a different set of angles. Many, if not all, of these patterns are based upon a ten-pointed star that results from the crossing of two regular pentagons.

2 Tiling with the decagram: a special case of the ten-pointed star polygon

Fig. 3 exhibits a method for creating a tiling design that is not based on previously mentioned regular tessellations [El-Said and Parman 1976]. In this method, the starting point is a (10, 3) star polygon – a figure created from connecting every third vertex in a set of ten equally spaced points on a circle in one direction in one stroke (the star inside the circle on the upper left corner of fig. 3). By extending some of the segments that constitute the (10, 3) star polygon and intersecting them with the lines perpendicular to some other segments, one achieves the construction of the rectangular frame and other necessary line segments inside the frame, as is illustrated in fig. 3. This rectangular tile tessellates the plane and creates a pleasing series of stars that are ordered in columns and in rows (fig. 4).

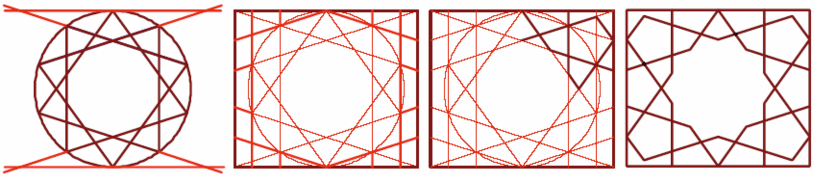

Fig. 3. Compass-straightedge construction of a decagram tile

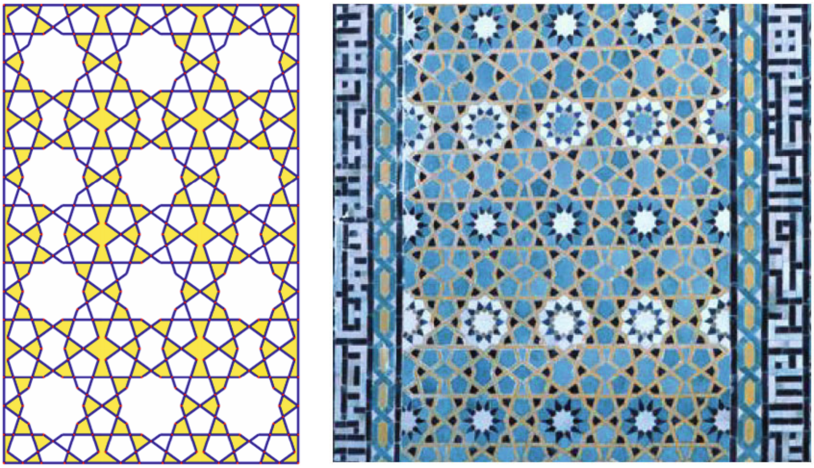

Fig. 4. a, left) A tessellation of the tile in fig. 3; b, right) Example of this tessellation on wall at Masjid-i-Jami, Kerman in Iran, executed in ceramic tile

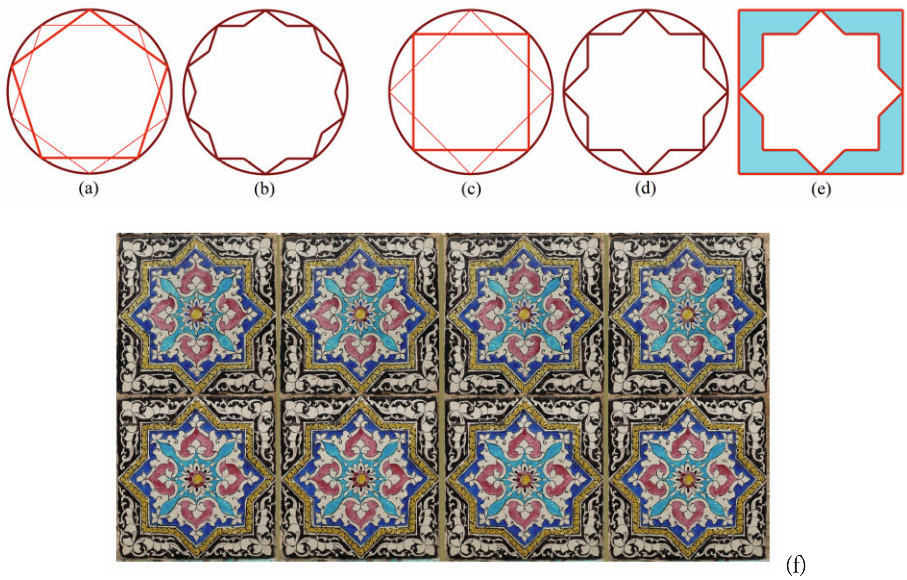

Fig. 5. a) Two concentric pentagons arranged at a 36° rotation from one another; b) the decagram that is thus generated; c) two concentric squares arranged at a 45° rotation from one another; d) the octagram that is thus generated; e) a square tile for the octagram-and-cross tessellation; f) an existing octagram-and-cross tessellation from a wall of Ark Karim Khan Zand in Shiraz, Iran

The special ten-pointed star polygon (called here a decagram for convenience), which is the visually dominant geometric shape of the tessellation presented in fig. 4, can be created independently through the rotation of two concentric congruent regular pentagons with a radial distance of 36° from each others' central angles (fig. 5a-b). Such a rotation with a radial distance of 45° for two squares creates the attractive eight-pointed star polygon (called an octagram) that appears in many geometric designs and tilings as one of the motifs in cross-octagram tessellations (fig. 5c-e). We may assume that creating a tile design using a decagram was a challenge for designers, compared to a less complicated, straightforward cross-octagram tessellation, defined in a square frame (fig. 5f).

Studying documents and monuments from the past reveals that in most cases the rectangular fundamental regions, similar to what is shown in the final image in fig. 3, were shapes recorded in scrolls (*tumār*) and booklets (*daftar*) as designs used in the execution of actual tilings on the surfaces of buildings, or as geometric experimentations using interlocking star-polygon patterns. In Persian architecture, such a fundamental region was called a knot: *aghd*, in Arabic, as is documented in *Interlocks of Similar or Complementary Figures* [Anonymous], or *girih*, in Persian, the term that will be used here.

The following resources from the past provide us with information about different girih patterns and their geometries, which were executed in ceramic mosaics and tiles.

3 Surviving historical documents

There are only a few documents that survive from the past, which allow us today to study the configuration of layouts of many interesting and intriguing ornamental designs so that we may understand their mathematics:

A. A treatise written by Būzjānī in the tenth century, *Kitāb fīmā yahtā ju ilayhi al-sāni' min a'māl al-handasa* (*A book on those geometric constructions which are necessary for a craftsman*): Abū'l-Wafā Būzjānī "was born in Būzjān, near Nishābur, a city in Khorāsān, Iran, in 940 A.D. He learned mathematics from his uncles and later on moved to Baghdad when he was in his twenties. He flourished there as a great mathematician and astronomer. He was given the title Mohandes by the mathematicians, scientists, and artisans of his time, which meant the most skillful and knowledgeable professional geometer. He died in 997/998 A.D." [Sarhangi, Jablan and Sazdanovic 2004: 282]. The treatise by Būzjānī does not include any ornamental designs. However, it presents a number of compass-straightedge constructions used in the creation of such designs [Özdural 2000].

B. *Interlocks of Similar or Complementary Figures* [Anonymous]. The treatise by Būzjānī was originally written in Arabic, the academic language of the Islamic world of the time, and was translated into Persian in several periods. In one of the translations, the aforementioned *Interlocks* treatise in its Persian original, was appended. Some researchers believe that the author of this document is an anonymous thirteenth- century mathematician/craftsman [Özdural 2000]. However, in a recent Persian book that includes a modern translations of both Būzjānī's treatise and this document, the author has been identified as the 15th century Persian mathematician Abūl-Es-hāgh Koobnāni [Jazbi 1997]. It is important to realize that *Interlocks* is the only known practical manual that provides "how-to" instructions for drawing two-dimensional girih patterns (similar to the right image in fig. 3). According to G. Necipoğlu, "The treatise shows that the girih mode was conceived as a system of proportionally related geometric patterns harmoniously interlocking with one another" [1995: 133-135].

C. The *Topkapi Scroll*, a scroll close to thirty meters long and about a third of a meter high, is a document attributed to Persian artisans and architects, during the reign of Timurid dynasty in the fourteenth or fifteenth century. This scroll is preserved in the collection of the Topkapi Palace Museum in Turkey. The scroll provides a wealth of knowledge for creating designs for wall surfaces and vaults in architecture and ornaments of its time, documenting efforts by architects and craftsmen in the late medieval Persian world. Belonging to a period of tradition of scrolls in which geometric designs and patterns used for various architectural purposes were recorded next to each other without any distinction and explanation of the creation process, it consists of 114 patterns ready for use by architects to create the mosaic patterns in many structures. A book, *The Topkapi Scroll*, documenting this scroll was published in 1995 with general discussion and extensive commentary about each design [Necipoğlu 1995].

D. The *Tashkent Scrolls* (sixteenth-seventeenth century documents that present the works of master masons of the post-Timurid period) were first kept in the Bukhara Museum after their discovery and later transferred to the Institute of Oriental Studies in Tashkent, Uzbekistan. "At the time of their discovery the mode of geometric design codified in these fragmentary scrolls was identified as the girih by traditional Central Asian master builders who still used such scrolls. This term refers to the nodal points or vertices of the web-like geometric grid systems or construction lines used in generating variegated patterns…" [Necipoğlu 1995: 9]

E. The Mirza Akbar Collection held at the Victoria & Albert Museum, London. This collection consists of two architectural scrolls along with more than fifty designs that are mounted on cardboard. The collection was originally purchased in Tehran, Iran, in 1876 for the South Kensington Museum (the precursor of the Victoria & Albert Museum) by Sir Caspar Purdon Clarke, who was Director of the Art Museum (Division of the Victoria and Albert Museum) from 1896 to 1905. Purdon Clarke purchased them after the death of Mirza Akbar who had been the Persian state architect of his period. Although not as old as the *Topkapi Scroll*, they do show a continuation of the drafting scroll tradition that continued into the modern era. [Necipoğlu 1995]

To have a significant grasp of the ideas behind a large number of designs in the above scrolls, to see some of them on the walls of existing monumental buildings of the past, and deeply understand the process of creating such designs in a traditional approach, we need to refer to a useful five-volume book in Persian, *Construction and Execution of Design in Persian Mosaics* written and illustrated by M. Maheroannagsh [1984]:

> Perhaps the most comprehensive and elegant recent book about Persian mosaics that includes geometric constructions of designs presented along with colorful images of their executions performed on different mediums on the wall, floor, interior and exterior of domes, doors and windows, and many more, is Construction and Execution of Design in Persian Mosaics. The author, who was a professional artisan, inherited his profession from his ancestors of several centuries, had the most access to original artisans' repertories of the past. The

ornamental qualities of these geometric constructions and their executions provide a joyful journey to the past for readers [Sarhangi, Jablan and Sazdanovic 2004: 282].

4 An interlocking decagram-polygon mosaic design

The existing tessellation on the wall of a Persian structure shown in fig. 6a includes a decagram motif as in the tiling shown in fig. 4. There are tiles of other shapes that constitute the tiling. In fact there are exactly five motifs (modules). fig. 6b presents these modules, which in Iran are called *muarraq*, an Arabic word. The Arabian-Andalusian word for these hand-cut pieces of glazed ceramic tiles is *zellij*. In this article they are called sāzeh (operative in Persian) module tiles.

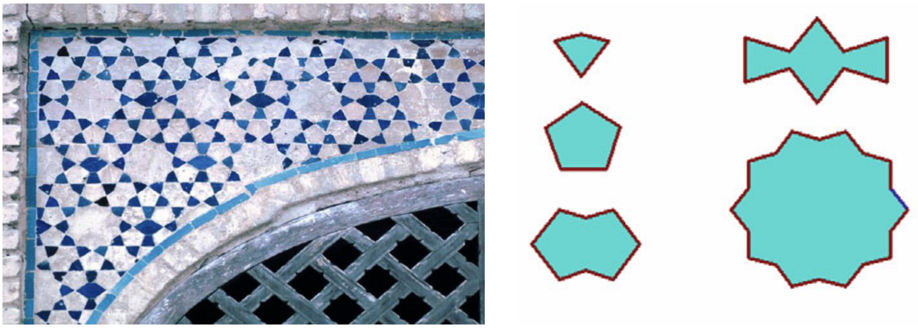

Fig. 6. a, left) Imamzeda Darb-i Islam, Isfahan in Iran; b, right) The five sāzeh module tiles.

These modules have their own specific Persian names: *Torange* (the quadrilateral tile), *Pange* (the pentagonal tile), *Shesh Band* (the concave octagonal tile), *Sormeh Dān* (the bow tie tile), and *Tabl* (decagram tile).

Comparing the two tessellations in figs. 4 and 6, one can see that, despite the fact that the individual sāzeh modules used in both tessellations are identical, they are very different tessellations.

The following solution for the geometric construction of the tiling in fig. 6, with a few minor revisions in the process, comes from the mosaic designer M. Maheroannaqsh [1984]. Please note that the division of a right angle into five congruent angles is not a part of the instructions provided, because it is considered an elementary step for designers. Interested readers who wish to learn about this division may read the construction technique for a regular decagon in [Sarhangi 1999] or search online.

Divide the right angle ∠A into five congruent angles by creating four rays that emanate from A. Choose an arbitrary point C on the second ray, counter-clockwise, and drop perpendiculars from C to the sides of angle ∠A. This results in the rectangle ABCD along with four segments inside this rectangle, each having one endpoint at A and whose other endpoints are the intersections of the four rays with the two sides of CB and CD of rectangle ABCD. Find E, the midpoint of the fourth segment created from the fourth ray. Construct an arc with center A and radius AE to meet AB on F and the second ray on G (the second segment is now part of the diagonal of the rectangle). Make a line, parallel to AD, passing through G, that intersects the first ray at H and the third ray at I. Line FH passes through point E and meets the third ray at L and line AD at J. Construct a line, passing through J, that parallels the third ray. Also construct line EI and find M, the intersection of this line with AD. From F make a parallel line to the third ray to meet the first ray at K. Construct segments GK, GL, and EM. Find N such that GI = IN by constructing a circle with center I and radius IG. Construct the line DN (which happens to be parallel to GK), to intersect the line emanate from J, to find P to complete the regular pentagon EINPJ. Line DN meets the perpendicular bisector of AB at Q. From Q construct a line parallel to FK to intersect ray MI at R and then complete the figure (fig. 7a). Using O, the center of the rectangle ABCD, as a center of rotation for 180°, one can make the fundamental region for the tiling in fig. 6 (fig. 7b). Figs. 7c, d show the motif and its tessellation for this girih.

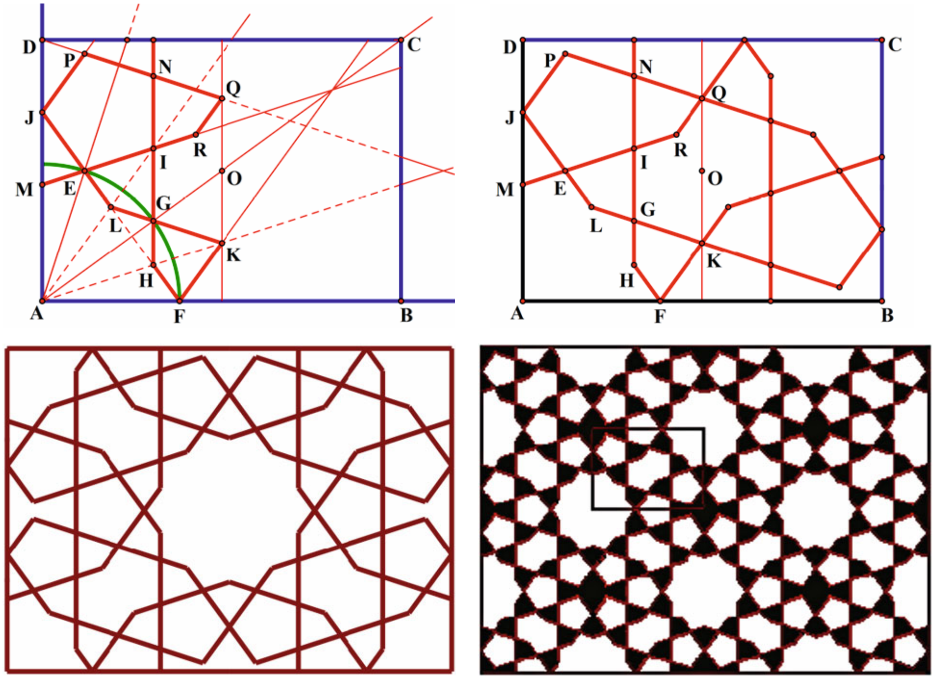

Fig. 7. Compass-straightedge construction of the mosaic design in fig. 6

5 Three compass-straightedge related interlocking pattern construction methods

5.1. The radial grid method

The construction method in fig. 7, which uses a radial grid approach, as a method used in medieval times, is supported by some images along with their construction details and instructions, recorded in the *Interlocks of Similar or Complementary Figures* [Anonymous].

Fig. 8, created by the author, shows another example for radial grid method. It shows a step-by-step construction of a girih for a ten-pointed star design and its tessellation. To follow the construction comfortably the points are labeled in alphabetic order according to their appearance in the construction. The first step, as in the previous construction in section 4, is the division of a right angle into five and then the selection of an arbitrary point C on a ray (now on the third ray). The first few points are found as intersections of the rays emanating from the right angle ∠A, the rays emanating from the opposite right angle ∠C, and the diagonal BD.

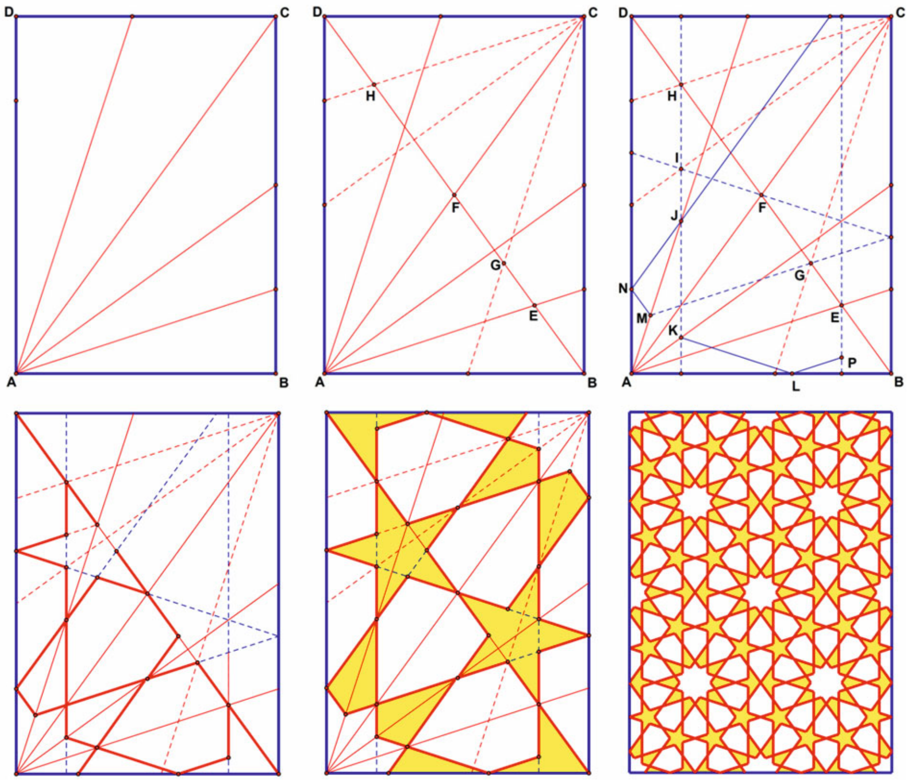

Fig. 8. Step-by-step construction of a girih and its tessellation based on radial grid approach performed by the author

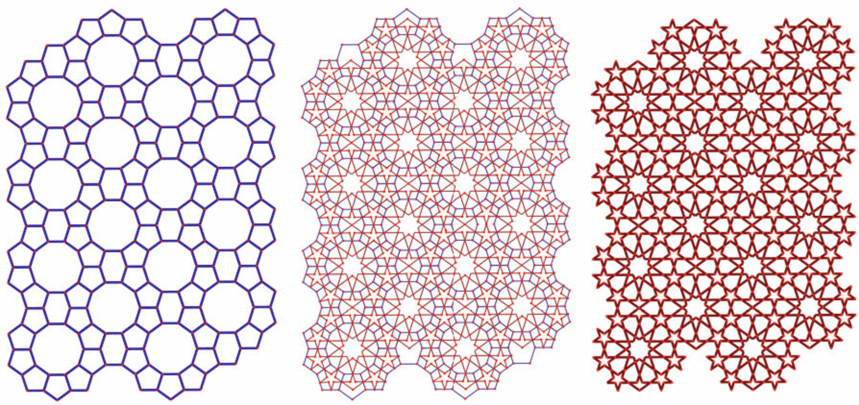

Fig. 9. a, left) A pattern created from laying three types of polygonal blocks, pentagons, hexagons, and decagons, edge-to-edge; b, center) using the midpoints of the sides of the polygons to create segments and properly decorate the tiles; c, right) discarding the block lines to exhibit the final pattern

5.2. The polygons in contact method

E. H. Hankin [1925] introduced another technique called "polygons in contact" (PIC), which is explained in recent articles [Bonner 2003; Kaplan 2005; Cromwell 2009; Bodner 2009]. This is another system for which there is evidence of historical use by designers [Bonner 2003]. Fig. 9 from left to right, exhibits this technique starting from the underlying polygonal network and ending in the final pattern, which is the same pattern in shown in fig. 8.

5.3. A method based on the (n, k) star polygons

Following [El-Said and Parman 1976] one may construct the same pattern using the technique introduced in fig. 3, which is based on the use of (10, 3) star polygon and extensions of some of its sides (fig. 10).

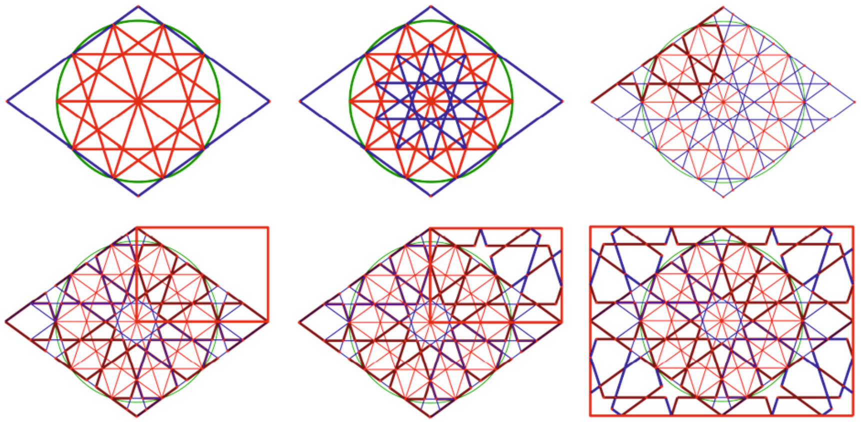

Fig. 10. The construction of girih for the tessellation in figs. 8 and 9 using the extensions of the lines that constitute the (10, 3) star polygon

6 A tessellation from Mirza Akbar architectural scrolls and its construction

The two different tessellations in figs. 4 and 6, each made from the same set of sāzeh modules, raise the question: Are there other tessellations that can be made from the same decagram and its interlocking polygons?

The image in fig. 11 is an exact rendering of a design illustrated in the Mirza Akbar collection [also see Bovill, 2012]. In this tessellation, the decagrams are farther apart from each other. Using the steps involved in the construction of the design in section 4, we can find a traditional radial solution for the tessellation in fig. 11.

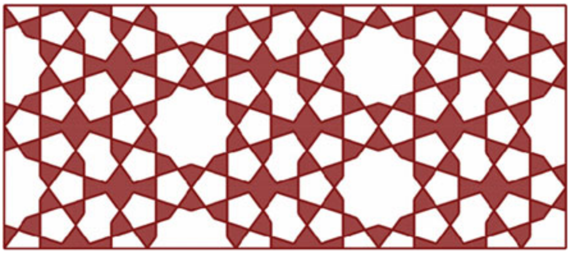

Fig. 11. A tessellation from the Mirza Akbar collection

It is not difficult to discover that the fundamental rectangle for this tilling has a longer length compared to the rectangle in fig. 7. So, starting with the radials that divide the right angle into five congruent angles, the arbitrary point P was selected on the first ray counterclockwise (rather the second ray in the previous problem). For the radius of the circle inscribed in the decagram one half of the segment created from the third ray, segment AM, was selected (unlike the fourth ray in the previous construction). Then a similar approach was taken to create the tiling in fig. 12. The following figure illustrates a step-by-step compass-straightedge visual solution to the problem by the author. Fig. 13 illustrates another approach, which is presented by El-Said and Parman [1976], for creating the same tiling as in fig. 12. The starting point is again a (10, 3) star polygon but with new extensions as illustrated in this figure.

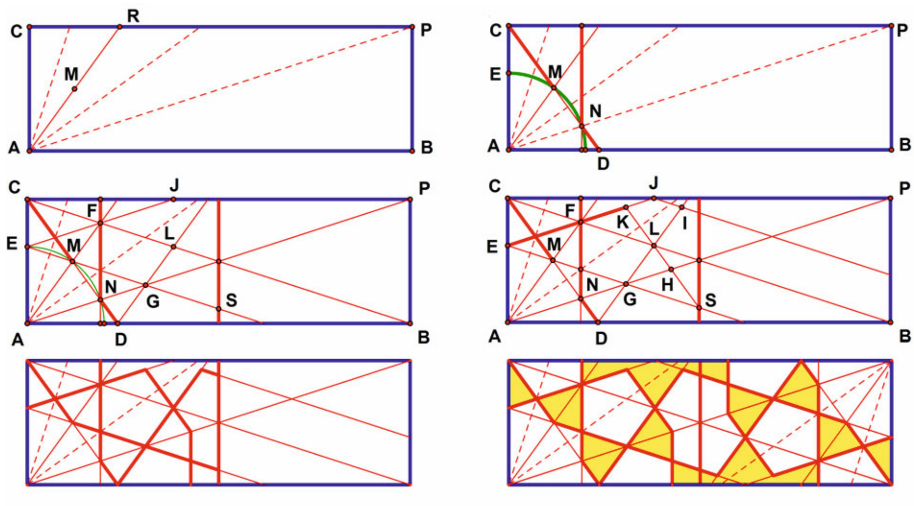

Fig. 12. Step-by-step compass-straightedge construction of the tessellation in fig. 11 as a visual solution to the problem solved by the author

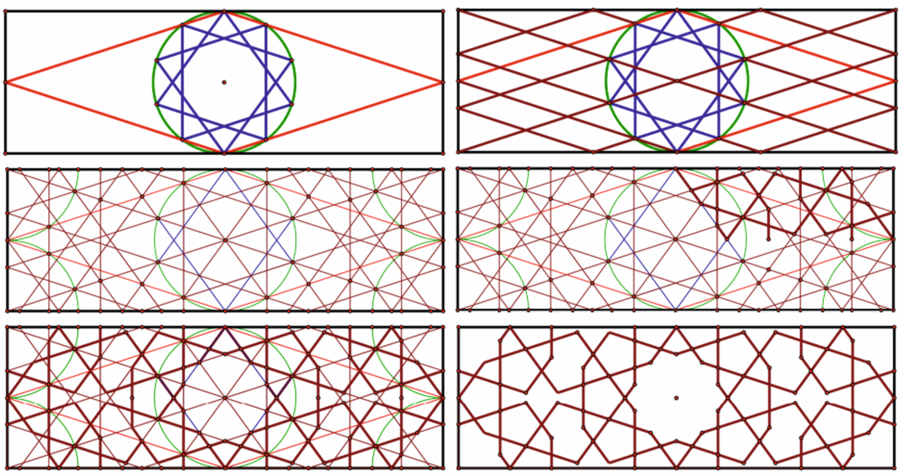

Fig. 13. The construction of girih for the tessellation in fig. 11 using the extensions of the lines that constitute the (10, 3) star polygon

7 Square girih for constructing an interlocking tessellation

Selecting an arbitrary point on any of the rays that divide the right angle into five congruent angles and dropping perpendiculars to the sides of the right angle from that point results in only two types of rectangles of different proportions:

- Selecting the arbitrary point C on the first ray and dropping the two perpendiculars BC and CD to the sides of right angle ∠A results in the rectangle ABCD (fig. 14a), where the relationship between its diagonal AC and side BC is AC/BC = $2\phi = 1+\sqrt{5}$, where ϕ is the Golden Ratio. Therefore, AB/BC = $\sqrt{5+2\sqrt{5}}$.

- Selecting the arbitrary point F on the second ray will result in the rectangle AEFG (fig. 14a), which is the Golden Rectangle: AE/EF = ϕ.

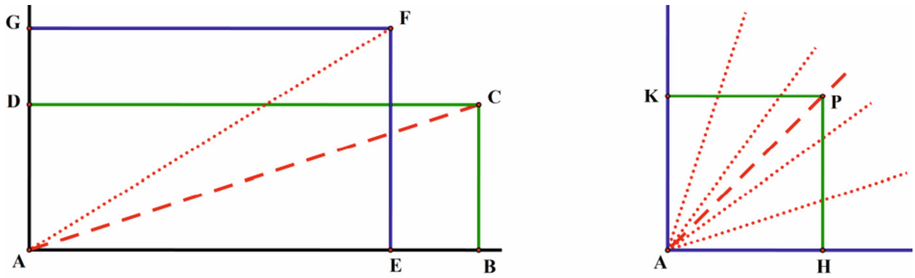

Fig. 14. The possible rectangles and square for girih construction based on the radial grid created from the division of a right angle into five

Now the question is whether using the same technique as mentioned above, are we able to come up with a new pattern composed from all five sāzeh modules in fig. 6b but now based on a square shape girih?

Obviously, none of the rays that divide ∠A into five congruent angles helps directly. Choosing P as an arbitrary point on the angle bisector of ∠A and constructing square AHPK cannot help us either (fig. 14b). Nevertheless, we are able to obtain a solution if we start with an arc with center A and an arbitrary radius. This arc cuts the sides of the right angle and the four rays at certain points that are used to find a solution. Following images in fig. 15, starting from the upper left and ending at the lower right, demonstrates a step-by-step solution by the author to this problem.

Fig. 16 is a tessellation that is created from the five sāzeh modules using the square girih in fig. 15. This artwork was exhibited at the 2012 Bridges Mathematical Art Exhibition at Towson University, Maryland, USA [Fathauer and Selikoff 2012].

Adding the tiling girih in fig. 15 to the previously mentioned girihs that are illustrated in figs. 4, 7, and 11, makes a set of four different mosaic patterns, each made from the aforementioned five sāzeh modules.

A curious reader may want to know whether more tessellations can be formed from this set of modules. The same curiosity may prompted the craftsmen-mathematicians of the past to look for new solutions that are not necessarily, at least in part, based on the compass-straightedge constructions.

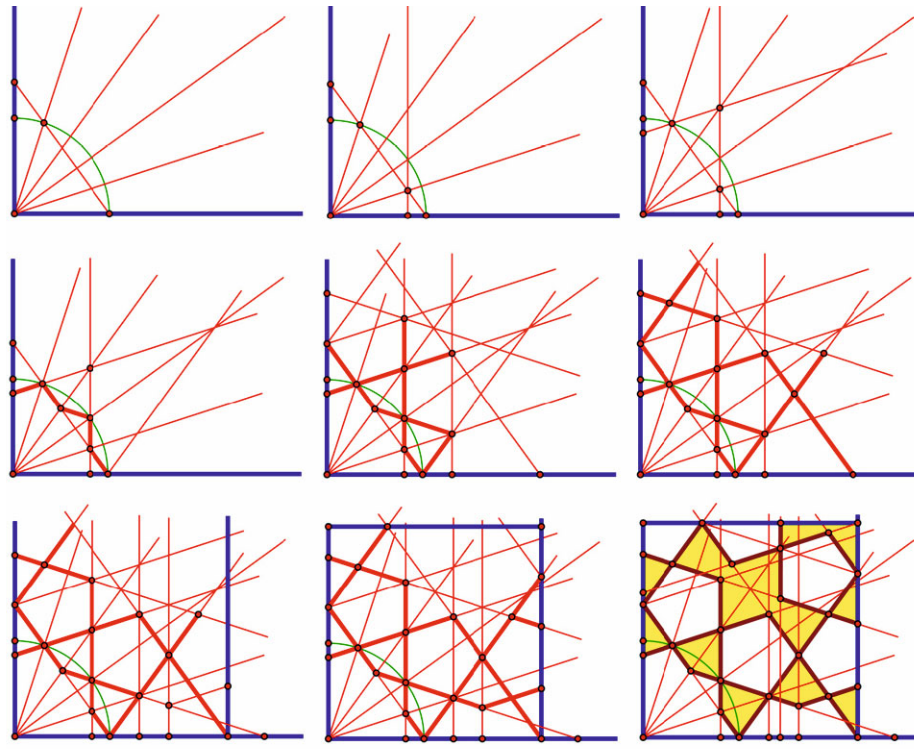

Fig. 15. Step-by-step compass-straightedge construction of a square shape girih for a decagram interlocking pattern as a visual solution to the problem solved by the author

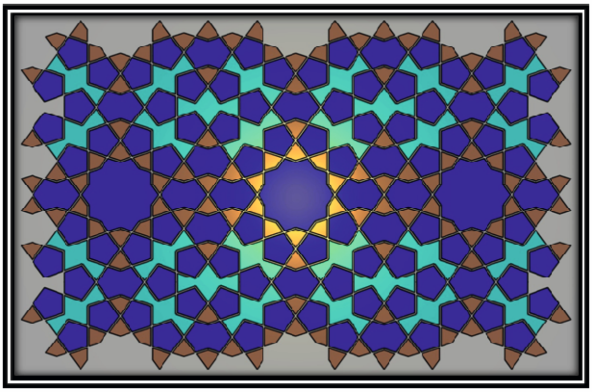

Fig.16. *Dah Par I* (September 2011) by the author, Bridges Mathematical Art Exhibition, Towson University, Maryland, USA, 2012 [Fathauer and Selikoff 2012]

8 A modularity approach to mosaic designs

Figs. 5d-e present the way the design for a cross-octagram tessellation can be constructed. However, to tessellate the plane using actual tiles there are two different approaches:

- Use fig. 5e as an actual tile and tessellate the plane using this tile (fig. 5f). We call such a tile a girih tile (that is, one decorated with lines that form the layout of a tessellation). The expression "girih tiles" was used in [Lu and Steinhardt, 2007]. to indicate the tiles that create the layout of a decagram interlocking pattern. Here it is adopted to generalize the idea to include more cases;

- Make two separate modules of the cross and octagram. Then cut the tiles based on these two modules (fig. 17). This is what we call the sāzeh modules for this tessellation.

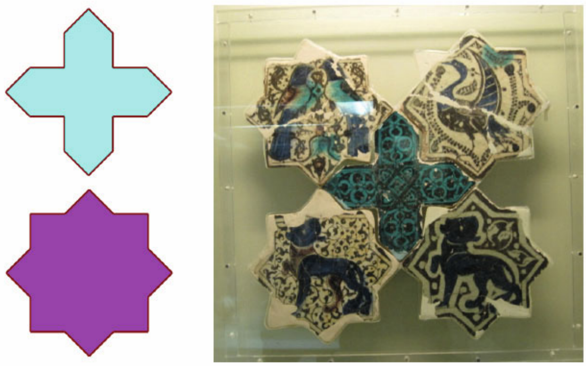

Fig. 17. Two individual sāzeh modules, cross and octagram, and photograph from a tiling in the Istanbul Archaeological Museum (photograph by author, August 2010)

This means it is possible (and indeed it has been a common practice) to find the layout of a tessellation using some tiles (girih tiles) but then use individual cut sāzeh tiles that are not necessarily the same as the girih tiles to tessellate. Fig. 18 shows the hand-cut pieces of glazed ceramic tiles, *zellij*, that are ready to be used for different tiling patterns in a market in Morocco. Each individual sāzeh tile (called "tesserae" in some literature) is cut and shaped from larger square-shaped colored and glazed tiles.

Fig. 18. Bags of individually cut sāzeh tiles(zellij in Arabian-Andalusian), Fez, Morocco. Source: Peter Sanders] Saudi Aramco World/SAWDIA

Modularity offers a method for creating the layout of a tessellation (that is, for conceptualizing but not necessarily making the individual actual tiles that compose the final tiling). The following sections demonstrate two ways the modularity technique has been employed to create the layout of mosaic patterns in contrast to the classical compass-straightedge method.

The modularity approach has been suggested as the means used in ancient cultures as old as the Paleolithic period as proposed by Slavik Jablan in his paper "Modularity in Art" [1980], which also appears as the last chapter of his book *Symmetry, Ornament and Modularity* [2002]. The possible use of modular tiles for creating the layouts of patterns in medieval Persia and surrounding areas was discussed in [Sarhangi, Jablan and Sazdanovic 2004 and Sarhangi 2008], where the authors also introduced gaps and overlaps as a tool for creating modular patterns. In recent years, several interesting and informative articles have appeared with more complex systems of modules discussed below.

8.1 Modularity based on color contrast

Fig. 19 shows *Kharragan I* (January 2011) an artwork based on a design from one of the eleventh-century tomb towers in Kharraqan, western Iran [Sarhangi 2010; Bier 2002; Bier 2012]. The artwork demonstrates two different approaches that are assumed to have been utilized centuries ago to create the layout of the pattern that appears in the center of the artwork. From left to right, the artwork exhibits the construction of the design based on a compass-straightedge construction (fig. 20). From right to left, we see another approach, the modularity method based on color contrast, to construct the same design using cutting and pasting of tiles in two colors (fig. 21). These two methods of constructions were presented at "A Workshop in Geometric Constructions of Mosaic Designs" during the 2010 Bridges Conference, Pécs, Hungary [Sarhangi 2010].

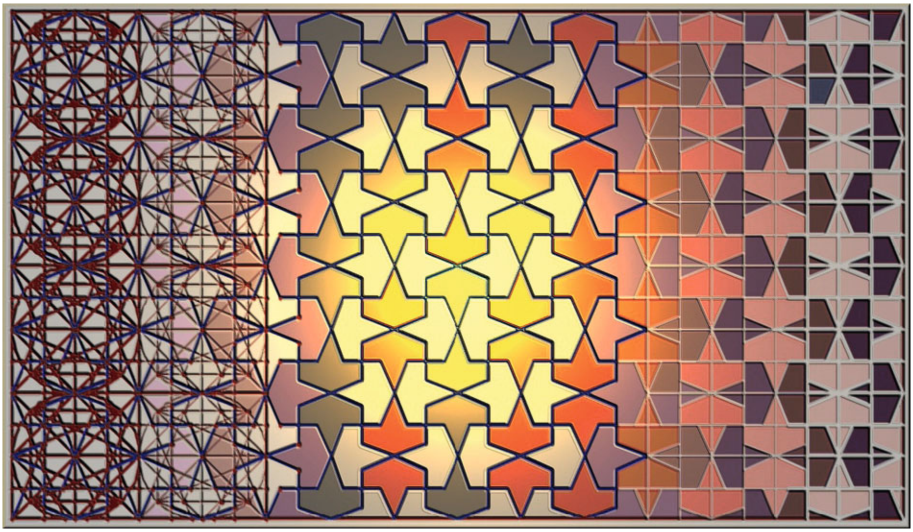

Fig. 19. *Kharragan I* (January 2011) by the author, Bridges Mathematical Art Exhibition, Coimbra University, Portugal, 2011 [Fathauer and Selikoff 2011]

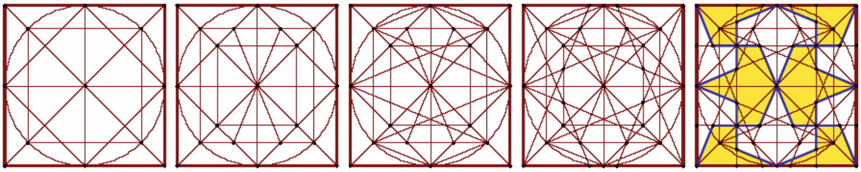

Fig. 20. Polygon construction approach for generating the grid for the "hat" tiling

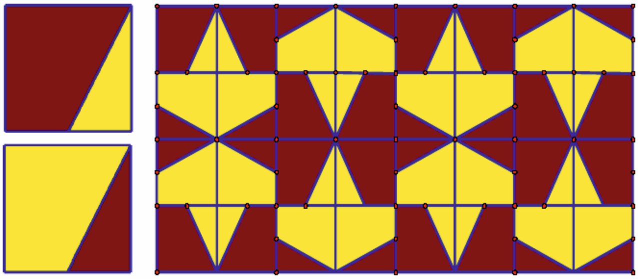

Fig. 21. Modularity approach for the "hat" tiling

The photograph shown in fig. 22 was taken by the author from an actual tiling in the entrance floor of the Enderun Library in the Topkapi Palace Museum. From the actual tile it is not possible to discover whether the layout has been formed by a compass-straightedge method (fig. 20) or the modularity approach (fig. 21). However, the color contrast in the actual tiling suggests the latter.

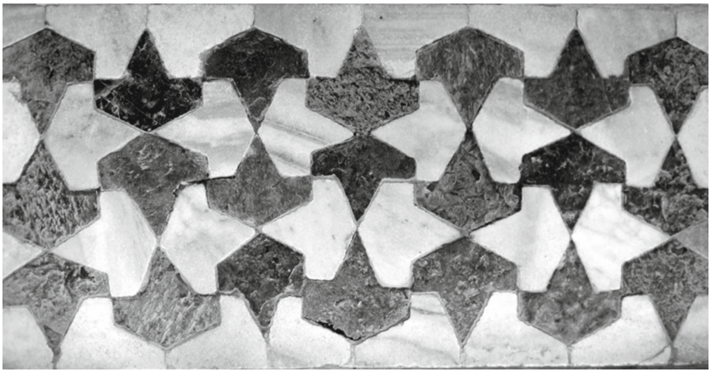

Fig. 22. A photograph of entrance floor of the Enderun Library in Topkapi Palace, Istanbul, Turkey

8.2. Modularity based on motifs formed from the combination of polygons

Fig. 23b shows *Hope* (December 2008), an artwork by the author [Akleman 2009], based on the modularity concept using two triangles, each composed from smaller triangles and rhombuses in three colors. The actual tiling adorns a wall of Bibi Zinab Mausoleum in Isfahan, Iran [Maheroannagsh 1984]. Notice that in fig. 23a, except for the diamonds in outermost vertices, the two compound triangles (girih modules) are in opposite colors. Using these two girih modules in a rotational fashion, results in the pattern in this artwork (fig. 23b). To make an actual tiling for this pattern a craftsman might use the cut sāzeh tiles illustrated in fig. 23c.

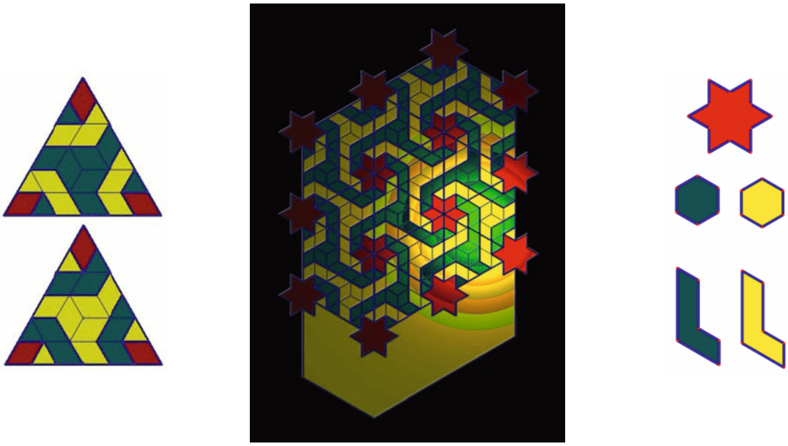

Fig. 23. a, left) The two modules that can be used to find the layout of a tessellation; b, center) The tessellation *Hope* (December 2008), made from these two modules; c, right) The sāzeh tiles that may have been used to create this pattern on a wall

Fig. 24b shows *Together* (November 2008) [Akleman 2009], another artwork based on the modularity concept using one single triangle, formed from smaller triangles and rhombuses in three colors; however, the number of required colors to make the tiling is two. The actual tiling of this pattern appears on a wall of Jāme Mosque in Natanz, Iran [Maheroannagsh 1984]. Fig. 24a is the girih module for finding the design. Fig. 24c shows the two sāzeh tiles used for the actual tiling. In the actual tiling, the physical copies of these two sāzeh modules have been set next to each other with a uniform gap between them filled by mortar.

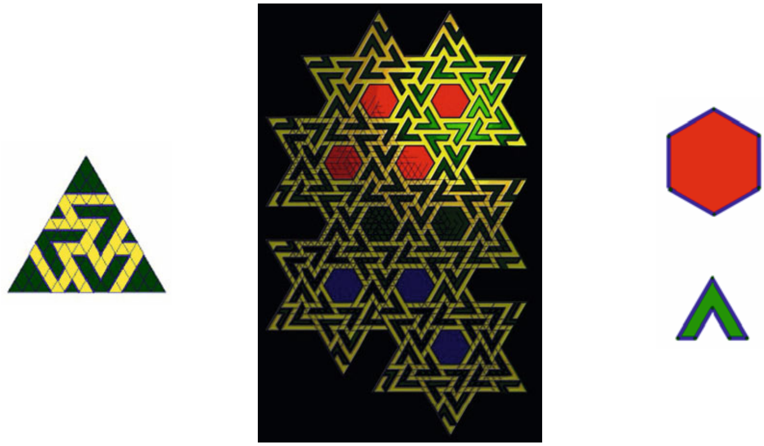

Fig. 24. a, left) The girih module used to find the layout of a tessellation; b, center) "Together" (November 2008); c, right) The sāzeh tiles that were used to execute this pattern

9 Modularity in interlocking star polygon mosaic designs

Jay Bonner [2003] has described several polygonal systems that generate Islamic geometric patterns using the polygons in contact technique]. The polygonal elements within these systems have associated patterns with lines that Bonner describes as having historical precedent. The variety of five-fold design represented in the mosaic patterns described herein uses the decagram and have 72°, 108°, 72°, 108° angles at each pattern vertex. This has been identified by Bonner as the medium pattern family. Within the five-fold System, medium patterns have 72° crossing pattern lines placed upon the midpoints of each edge of the repetitive module. fig. 25a illustrates the ten polygonal modules that form the five-fold system.

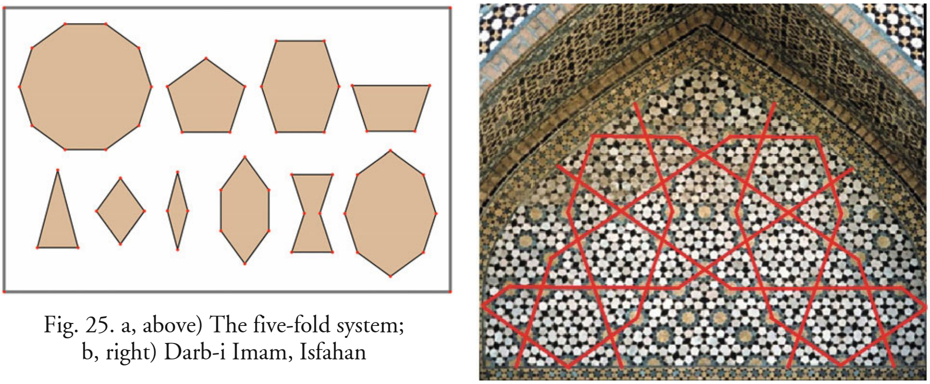

Fig. 25. a, above) The five-fold system; b, right) Darb-i Imam, Isfahan

Looking at fig. 25b (Darb-i Imam, Isfahan) Bonner noticed a set of lines that connect the centers of decagrams to form other tessellations with larger composite tiles (For the readers of this article these lines have been made bold to be more visible). Bonner used this figure to introduce self-similarity and introduced the term "sub-grid" to explain medieval Persian mosaic designs.

An informative book that appeared in recent years about mosaic designs in their historical context is *The Topkapi Scroll* [Necipoğlu 1995], which reproduces in facsimile at half-scale all of the images illustrated on the scroll. Bonner used image no. 28 from the scroll (fig. 26) as another example for the five-fold self-similar Type A:

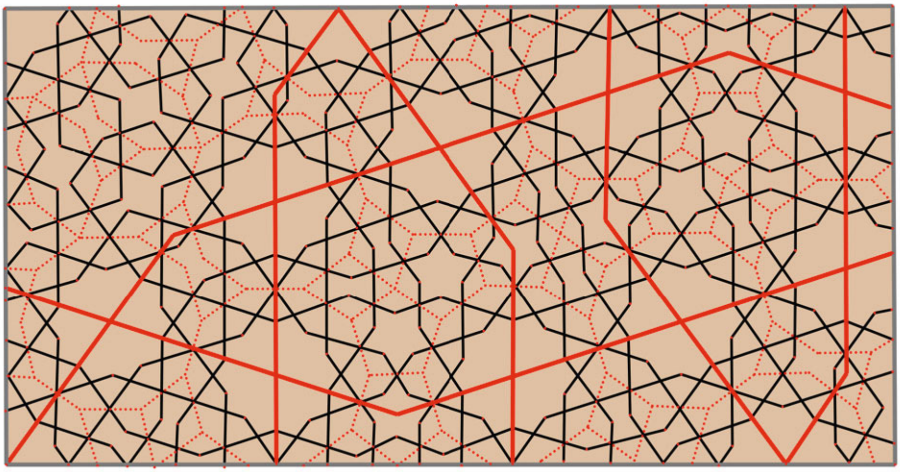

Fig. 26. A rendering of image no. 28 in the Topkapi Scroll

> Pattern 28 in the Topkapi scroll is a 5-fold self-similar Type A design that also depicts the underlying polygonal sub-grid used in the creation of the secondary design…. That this very particular technique was used historically is confirmed in the Topkapi scroll. Pattern no. 28 from this scroll makes use of small red dots to distinguish the underlying polygonal sub-grid of the secondary pattern [Bonner 2003].

Lu and Steinhardt [2007] also noticed these red dotted lines and proposed that they illustrate a new set of tiles, where the black solid lines decorate these new tiles (fig. 27). They realized that this new set could be used as a set of modules, similar to the modules that were presented in the previous section, but now in more complex and fascinating forms, for finding new interlocking star polygon patterns. This would eliminate the difficulties involved in compass-straightedge constructions and, in fact, opens the door to creating many more interesting mosaic patterns. Lu and Steinhardt named this new set girih tiles, the name adopted in this paper for the modules that form the layout of a mosaic tiling.

The edge-to-edge modular methodology that Lu and Steinhardt propose corresponds to five of the medium family design modules from the five-fold system presented in [Bonner 2003] that are illustrated in an earlier work, an unpublished manuscript by Bonner [2000]. Lu and Steinhardt posit that,

> … by 1200 C.E. there was an important breakthrough in Islamic mathematics and design; the discovery of an entirely new way to conceptualize and construct girih line patterns as decorated tessellations using a set of five tile types, which we call 'girih tiles'. Each girih tile is decorated with lines and is sufficiently simple to be drawn using only mathematical tools documented in medieval Islamic sources. By laying the tiles edge-to-edge, the decorating lines connect to form a continuous network across the entire tiling [Lu and Steinhardt 2007: 1106].

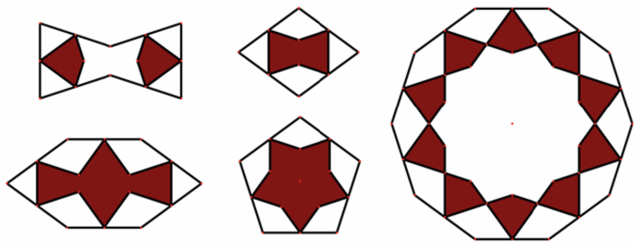

Fig. 27. The five girih tiles used to create decagram interlocking designs

Using three modules from this set enable us to construct the four aforementioned tessellations that were constructed using a compass and straightedge. In fig. 28, from upper left to lower right, one sees how the composition of three girih modules from the five modules in fig. 27 can generate the pattern designs illustrated above in fig. 4, fig. 7, fig. 11, and fig. 15, respectively.

Using the girih modules in fig. 27, a tile designer could compose much more complex repeating patterns. Even two of them are sufficient to make an attractive interlocking star polygon tiling, where the same pattern would require a very long process using a compass-straightedge construction (fig. 29).

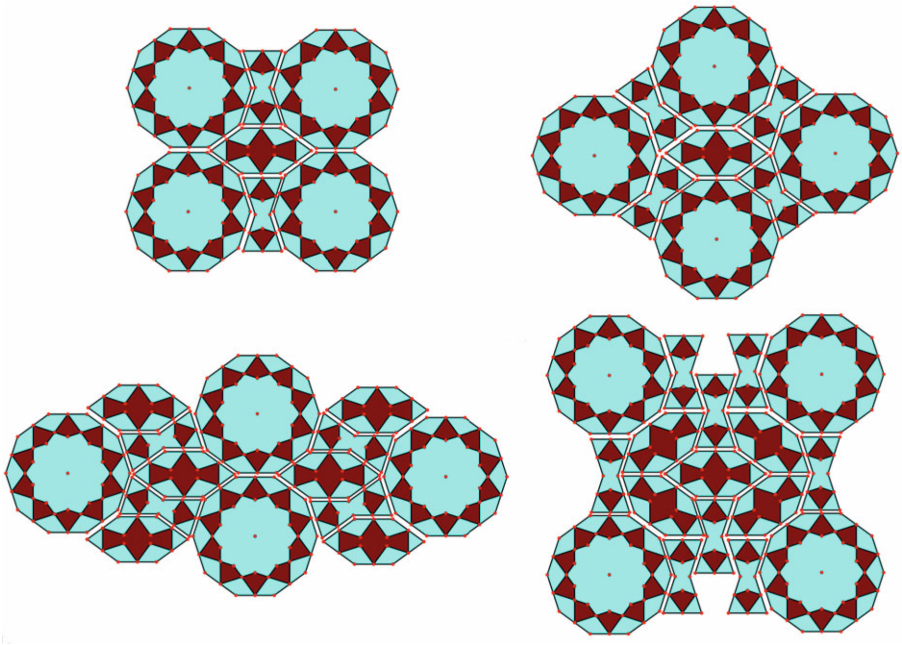

Fig. 28. A modular approach to constructing aforementioned patterns

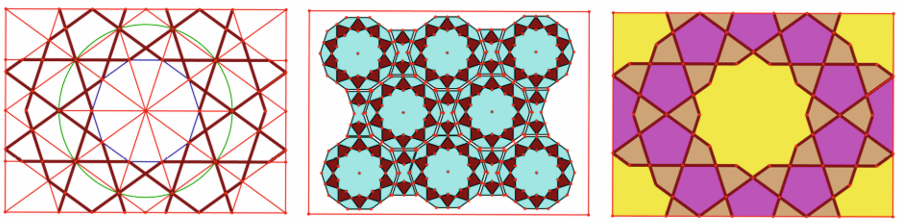

Fig. 29. Compass-straightedge method (a, left) and modular approach (b, center) for creating a pattern (c)

The design in fig. 25b is highly complex and much more complicated than the four tessellations in fig. 28. However, using the girih tiles set, one can construct it much more conveniently and quickly than by employing a compass and straightedge. Fig. 30 shows how one can construct the larger tessellation tiles on the Darb-i Imam using the three girih tiles used in fig. 28. Then the execution of the entire tessellation on a wall using hand-cut sāzeh tiles would be only a matter of time.

It is interesting to note that the larger tessellation on the Darb-i Imam in fig. 25b is indeed a part of the tiling created from the two girih modules of decagon and non-convex hexagon used in the tilling image in fig. 29. Fig. 31 illustrates this tessellation and the part that has been used in Darb-i Imam.

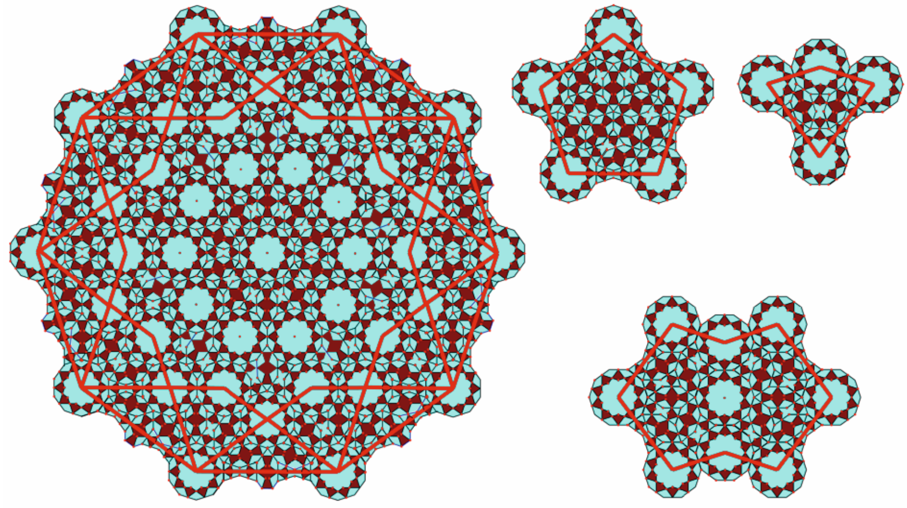

Fig. 30. Details of larger tiles in Darb-i Imam using girih tiles

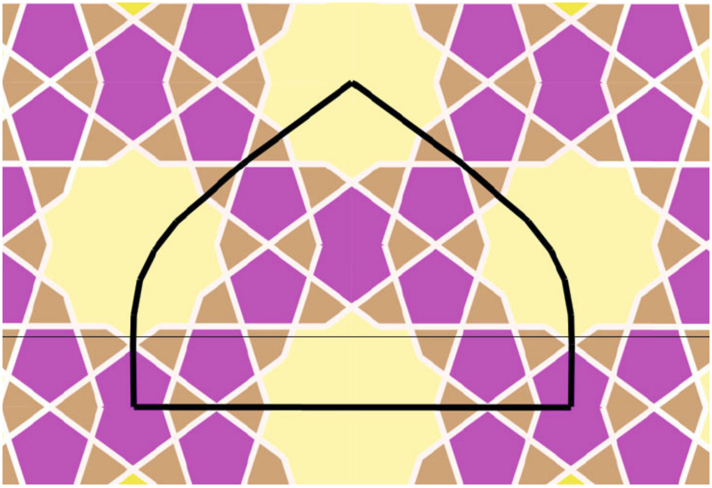

Fig. 31. The larger tessellation at Darb-i Imam in relation to the pattern in fig. 29

10 Quasi-periodic interlocking star polygon designs

A periodic tiling is one in which the image of the tiling under an appropriate translation based on some vectors coincides with the original one without using rotational or reflective symmetries (the tiling should be invariant under translations by vectors in a two-dimensional lattice). Fig. 32a is the hat tiling that is formed from the two modules in fig. 21. If the tessellation translates so that the image of A under this translation coincides with B, then the entire image of this tessellation will coincide with the original. The same property exists if the tessellation translates under vector AC. Therefore, fig. 32a demonstrates a periodic tiling.

A non-periodic tiling is a tiling that does not follow the rule of periodicity. Consider, for example, the same two hat modules shown in fig. 21 arranged randomly to tile the plane. This simple rule, as stated above, generates a tiling that is non-periodic (fig. 32b).

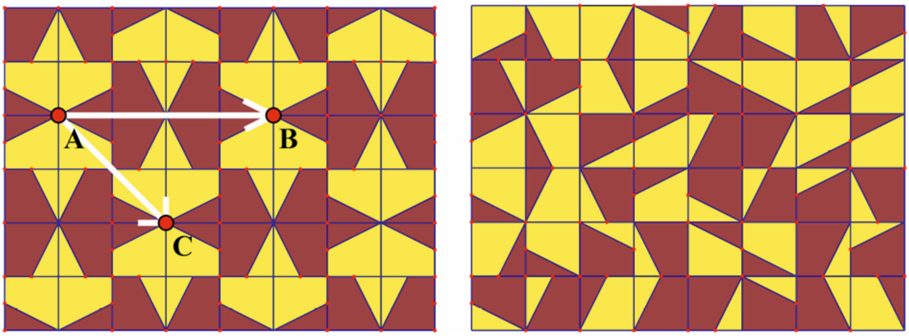

Fig. 32. A periodic tiling (a, left) and a non-periodic tilling (b, right) obtained from the hat modules

The four tessellations in fig. 28 offer further examples. Each is a periodic pattern, which may be compared to the decagon in fig. 30 that can be expanded in all directions as a non-periodic tiling. Note that each of these tilings, the four illustrated in fig. 28 and the expanded decagon shown in fig. 30, are made using the same three girih modules, comprising the decagon and two hexagons in fig. 27.

An interesting question in this regard that appeared in mathematics literature of 1960s was: “Are there sets of tiles that tessellate the plane only non-periodically?” [Gardner 1977].

A quasi-periodic, or aperiodic, tiling is one generated from a quasi-periodic set of tiles, a set that tessellate the plane only non-periodically. Mathematically speaking, a tiling of the plane is aperiodic if and only if it consists of copies of a finite set of tiles that only produce non-periodic tilings. So the above question can be rephrased as: “Does a set of aperiodic tiles exist?”

In 1961 the logician and mathematician Hao Wang claimed that any set of tiles that can tile the plane non-periodically can tile it periodically as well (that is, a set of quasi-periodic tiles does not exist). Robert Berge, a student of Wang, using Wang dominos, the tiles that were invented by Wang, showed that Wang’s conjecture is not correct. He discovered that there is a set of Wang dominoes that tiles only non-periodically. Berger constructed such a set, using more than 20,000 dominoes. Later he found a much smaller set of 104; Donald Knuth was able to reduce the number to 92. Karel Culik discovered a set of 13 tiles that can tile the plane, but only non-periodically. Robinson constructed six tiles that force non-periodicity. In 1977 Robert Ammann found a different set of six tiles that also force non-periodicity [Gardner 1977].

In 1973 the mathematical physicist Roger Penrose found a quasi-periodic set of six tiles: “In 1974 he found a way to reduce them to four. Soon afterward he lowered them to two” [Gardner 1977]. The two tiles that he discovered, called “kite” and “dart,” can only tessellate non-periodically (fig. 33b). These two tiles form a rhombus that is the wing of a five-folded star that can be constructed using the (10, 3) star polygon (fig. 33a). In order to tile aperiodically using kites and darts one should also connect the arcs of the same color printed on the tiles properly to create continuous curves (closed or open). The curves prevent the two tiles from forming a rhombus. John H. Conway found that a set of aperiodic tiles (Ace, Short bow tie, and Long bow tie) that are made from the Penrose tiles, can tessellate the plane in a faster and more stable fashion (fig. 33c) [Gardner 1977].

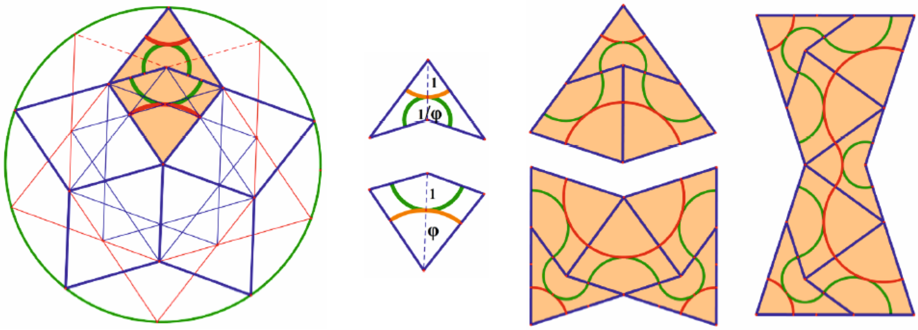

Fig. 33. a, left) The kite and dart in a five-pointed star; b, center) The tiles with printed curves that are in the proportions of $1/\phi$, 1, and ϕ, where ϕ is the Golden ratio; c, right) The new set of non-periodic tiles suggested by Conway

It can be proven that one needs ϕ times as many kites as darts in an infinite Penrose tiling. But ϕ as the ratio of the number of kites over the number of darts is an irrational number. This irrationality is the basis for a proof by Penrose that the tiling is nonperiodic: If the Penrose tiling were periodic, then the ratio of kites to darts would have to be rational.

Fig. 34a shows a Penrose tiling created from kites and darts of a certain size. Fig. 34b shows how one can use the vertices of the previous tiling to generate a new tiling with larger darts and kites, a phenomenon called inflation. One can continue forming new tilings using inflation with each new generation of tiles larger than the previous iteration. Deflation is the reverse process of inflation. Using inflation one can prove that the number of Penrose tiling is uncountable.

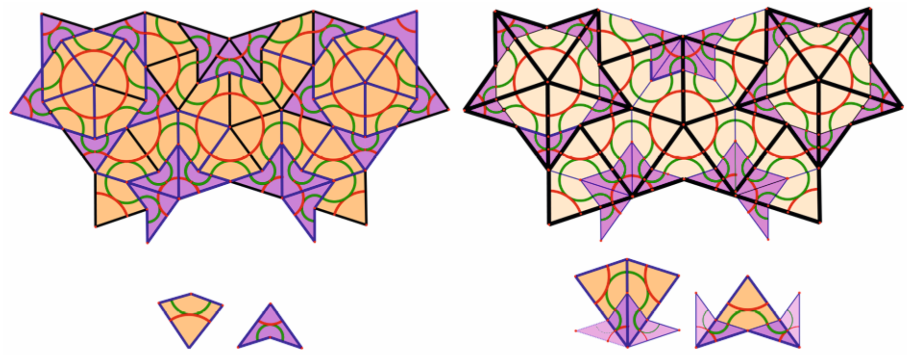

Fig. 34. a, left) A Penrose tiling; b, right) The inflation

A Penrose tiling has many remarkable properties, most notably:

- It is non-periodic, which means that it lacks any translational symmetry. Stated more informally, a shifted copy will never match the original;

- It is self-similar, so the same patterns occur at larger and larger scales. Thus, the tiling can be obtained through "inflation" (or "deflation") and any finite patch from the tiling occurs infinitely many times;
- It is a quasi-crystal: implemented as a physical structure a Penrose tiling will produce Bragg diffraction and its diffractogram reveals both the fivefold symmetry and the underlying long range order [Wikipedia].

The five-pointed star in fig. 33 has been used in Persian architectural designs on the wall and on the dome interiors [Sarhangi 1999]. Fig. 35a shows this star as an actual tiling. Fig. 35b shows the compass-straightedge details of the construction that exhibits a subdivision rule for creating self-similar smaller generation of stars [Sarhangi 1999].

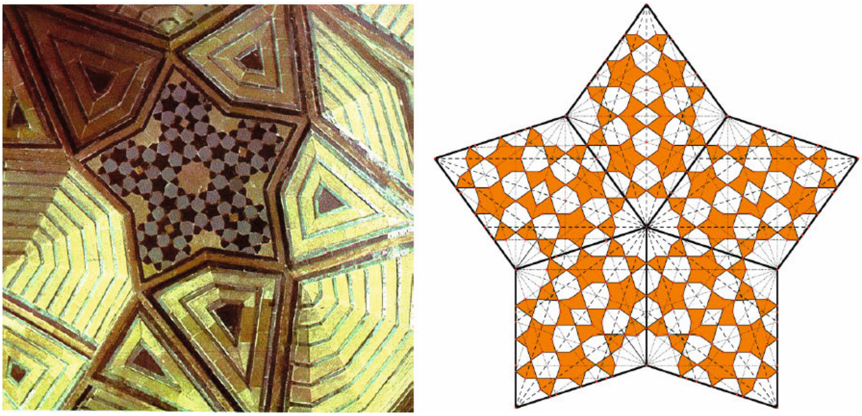

Fig. 35. An actual tiling for star in fig. 33a and the details of its construction

Emil Makovicky, a crystallographer, was among the earliest individuals who studied mosaic designs on Persian architecture to recognize the existence of quasi-crystalline patterns [Makovicky 1992]. He studied the Gonbad-e Qabud in Maragha, Iran [Bier 2012]. This decagonal Seljuk building of the late twelfth century has attracted the attention of many mathematicians and designers in recent years. For his course of study, Makovicky considered three tiles that he named “Maragha-type” tiles: pentagons, butterflies, and rhombuses with marked acute vertices.

To study the quasi-crystalline properties of some Persian structures based on the set of compounded tiles created by Conway shown in fig. 33c, Lu and Steinhardt proposed three new tiles that resemble the three aforementioned girih tiles in fig. 27 (fig. 36a). These new three tiles generate aperiodic tilings; nevertheless, considering the decorations on each tile, which are Penrose curves, they are different in nature from the three girih tiles in fig. 27, as they possess fewer symmetries. For this set (and not the set that constitute the Penrose kite and dart) we can tessellate the plane using ten-fold rotational symmetry (fig. 36b). In this fashion it is reasonable to accept that we are able to expand the cartwheel in fig. 36 to a bigger structure to construct the decagon in fig. 30 (without considering the decorations on the individual tiles) and continue to infinity. We may also choose an opposite approach in this regard: we start with a decagon and subdivide it based on the rules exhibited visually in fig. 30 and then replace the small girih tiles inside of the decagon in fig. 30 with the tiles in fig. 36a and continue this process indefinitely. This pattern that is generated from the quasi-periodic tiles in fig. 36a is a quasi-crystalline pattern with respect to the original Penrose kites and darts. Lu and Steinhardt showed that considering the tiles in fig. 36a as units, this subdivision process, using the matrix notation, can be expressed as a matrix that exhibits the frequencies of appearance of the three tiles in each step of subdivision. The eigenvalues of this matrix, which represent the ratio of tile frequencies in the limit of an infinite subdivision process, are irrational, indicating that the pattern is not periodic. Therefore the pattern generated from this subdivision is quasi-periodic if we use the three tiles in fig. 36a, and is non-periodic if we exchange these three tiles with their correspondence girih tiles in fig. 27.

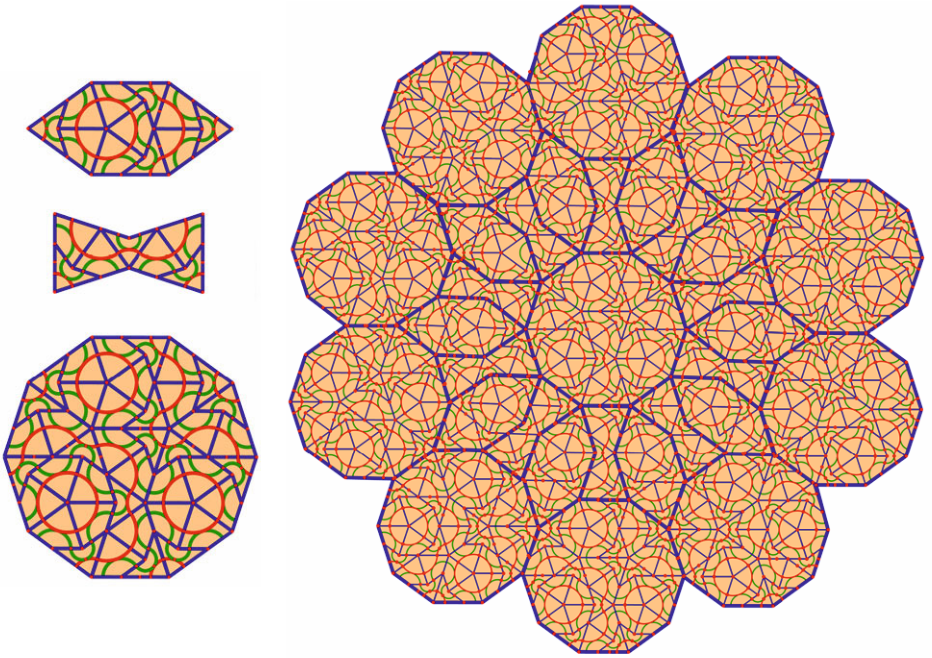

Fig. 36. a, left) The three tiles made from kites and darts that resemble the three girih tiles; b, right) The start of edge-to-edge construction of a cartwheel that resembles the decagon in fig. 30

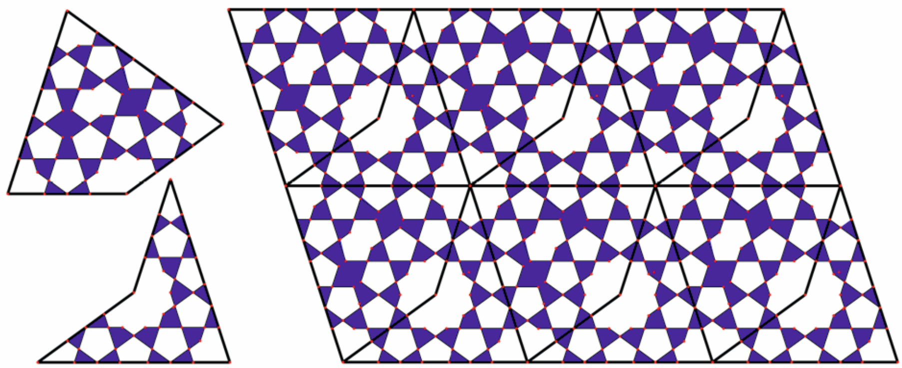

Fig. 37. a, left) Rigby kite and dart; b, right) A periodic pattern formed by Rigby tiles

To generate non-periodic mosaic patterns, Jean-Marc Castera created a set of four decorated module tiles [Castera 2003], and later introduced more decorated tiles in this regard.

In 2006, in an independent effort for making Penrose interlocking star polygons, John Rigby [2006] proposed a way to cover the surface of kites and darts with appropriate patterns to obtain a set of sāzeh tiles to generate various interlocking patterns (fig. 37a). The patterns that decorate these two tiles are not in an equivalent class of symmetries with the Penrose curves. This is due to the fact that the Rigby tiles can tessellate the plane periodically because the kites and darts can form rhombuses (fig. 37b). Nevertheless if we avoid forming any rhombuses in a Rigby tiling then the pattern will be non-periodic.

As described above, it is important to emphasize that the Penrose tiles cannot produce either global or local ten-fold rotational symmetry. They only yield five-fold symmetries. There are uncountably many Penrose tilings. Even though a Penrose tile has infinite centers of local five-fold rotational symmetry, none has global five -fold rotational symmetry, with two exceptions. The two tessellations that have perfect global five -fold rotational symmetry are called "Sun" and "Star" (fig. 38). These two tessellations are the dual of each other in the sense that one is the inflation of the other.

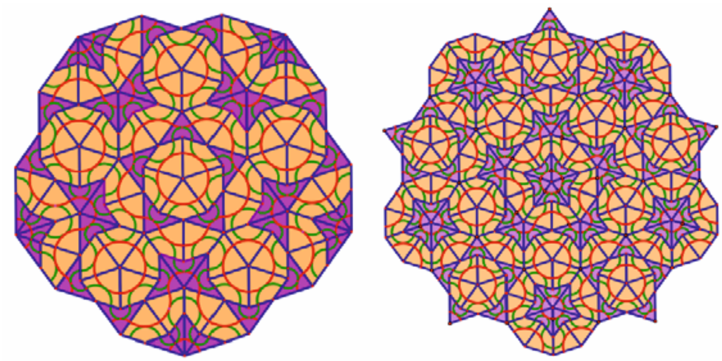

Fig. 38. left) Penrose Sun; right) Penrose Star

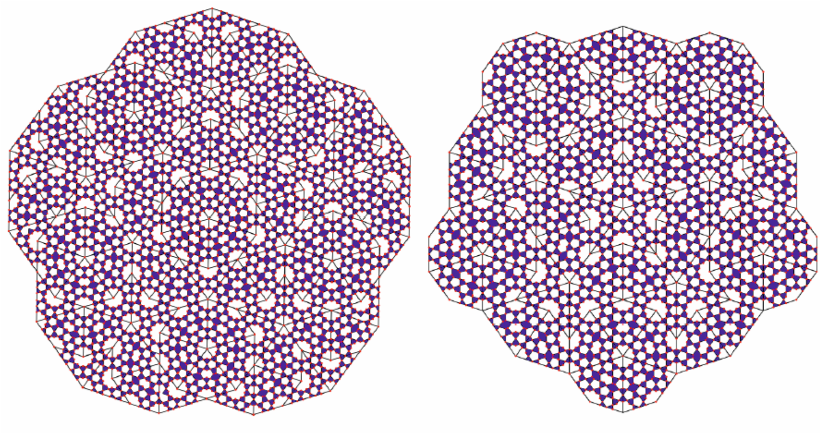

Fig. 39. Penrose Sun and Star with Rigby kites and darts

Fig. 39 shows the Penrose Sun and Star where we use Rigby kites and darts for their formations. Looking at the Sun in fig. 39a, an observer notices a striking similarities between the center of the cartwheel in fig. 30a that can be expanded as a perfect non-periodic (but not quasi-periodic) pattern holding a ten-fold rotational symmetry, and the center of the Sun in fig. 39a, which is modeled based on a quasi-crystalline pattern with perfect five-fold rotational symmetry.

Conclusion

Medieval Persian artisans, architects, and mathematicians must have had a relatively strong background in geometry to manifest their findings not only in the structures of domes and buildings, but also in the designs and patterns that adorn the walls of these structures.

It seems that some of the earlier geometric designs found in ancient times have been created through trial and error combinations of cut-tile pieces. This technique is called modularity in some literature including this article.

Later, as the center of knowledge moved from Greece and Byzantine to the East, Persians used compass-straightedge extensively to create geometric patterns. The few treatises and scrolls that have come down from the past exhibit this point clearly.

Later, designers used the tessellations generated from simple polygons as a base for constructing exquisite mosaic designs on them. Soon after that, the combination of their knowledge in modularity and in creating complex patterns using compass-straightedge resulted in a new level of the modularity method, which generated many highly complex and elegant designs that are difficult, if not impossible, to execute using compass and straightedge alone.

Looking at these designs on the walls of existing structures and historical scrolls one may come to the conclusion that the designers of the past were looking to maximize symmetries, especially local and global rotational symmetries, and in particular, five-fold and ten-fold rotational symmetries. It is amazing that some of their solutions as expressed in mosaic designs, have attracted the attention of modern crystallographers and mathematicians, who have found similar patterns to answer contemporary questions in modern mathematics.

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Reza Sarhangi is a professor of mathematics at Towson University, Maryland, USA. He teaches graduate courses in the study of patterns and mathematical designs, and supervises student research projects in this field. He is the founder and president of the Bridges Organization, which oversees the annual international conference series "Bridges: Mathematical Connections in Art, Music, and Science" (www.BridgesMathArt.Org). Sarhangi was a mathematics educator, graphic art designer,

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