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Research

From Sultaniyeh to Tashkent Scrolls*: Euclidean Constructions of Two Nine- and Twelve-Pointed Interlocking Star Polygon Designs*

Abstract. In this paper we will explore two nine- and twelve-pointed Islamic star polygon patterns consisting of "nearly regular" nine-pointed, regular twelve-pointed and irregularly-shaped pentagonal star polygons. The two designs are similar in that they may both be classified mathematically as being p6m patterns with the major star polygons placed in identical locations within each layout; however, the structure of the major stars is quite different. Both of the patterns considered here are of Persian origin. The first design may be found as a repeat unit sketch of the *Tashkent Scrolls*, and exists as a Timurid-style stone inlay and mosaic tiling in India. The second pattern may be found as Plate 120 of Bourgoin's *Arabic Geometrical Pattern and Design* and exists as a stucco/plasterwork ceiling in the Mausoleum of Sultan Oljaytu in Sultaniyeh, Iran, as well as numerous other locations across the Islamic world. Both patterns may be recreated via plausible Euclidean "point-joining" constructions (that is, using only the methods available to medieval artisans) in an attempt to ascertain how the original designers of these patterns may have determined the proportion and placement of the stars.

Introduction

Islamic art developed over a period of a thousand years, as Islamic rule expanded to encompass regions previously under the influence of other empires. As the cultures intermingled, the styles and techniques of the conquered craftsmen and master builders were assimilated into the art. Roman, Byzantine, Persian and Central Asian art styles – such as the Roman and Byzantine use of geometric patterns, especially the "heavenly stars," as well as Persian arches, domes and squinches from the Sassanid dynasty – were incorporated and refashioned to create motifs, shared themes and unifying elements in the Islamic art and architecture found from Spain to Central Asia.

Attributes that make Islamic ornament readily recognizable, regardless of the locales or times from which it originates, include the covering of surfaces with calligraphy and endlessly repeating geometric or highly-stylized vegetal patterns. Some of the design principles and aesthetic choices of a particular region, and their subsequent resulting patterns, were perpetuated down through the centuries and also disseminated widely from one region to another by exchanges of drawings and the movement of artisans [Necipoğlu 1992: 48]. Gülru Necipoğlu, the author of *The Topkapı Scroll – Geometry and Ornament in Islamic Architecture: Topkapı Palace Museum Library MS H. 1956* [1995], has written:

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> [the] Maghribi tradition was originally inspired by architectural developments in the eastern Islamic world that had traveled westward between the eleventh and thirteen centuries all the way to Spain [Necipoğlu 1995: 23].

That being said,

> [t]he diffusionist explanation is, however, incomplete. If certain forms or techniques spread throughout different provinces of the Muslim world, it was also because they reflected the values proper to Islam, this term being taken in its broadest sense to designate not only a religion but also a model of society, a way of thinking and even a philosophical system [Clévenot 2000: 21].

One of the two star polygon designs to be discussed in this paper is an example of a pattern that may be found rather ubiquitously across the Islamic world in a wide variety of media such as stucco, wood, masonry and ceramic tiling (and whose creation also spans a few hundred years). Consisting mainly of twelve-pointed stars each surrounded by six nine-pointed and twelve pentagonal stars, it was cataloged by Jules Bourgoin as Plate 120 of his *Arabic Geometrical Pattern and Design* [1973], a rich source of over 200 Islamic patterns, first published in 1879 and based upon drawings, with analyses, of Islamic monuments in Cairo and Damascus. Instances of this pattern, found as decorative ornamentation on monuments dating to medieval times in Iran, Egypt, Turkey, and Spain, may be discovered in the *Pattern in Islamic Art: The Wade Photo Archive*, a collection of over 4000 images of Islamic patterns, available online [Wade]. More specifically, this layout may be found in a stucco/plasterwork ceiling in the Mausoleum of Sultan Oljaytu in Sultaniyeh, Iran, dated 1313, from the Ilkhanid dynasty [Wade: Ira 2701] (fig. 1a). This mausoleum, considered a structural masterpiece and protected by the World Heritage Convention of the United Nations Educational, Scientific and Cultural Organization (UNESCO), "represents an outstanding achievement in the development of Persian architecture" and is "characterized by its innovative engineering structure, spatial proportions, architectural forms and the decorative patterns and techniques" (see http://whc.unesco.org/en/list/1188).

This prevalent pattern may also be found in a panel of a wooden *minbar* (pulpit) of the Mosque of al-Mu'ayyad in Cairo, Egypt, dated 1420, from the Mamluk dynasty [Wade: Egy 1216] (fig. 1b); a carved masonry stone relief in the Medrese of Dunbar Bek in Egridin, Turkey, dated 1302, from the Seljuk dynasty [Wade: Tur 0926] (fig. 1c); and, a thirteenth- to fifteenth-century century Nasrid dynasty ceramic tiling of the Alhambra in Granada, Spain [Wade: Spa 1008] (fig. 1d).

Nonetheless, variations in construction techniques and distinctly different decorative styles within Islamic art may be perceived, and thus may be designated as belonging to particular regions or dynastic traditions. For example, the Islamic architecture of Northern India, which from the sixteenth century onwards was ruled by the Mughals, originated in Persia [Volwahsen 1994: 3] and combined both Persian and Hindu traditions.

Fig. 1a. A portion of a stucco/plasterwork ceiling in the Mausoleum of Sultan Oljaytu in Sultaniyeh, Iran, dated 1313, from the Ilkhanid dynasty [Wade: Ira 2701] (photograph cropped by this author)

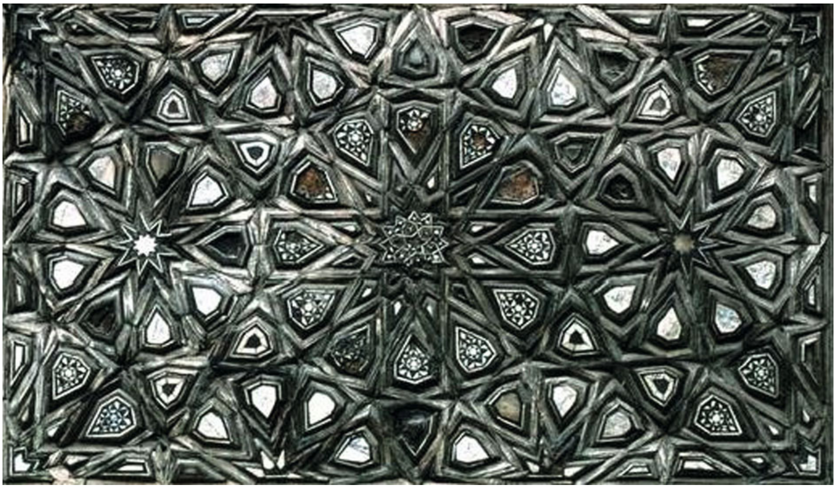

Fig. 1b. A portion of a wooden minbar panel of the Mosque of al-Mu'ayyad in Cairo, Egypt, dated 1420, from the Mamluk dynasty [Wade: Egy 1216] (photograph cropped by this author)

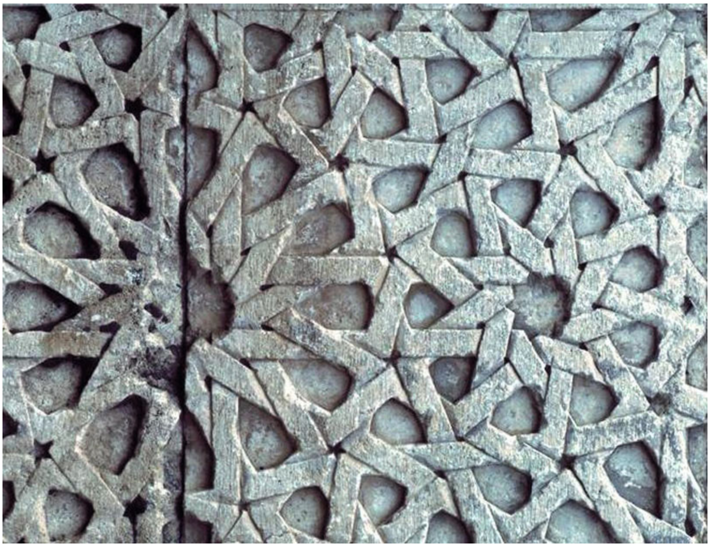

Fig. 1c. A portion of a carved masonry/stone relief in the Medrese of Dunbar Bek in Egridin, Turkey, dated 1302, from the Seljuk dynasty [Wade: Tur 0926] (photograph cropped by this author)

Fig. 1d. A portion of a thirteenth-fifteenth century Nasrid dynasty tiling of the Alhambra in Granada, Spain [Wade: Spa 1008] (photograph cropped by this author)

The second pattern to be discussed in this paper may be found at the Masjid-i Jami in Fatehpur Sikri, India which was constructed by Sheikh Salim from 1571-1574 and whose architecture was based on Timurid forms and styles. It appears as an extant stone inlay and mosaic tiling, illustrated in [Wade: Ind 0729] (fig. 2). In fact, this design may be generated from a *repeat unit* sketch belonging to a collection of fragmentary design drawings known as the *Tashkent Scrolls,* which "almost certainly reflect the sophisticated Timurid drafting methods of the fifteenth century, if not earlier" [Necipoğlu 1992: 50]. The scrolls, so named because they are now housed in Tashkent at the Institute of Oriental Studies at the Academy of Sciences of the Uzbek SSR, are attributed to an Uzbek master builder or a guild of architects practicing in sixteenth-century Bukhara, a city located in the Greater Khurasan of Sassanid and medieval Persian periods [Necipoğlu 1995: 7].

Fig. 2. A portion of a stone inlay and mosaic tiling of the Masjid-i Jami in Fatehpur Sikri, India, dated 1571, Mughal dynasty [Wade: Ind 0729] (photograph cropped by this author)

Many of the drawings of the *Tashkent Scrolls* are similar to those found in the much older, larger and better preserved fifteenth-century *Topkapı Scroll*, a 96-foot-long architectural scroll now housed at the Topkapı Palace Museum Library in Istanbul. Both the *Topkapı Scroll* and the *Tashkent Scrolls* contain geometric design sketches with no measurements or accompanying explanations for how to create them. In addition, they "belong to the same region, extending from Iraq and Iran to Central Asia" [Necipoğlu 1995: 23]. The drawings in these scrolls are also similar to sketches found in the collection of design scrolls and working drawings (now in the Victoria and Albert Museum in London) that belonged to Mirza Akbar, an official state architect of nineteenth-century Iran.

Comparing the Victoria and Albert scrolls with the earlier ones in Tashkent demonstrates the amazing continuity of architectural drafting methods in Iran up to the modern era, a continuity reflecting the absence of radical changes in style and building technology [Necipoğlu 1992: 52].

It is interesting to note that neither of the two nine- and twelve-pointed Islamic star polygon designs to be discussed in this paper may be found in the *Topkapı Scroll.*

To highlight the similar placement of the major 12-pointed and 9-pointed stars, as well as the other polygons filling the interstitial spaces, portions of these two patterns (in skeletal form without additional embellishments) have been recreated and colored by the author using the Geometer's Sketchpad® software program (the electronic equivalent of a compass and straightedge). Fig. 3a shows the design that appears as Plate 120 of Bourgoin, hereafter referred to as *B120*; fig. 3b shows the design hereafter referred to as *TS*, which may be generated from a *Tashkent Scroll repeat unit* drawing.

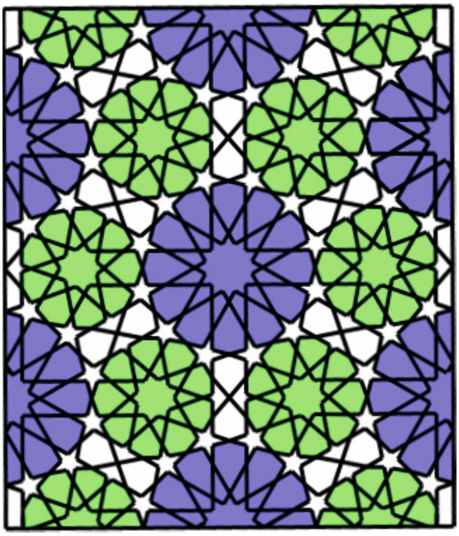

Fig. 3a. Author's reconstruction of the B120 design

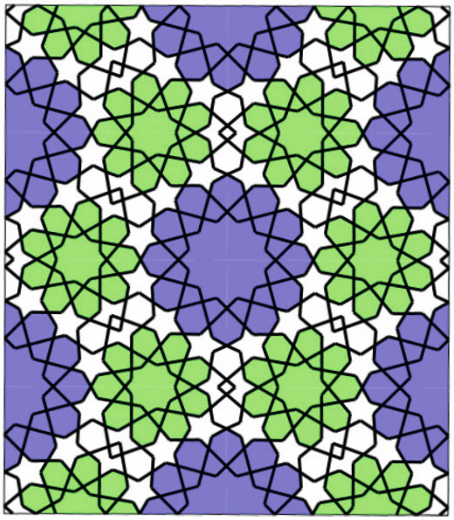

Fig. 3b. Author's reconstruction of the TS design

The placement of the major stars in these two patterns is identical and so (in skeletal form) they may both be mathematically classified as belonging to the same crystallographic group; that is, they are both p6m designs, whereby the design elements exhibit six-fold rotational symmetry and mirror reflections. The reflection and rotational symmetry is explicitly denoted in fig. 24 at the end of this present paper. Notice, however, that although the major stars have the same number of points, they are structurally quite different, as explained more fully in a later section.

Creating geometric Islamic patterns

How did artisans and master builders centuries ago create elaborate Islamic star polygon designs? There are few written records to answer this question definitively and it is quite likely that several different methods were actually employed. It most likely did involve "a considerable amount of geometrical knowledge," suggesting that some degree of mathematical literacy may have existed among artisans, or at least among the master builders, architects and master engineers [Berggren 1986]. For initially creating new designs, the traditional tools of the medieval period – a compass, straightedge, and set square – may well have been used by those master builders versed in construction techniques.

Theoretical mathematicians (al-Sijzī, Abū Nasr al-Farabī, Abū'l-Wafā al-Buzjani, Al-Kashi, Umar al-Khayyami, and Abu Bakr al-Khalid al-Tajir al-Rasadi, among others) developed and wrote about construction techniques useful to artisans interested in creating geometric ornamentation [Özdural 1995; 2000]. Manuals were written as a result of the meetings between these two groups, including one by al-Buzjani entitled, *Kitāb fīmā yahtā ju ilayhi al-sāni' min a'māl al-handasa* (*A book on those geometric constructions which are necessary for a craftsman*) which provided simplified instructions on how to perform basic Euclidean constructions using these traditional tools for:

> … the construction of a right angle; the bisection of a square or circle; the division of a right angle into equal parts; the trisection of an angle; drawing a line parallel to, perpendicular to, or at a certain angle to a given line; determining the center of a circle or its arc; dividing the circumference of a circle into equal arcs; dropping a tangent to a circle from a given point; drawing a tangent to a circle through a point on it; and trisecting the arc of a circle. … From these general problems al-Buzjani moved on to the construction of regular polygons inscribed in circles, other constructions involving circles and arcs, and the constructions of polygonal figures inscribed in various figures. The circle is used in al-Buzjani's treatise to generate all of the regular polygons in a plane [Necipoğlu 1995:138].

These geometric constructions could have formed the basis for creating many of the geometric Islamic patterns of the time.

Appended to a copy of a Persian translation of al-Buzjani's manuscript in the Bibliothèque Nationale in Paris, is an anonymous, twenty-page Persian manuscript, *Fī tadākhul al-ashkāl al-mutashābiha aw al-mutawāfiqa* (On Interlocking Similar and Congruent Figures) usually referred to by its shorter title, *A'māl wa ashkāl* (Constructions and Figures). Believed to have been prepared by an artisan sometime during the eleventh to thirteenth centuries, it is the only known practical manual that provides "how to" instructions for drawing the repeat units of 61 planar geometric patterns. The instructions in Persian usually start with the standard phrase: 'The way of drawing and the ratio or proportion of this construction is as follows' [Necipoğlu 1995]. Abu Bakr al-Khalid al-Tajir al-Rasadi, an otherwise unknown mathematician, is mentioned twice in the *A'māl wa ashkāl*: "Master craftsmen had questioned [Abu Bakr al-Khalid] about the different ways in which a particular geometric construction could be drawn; one of his solutions is explained in an accompanying diagram" [Necipoğlu 1995: 168]. *A'māl wa ashkāl* also contains a second geometric construction by Abu Bakr al-Khalid, which shows how to draw a pentagon inscribed with a five-pointed star by using the chord of an arc as the module [Necipoğlu 1995]. Even though the *A'māl wa ashkāl* provided very complicated, scientifically correct geometric constructions, it also provided simpler methods for constructing the same patterns [Necipoğlu 1995].

In addition, the use of memorized grids may also have been used for recreating familiar and well-established patterns already known to the artisans. Architectural scrolls provide evidence that polygonal and radial grids composed of concentric circles may have been used by master builders in Persia between the tenth and sixteenth centuries. These grids consisted of black and/or red inked construction lines along with uninked "dead" lines lightly scratched on the paper with a sharp metal tool (such as the pointed end of a compass). According to a note written on one of Akbar's nineteenth-century scrolls, "The uninked drypoint tracing indicates the basis of the formation of figures" [Necipoğlu 1995: 14].

Since meetings were held between mathematicians and craftsmen starting in the tenth century, and "practical geometry" manuals instructing artisans how to perform basic geometry constructions were written, it seems plausible that some master builders were capable of using geometric construction techniques to generate patterns. With this in mind, we propose the following plausible Euclidean "point-joining" compass-and-straightedge reconstructions for each of the two designs in an attempt to discover how the original designer of these patterns may have determined, without mensuration, the proportion and placement of the star polygons. We start first with the construction of the regular polygons, or circles partitioned into congruent arcs, in which the stars will be situated.

Constructing star polygon designs inscribed within regular (or approximately regular) polygons

One of the most straightforward techniques for creating geometric "heavenly star" patterns found throughout the Islamic world is to initially construct a p-gon (a polygon with p sides) inscribed in a circle and then draw in the corresponding regular p-pointed regular star polygon by methodically joining the qth vertices of the p-gon with line segments (diagonals) or by methodically joining midpoints of the p-gon's edges. (Recall that a regular polygon is one in which all the sides have the same length and all the internal angles formed by two adjacent edges have the same measure.) A figure formed in this way is mathematically designated as a $\{p/q\}$ star polygon, where p and q are relatively prime positive integers, with $q < p/2$. If p and q are not relatively prime the star polygon is considered *improper*, or a *star figure* instead. For example, to form the regular {5/2} star polygon, every second vertex of a regular pentagon is connected with line segments as shown in fig. 4a. By connecting alternate midpoints of edges of the same regular pentagon, a smaller regular pentagonal star figure may be formed, as shown in fig. 4b. Hence, star polygons or star figures with n points may be created within any n-gon, where every second (or third or fourth or k^{th}, where $2 \le k \le n/2$) vertex (or midpoint) is connected. (Note that portions of the segments have been highlighted to form the more decorative stars).

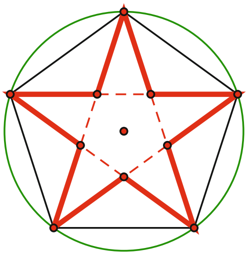

Fig. 4a. A {5/2} star

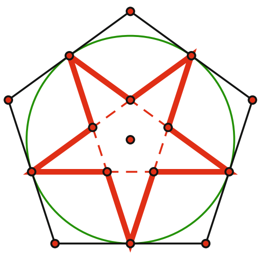

Fig. 4b. A {5/2} star

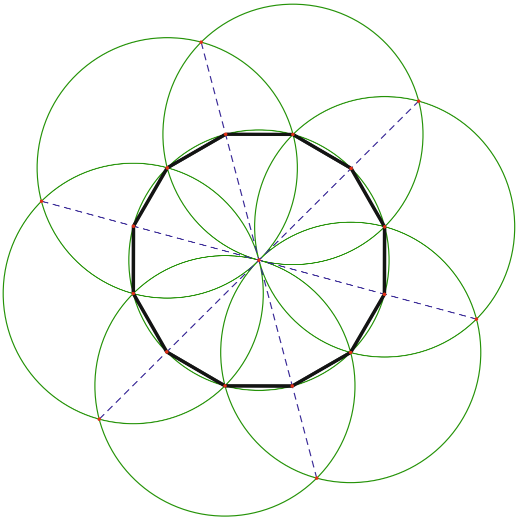

Fig. 5a. Construction of a 12-gon from seven congruent circles

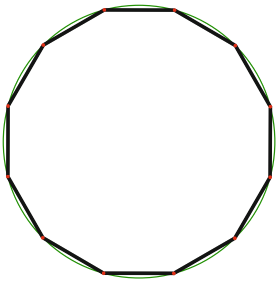

Fig. 5b. The 12-gon inscribed in a circle

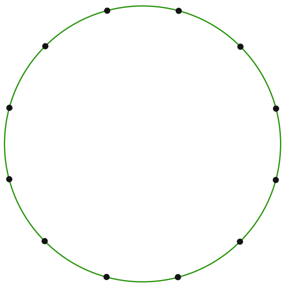

Fig. 5c. The circle 12-gon from seven congruent circles partitioned into twelve congruent arcs

The most common regular star polygon designs have n points, where n is 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, and so on. It should be noted that these regular n-gons are *constructible* in the Euclidean sense; that is, they may be constructed using only a compass and straightedge. For n = 7, 9, 11, 13, 14, 18…, the regular n-gons (and likewise, the corresponding regular n-star polygons) may only be constructed approximately using these tools (for more about this, see the Wikipedia page at http://en.wikipedia.org/wiki/Constructible_polygon and [Sarhangi 2007]). Of these *non-constructible* star polygons, master builders in the eastern regions of the Islamic world often created star patterns with n = 7, 9, 14 and 18.

To create the twelve-pointed stars for both the *TS* and *B120* designs, we first need a regular 12-gon, which may be produced from the construction of six congruent ("outer") circles passing through a common point, which is the center of a seventh "inner" circle. The intersection of these six outer circles with the inner circle produces six equally spaced points on the inner circle. Two adjacent outer circles also intersect to form points outside the inner circle. The six outer intersection points formed in this manner may now be connected with three line segments that pass through the inner circle's center in such a way as to produce six additional points of intersection on the inner circle that are midway between the already extant points. Together, these twelve points on the inner circle may now be connected with line segments to form an inscribed regular 12-gon, as shown in figs. 5a and 5b. If we erase the segments forming the 12-gon, we are left with a circle partitioned into twelve congruent arcs (fig. 5c).

To produce the *TS* 12-star design, connect every fourth point along the circle with line segments to form a {12/4} star figure, highlight the desired segments (fig. 6a) and then erase those no longer needed (fig. 6b). Similarly for the *B120* 12-star design, connect every fifth point along the circle with line segments to form a {12/5} star polygon, highlight the desired segments (fig. 6c) and then erase those no longer needed (fig. 6d). It is evident that the 12-stars depicted in figs. 6b and 6d have clearly different structures; that is, the length of line segments and the angles formed when the segments meet differ markedly from one star to the next. More importantly, the star in fig. 6d is a {12/5} star polygon but the one in fig. 6b is a {12/4} star figure resulting from four overlapping equilateral triangles.

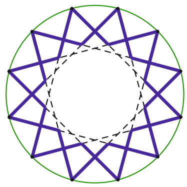

Fig. 6a. A {12/4} star

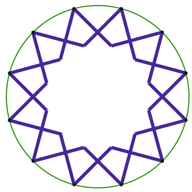

Fig. 6b. The *TS* 12-star design

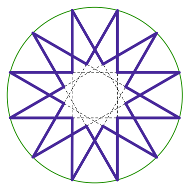

Fig. 6c. A {12/5} star

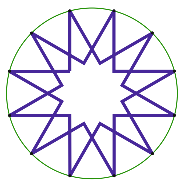

Fig. 6d. The *B120* 12-star design

Likewise, to create the nine-pointed stars of the *B120* and *TS* designs, we first need to construct an approximately regular 9-gon, or a circle partitioned into nine congruent arcs. As previously mentioned, it is impossible to construct a regular 9-gon using only a compass and straightedge. The actual construction of the two 9-gons – one for each design – will be shown in their upcoming sections, respectively. However, if given a 9-gon, one could partition a circumscribing circle into nine congruent arcs and then create the *TS* 9-star design by connecting every third point of nine along the perimeter with line segments to form a {9/3} star figure, highlight the desired segments (fig. 7a) and then erase those no longer needed (fig. 7b).

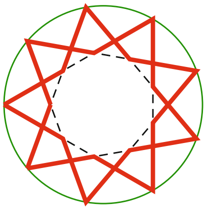

Fig. 7a. A {9/3} star

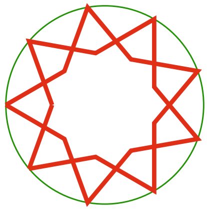

Fig. 7b. The *TS* 9-star design

To create the *B120* 9-star design within a given 9-gon, partition a circle into eighteen congruent arcs and connect every seventh point with line segments, thereby creating a {18/7} star (fig. 8a). By highlighting every other star point, as shown in fig. 8b and erasing the remaining dashed line segments yields the 9-pointed star polygon with the same proportions as those found in the *B120* design (fig. 8c). It is also apparent that the 9-stars depicted in figs. 7b and 8c have clearly dissimilar structures; that is, the length of line segments and the angles formed when the segments meet differ noticeably from one star to the next. More importantly, the star in fig. 7b is a {9/3} star figure but the one in fig. 8c is the result of highlighting half of the line segments comprising an {18/7} star.

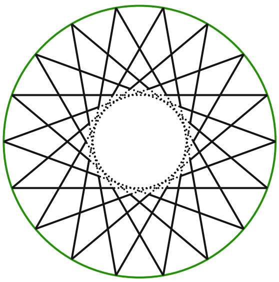

Fig. 8a. A {18/7} star design

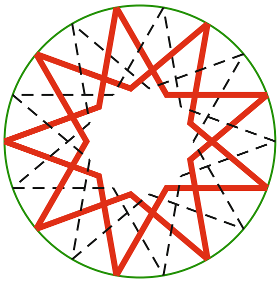

Fig. 8b. Half of the {18/7} star highlighted

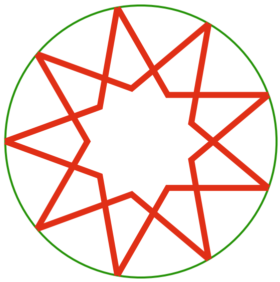

Fig. 8c. The *B120* 9-star design

Note also that the line segments comprising the *B120* 12-star image (fig. 6d) and those of the *B120* 9-star image (fig. 8c) have sets of parallel segments. This is not the case for the line segments comprising the *TS* 12-star image (fig. 6b) and the *TS* 9-star image (fig. 7b). Before we continue with the construction of the designs – one of which may be generated from a *repeat unit* which is then replicated using symmetry operations, and the other which will be constructed without regard to a repeat unit – let's first consider the source of each design.

The Tashkent Scrolls (TS) design

The source of the *TS* design, as previously mentioned, is a sketch belonging to the collection of post-Timurid design scrolls known as the *Tashkent Scrolls*, which "appear to have been a catalogue of ideal two- and three-dimensional designs … identified by practicing Persian builders as *girih*" [Necipoğlu 1992: 50]. Some of the drawings, contained within rectangular or square regions, represent only a small portion, or *repeat unit,* of an overall geometric pattern that may be produced by the symmetry operation of reflection, across the edges of the rectangle or square, or rotation, about a corner of the square. Beneath some of the illustrations drawn in black ink are also "dead" construction lines, arcs and circles that were scratched on the paper with a compass or other sharp implement "so as not to detract from the inked patterns they generate" [Necipoğlu 1995: 31]. As is also the case for the *Topkapı Scroll*, the sketches which are

> [u]naccompanied by explanatory texts or measurements … appear to have served as an *aide-mémoire* for master builders who were already familiar through experience and empirical knowledge with the language of the depicted forms. [Necipoğlu 1992: 50].

Here I have created a facsimile of the *TS* sketch by tracing the image and producing its copy using the Geometer's Sketchpad® software (fig. 9a). The repeat unit contains two halves of nine-pointed star polygons along its horizontal edges and two quarters of twelve-pointed star polygons in the upper left and the lower right corners of the rectangle. Between the 9- and 12-star polygons (which include the petals) there are six irregularly-shaped pentagonal stars, two non-regular hexagons (similar in size and shape to the star petals) and two overlapping arrow-like polygons that produce a parallelogram in the overlapping region they share at the center of the rectangle. Fig. 9b shows a second facsimile (also created by me) of the *TS* design's overlay where the inked pattern (as dashed line segments) is superimposed on the uninked "dead" construction line segments, circles and arcs (in bold). Note that the "nearly regular" 9-stars, the regular 12-stars and the irregularly shaped pentagonal stars drawn in black ink in fig. 9 are constructed using inscribed and circumscribed circles. The remaining polygonal shapes may be achieved by joining certain intersection points with line segments. Also, notice that the pattern in the repeat unit has 180° rotational symmetry about the center of the repeat unit's rectangle (located within the parallelogram of the two arrow-like shapes), which is useful in the construction of the pattern.

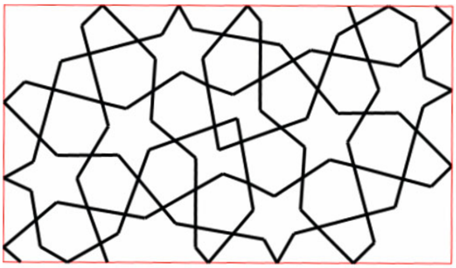

Fig. 9a. Author's reconstruction of the TS design produced using Geometer's Sketchpad®

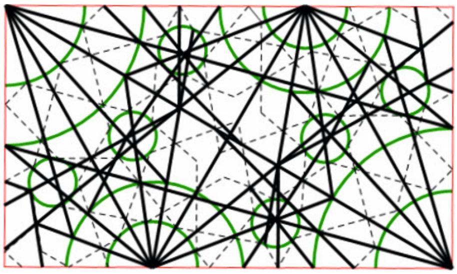

Fig. 9b. Author's construction of an overlay of the TS design produced using the Geometer's Sketchpad®

The *TS* design's repeat unit is interesting for several reasons. First, the major stars in the vast majority of Islamic star designs usually have an even number of points, and this has major stars with both an even number of points (the twelve-pointed stars) and an odd number of points (the nine-pointed stars). Second, the repeat unit contains 9-star polygons, presumably created from an underlying grid consisting of 9-gons, which are *not constructible* using the traditional drafting tools of straightedge and compass, so most likely an approximation was utilized to create them. Third, the position of the stars in the repeat unit is remarkable as well. Usually the major stars appear centered at either corner vertices (as the twelve-pointed stars are) or at the midpoints of the edges in the repeat units of Islamic star designs. But for the *TS* design, the centers of the major nine-pointed stars are not located in these standard positions.

Construction of the *TS* design repeat unit rectangle

In this section, we outline a method to construct the rectangle containing the polygons that comprise the *TS* design's repeat unit. Starting with a line segment of any length, construct two circles centered about the two endpoints and sharing the same radius. Using one of the points of intersection between these two circles as its center, construct a third circle through the endpoints of the line segment. Connect the centers of these three circles with line segments to form an equilateral triangle (fig. 10a). Extend the horizontal segments until they intersect and terminate on these circles. Also construct a segment that connects two of the points of intersection of two of the circles to form a second equilateral triangle (fig. 10b). Finally, construct two perpendicular line segments through existing points that terminate at the horizontal segments (fig. 10c). Note that the two resulting right triangles at either end of the rectangle are 30-60-90 triangles, halves of equilateral triangles. Erasing the circles and any unneeded segments yields the *TS* design's repeat unit rectangle (fig. 10d).

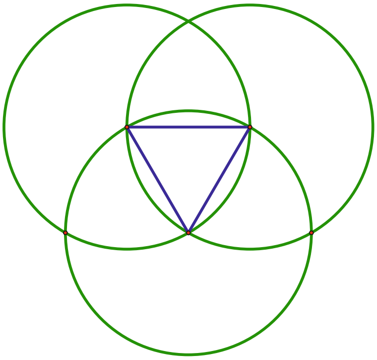

Fig. 10a

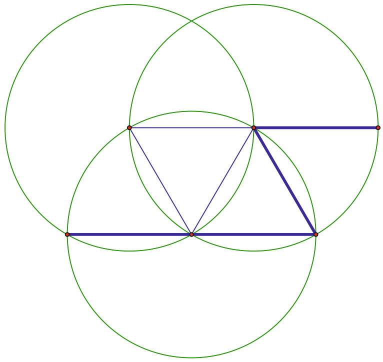

Fig. 10b

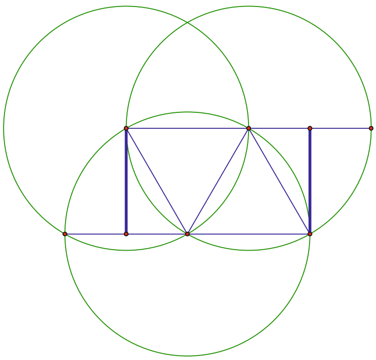

Fig. 10c

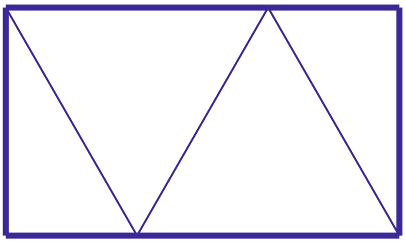

Fig. 10d

Construction of the twelve-pointed star polygons of the *TS* design

Constructing a diagonal line segment between the upper left corner and the lower right corner of the rectangle creates two additional, larger 30°-60°-90° right triangles and four smaller ones (congruent to the first two right triangles formed when the perpendiculars were constructed) from the halving of the equilateral triangles (fig. 11a). Three of the small right triangles share at their 30° angles the upper left corner point of the rectangle. Likewise, the lower right corner point of the rectangle is the vertex that the other three, small right triangles share at their 30° angles. We bisect all six of these 30°angles to produce the grids shown in figs. 11b and 11c.

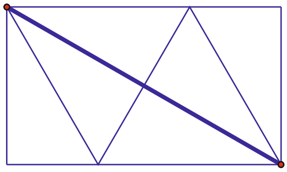

Fig. 11a

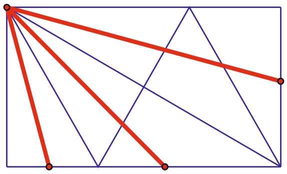

Fig. 11b

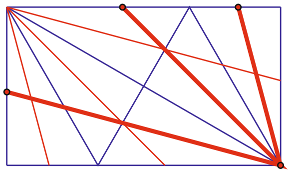

Fig. 11c

Next we construct congruent circles centered at the upper left corner and the lower right corner, using the existing points on the vertical edges of the rectangle (fig. 12a). To determine half of the vertices of a 12-gon, construct the three points of intersection formed between the upper left circle and existing segments within the rectangle, as well as three points formed when these same segments are extended until they intersect the portion of the circle outside the rectangle. The construction of six angle bisectors produces the radii needed to construct the remaining six points of intersection that divide this circle into twelve congruent arcs. To form a regular 12-gon, construct line segments between these twelve points, two adjacent ones at a time, as shown in the upper left corner of the rectangle in fig. 12b.

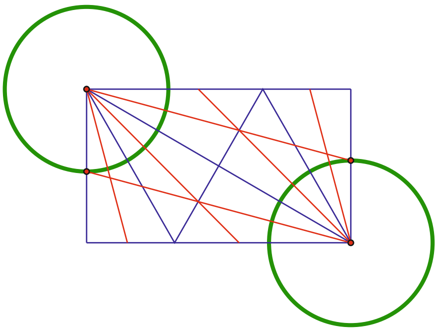

Fig. 12a

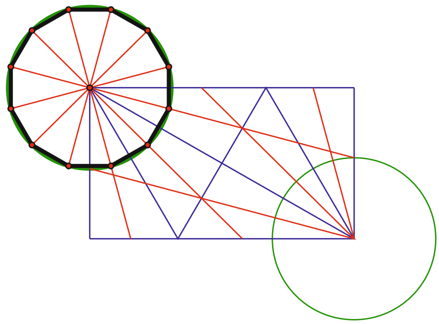

Fig. 12b

Construct a {12/4} star figure by joining every fourth point of the 12-gon – this creates four overlapping equilateral triangles, as shown in fig. 12c, on the following page. (The circle is erased in this figure to more clearly illustrate the 12-gon.) By erasing some segments and highlighting others, we generate the more decorative twelve-pointed star polygon image centered on the upper left corner of the rectangle (fig. 12d). Note that the 12-gon is erased leaving the circle partitioned into twelve congruent arcs. In a similar manner, a twelve-pointed star may also be constructed, centered on the lower right corner (fig. 12e).

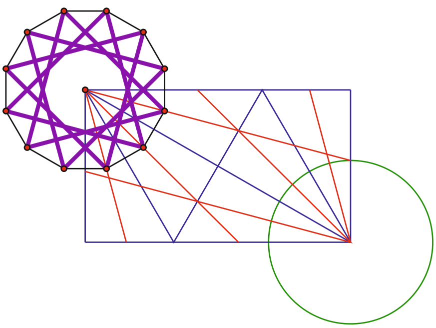

Fig. 12c

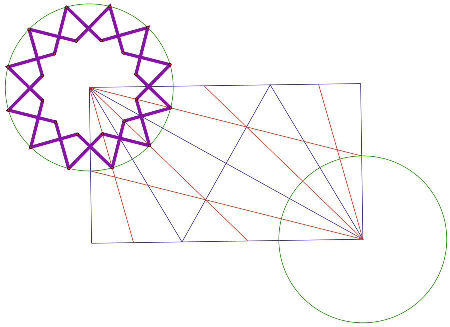

Fig. 12d

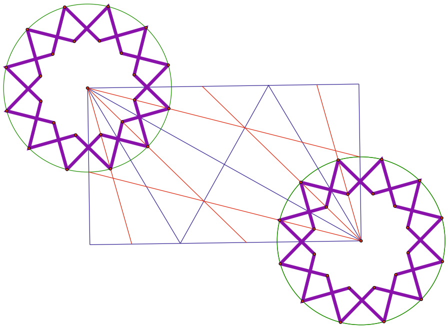

Fig. 12e

Construction of the nine-pointed star polygons of the *TS* design

Before we can construct the nine-pointed star, we need to draw two additional line segments – one in the lower left corner and one in the upper right corner of the rectangle – by connecting existing points on adjacent edges of the rectangle. Next, construct a large parallelogram using points of intersection of the rectangle's diagonal with the circles containing the twelve-pointed stars and existing points on the horizontal edges of the rectangle (fig. 13a). The nine-pointed stars will be centered on the two vertices of the parallelogram that are coincident with the horizontal edges of the rectangle. Finally, we may construct angle bisectors (fig. 13b).

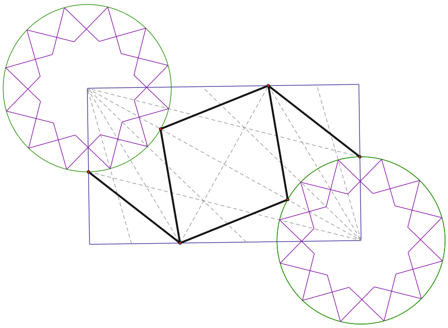

Fig. 13a

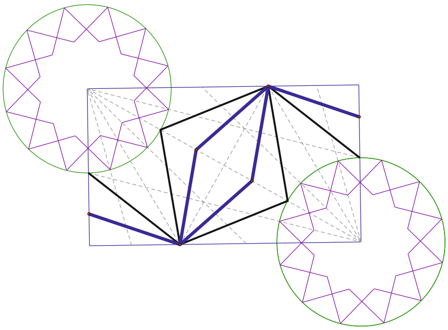

Fig. 13b

We are now ready to construct the nine-pointed stars. First, construct two congruent circles centered on the two vertices of the parallelogram that are coincident with the horizontal edges of the rectangle and through the points of intersection where the two segments emanating from the 12-stars (that are closest to the rectangle's width) intersect the horizontal edges of the rectangle. Then, construct four points of intersection (within the rectangle) between the upper right circle and the existing line segments (fig. 14a). To obtain four additional points on this circle outside the rectangle, extend four existing segments (one emanating from the lower right corner of the rectangle, two emanating from the angle bisector segments within the parallelogram, and a fourth from an angle bisector in the upper right corner of the rectangle). These nine points now divide the circle into nine nearly congruent arcs, which, if joined by line segments, would form an approximately regular 9-gon. (In an attempt to minimize the clutter in the diagram, we did not construct the inscribed 9-gon but instead just show the partitioned circle.) A {9/3} star figure may be created by joining with line segments every third point of the nine points found along the circle, produced from three "nearly equilateral" triangles (fig. 14b).

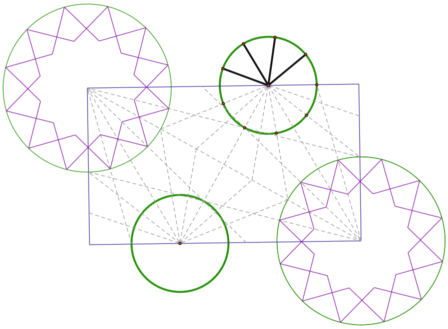

Fig. 14a

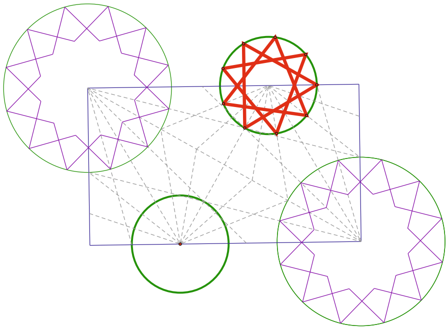

Fig. 14b

By erasing some segments and highlighting others, we generate the more decorative star image (fig. 14c). We may repeat the same procedure to create a second nine-pointed star in the second circle along the lower horizontal edge of the rectangle (fig. 14d).

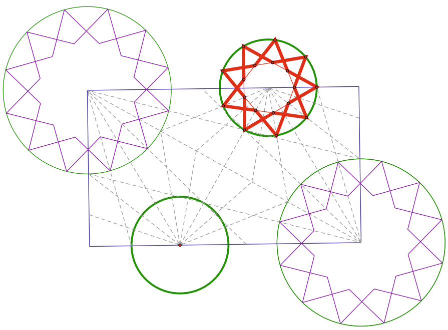

Fig. 14c

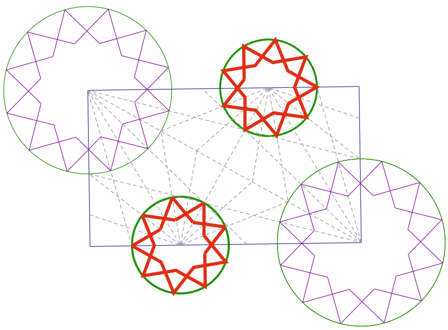

Fig. 14d

Construction of the polygons found within the interstitial space of the *TS* design

With the construction of the major 12- and 9-stars completed, we only need now to fill in the interstitial space with the irregularly-shaped pentagonal stars, hexagons and arrow-like shapes. By extending some of the line segments of the major stars until they meet existing segments (their points of intersection are denoted in fig. 15a by the dots), we can construct large circular arcs centered on these stars and through these points to determine additional segments also obtained by extending existing line segments, also shown in fig. 15a (these points of intersection are drawn without the dots). Note that the original circular arcs circumscribing the major stars were erased for clarity. Line segments may now be drawn between these points, as shown in fig. 15b.

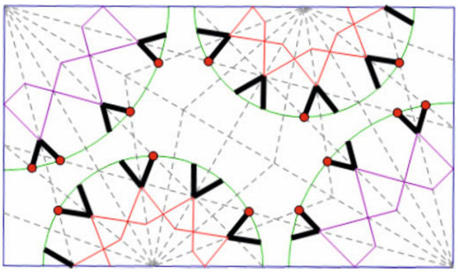

Fig. 15a

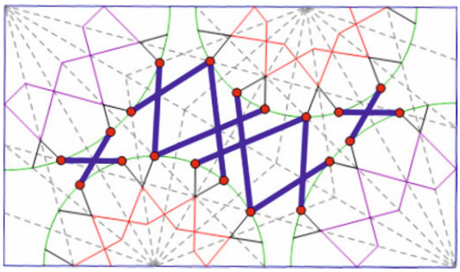

Fig. 15b

Additional points of intersection, which were constructed from some of these segments, may now be used to determine the radius of two new circular arcs drawn in the upper left and lower right corners of the rectangle, about the 12-stars. (The previously used arcs were erased for clarity.) The points of intersection between these new arcs and the edges of the rectangle yield a few additional points and hence line segments (fig. 15c). Similarly, two more circular arcs may now be drawn about the centers of the 9-stars to also determine additional points and line segments (fig. 15d).

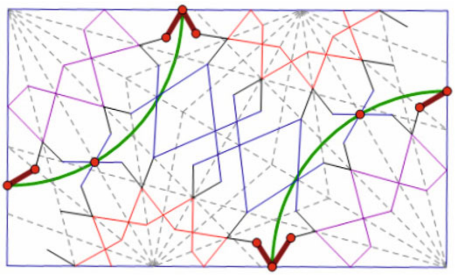

Fig. 15c

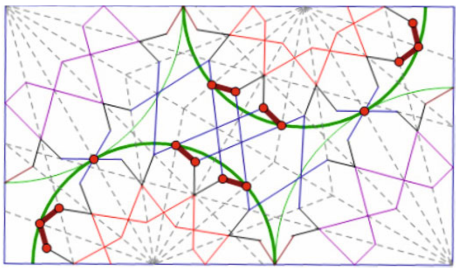

Fig. 15d

Extending some of the segments constructed in the previous step allows for the completion of the pentagonal stars and hexagons (fig. 15e). Lastly, short line segments in bold are drawn through the existing points on the vertical edges and parallel to the non-bold segments of the 9-stars (fig. 15f).

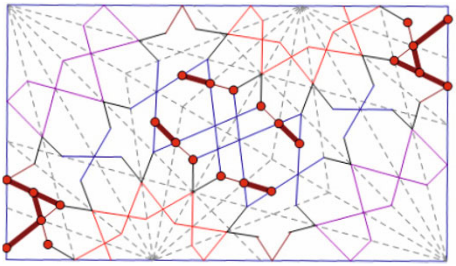

Fig. 15e

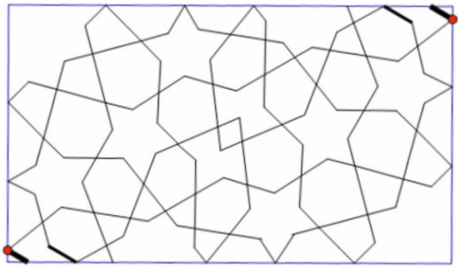

Fig. 15f

Erasing the unneeded segments yields the completed repeat unit in skeletal form (fig. 15g) and a colorized version can be created using the Microsoft Paint program (fig. 15h).

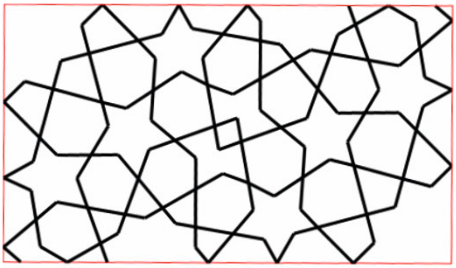

Fig. 15g

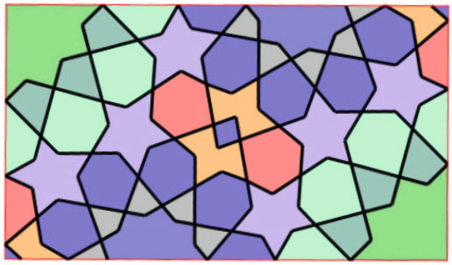

Fig. 15h

Sixteen copies of the colorized repeat unit, replicated by reflection across the edges of the rectangle, are shown in fig. 16.

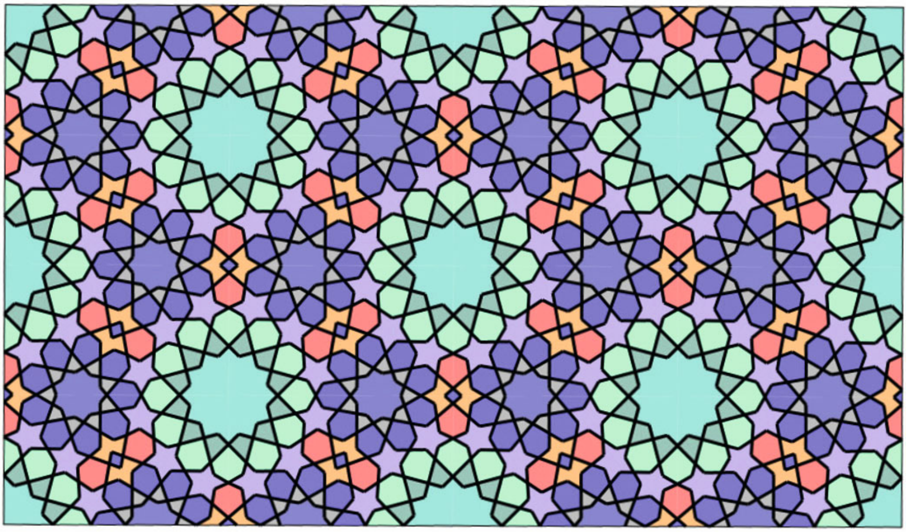

Fig. 16. Sixteen copies of the *TS* repeat unit, colored and replicated by reflection across the rectangle's edges

The left side of fig. 17 shows a portion of the photograph reproduced in fig. 2 above [Wade: Ind 0729], cropped and rotated so that it has the same orientation as my reconstructed *TS* design. The interior of the stars in the Ind 0729 pattern contain additional embellishment, and the arrow-shapes do not overlap to form a parallelogram, as is the case for the *TS* repeat unit, but otherwise the skeletal patterns are identical.

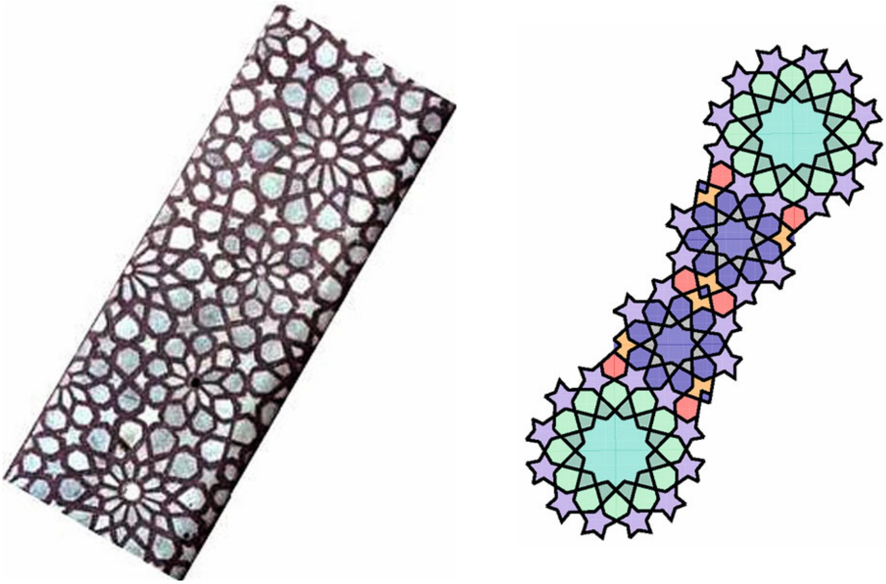

Fig. 17. left) A portion of the Masjid-i Jami in Fatehpur Sikri, India [Wade: Ind 0729] (photograph cropped and rotated by this author); right) the author's cropped reconstruction

Construction of the B120 *design*

One source for the next nine- and twelve-pointed star design is Plate 120 of Bourgoin's *Arabic Geometrical Pattern and Design.* Bourgoin copied these designs from Islamic monuments in Cairo and Damascus and lightly sketched some circles and lines that underlie his constructions. But this is after the fact, an analysis of a completed design. He does not indicate at all how the design might have been achieved. So in this section, I will outline a plausible Euclidean construction that produces the *B120* design without regard to a repeat unit, as was the case for the *TS* design. Instead the focus is on the completed design and how to properly position the major stars within it.

Construction of the twelve-pointed star polygons of the *B120* design

First, divide a circle of any size into twelve arcs of equal measure using the following method. Construct a square, and then using the diagonals to determine the center, inscribe it within a circle. Construct three additional, smaller squares, using the midpoints of each larger square as the vertices of the next smaller square (fig. 18a). Erase the segments comprising the three outer squares, reserving the smallest innermost square and then construct six ("outer") circles all congruent to the original ("inner") circle. The outer circles are all coincident with the center of the inner circle and their intersections with the inner circle define six of the twelve points needed. Two adjacent outer circles also intersect at points outside the inner circle, and altogether there are 6 of these outer intersection points. Connect the opposite outer points with three line segments that pass through the inner circle's center point. These segments will intersect the inner circle forming the six additional points needed to divide the inner circle into twelve arcs of equal measure, as shown in fig. 18b. Erase the six unneeded outer circles and the three line segments. To construct twelve additional points midway between the twelve existing points already on the circle, bisect the angles formed by rays emanating from the inner circle's center point and two adjacent points on the circle (fig. 18c). The twelve bisectors will produce the needed twelve points.

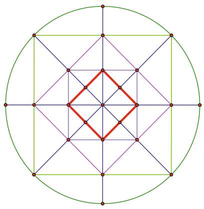

Fig. 18a

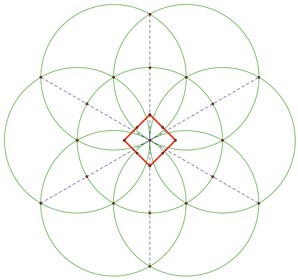

Fig. 18b

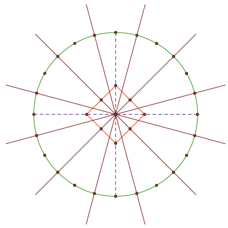

Fig. 18c

Second, construct a {12/5} star by following these steps. Erase the twelve angle bisectors and unneeded points. Connect every second point of the 12 remaining on the circle with line segments to form a {12/2} star figure (fig. 19a). These segments form two overlapping regular hexagons. Draw line segments through adjacent midpoints of the small square (at the center of the circle) until they intersect the segments forming the outer {24/2} star figure. Two such segments are shown in fig. 19b, with all of them shown in fig. 19c. Erasing the segments comprising the inner square and other segments now no longer needed, while highlighting others, yields the more decorative star shown in fig. 19d. Highlighting segments that complete the petals of the star and erasing some unneeded segments yields the image in fig. 19e. Erase the original circle to yield the motif we will call the completed 12-star, which includes the hexagonal petals (fig. 19f).

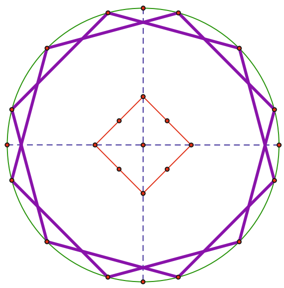

Fig. 19a

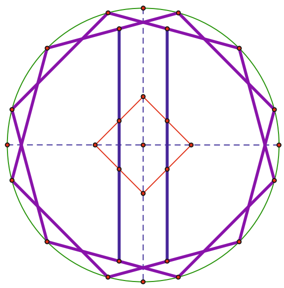

Fig. 19b

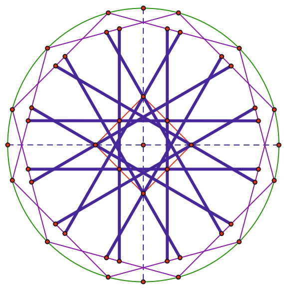

Fig. 19c

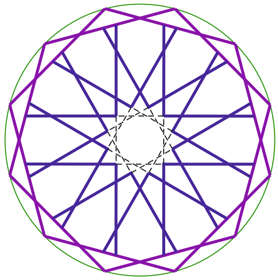

Fig. 19d

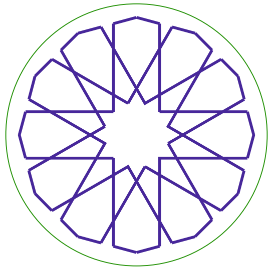

Fig. 19e

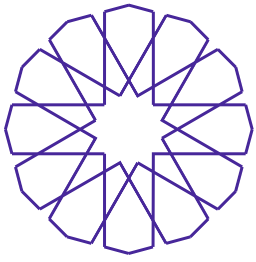

Fig. 19f

Construction of the nine-pointed star polygon of the *B120* design

To construct a nearly regular 9-gon, and within it an {18/7} star (which in turn will produce the requisite 9-star), follow these steps. Starting with a 12-star (fig. 19f), construct a circumscribing circle about it. (Note that this is ***not*** the same circle that was used to create the 12-star in the original construction and that is shown in fig. 19e). Then construct an equilateral triangle whose edges must measure three times the length of the radius of this circumscribing circle (fig. 20a). Centered on each of the remaining two vertices of the triangle, construct two additional 12-stars in the manner previously described and erase the circles (fig. 20b).

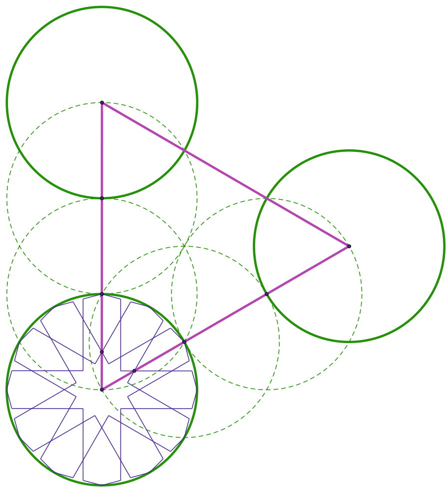

Fig. 20a

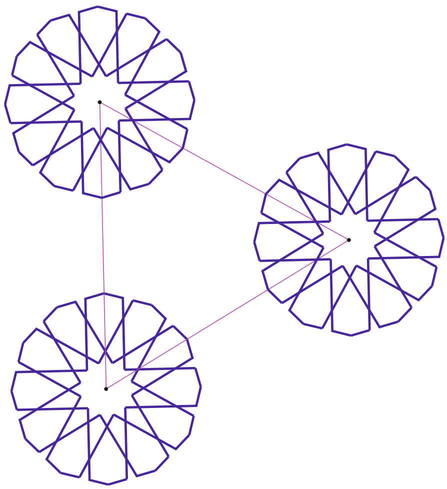

Fig. 20b

We may now create an approximately regular 9-gon in the interstitial space by constructing the nine line segments between extant points (fig. 21a). Six of the interior angles of the 9-gon shown in fig. 21b measure approximately 140.10° (as measured by the Geometer's Sketchpad® software program) and the remaining three interior angles measure 139.79°. The theoretical angle measure of a regular 9-gon would be exactly 140°. Hence the measure of these constructed angles differs by approximately .10 to .21 of a degree from the exact (theoretical) angle measure.

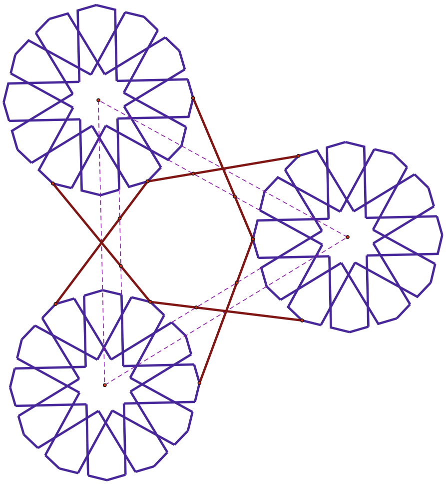

Fig. 21a

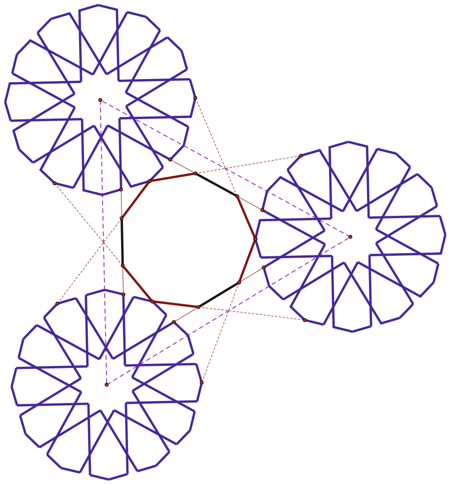

Fig. 21b

It is within this 9-gon that we will now construct the 9-star. Extend three line segments emanating from the centers of the three 12-stars, parallel to existing segments, until they intersect at the midpoints of the segments joining the centers of these 12-stars. These segments intersect at the center of the 9-gon. Also extend six line segments parallel to these three line segments until they intersect each other near the center of the 9-gon (fig. 22a). Construct two additional segments through existing points and construct a circle centered within the 9-gon and through the point of intersection of these two segments (fig. 22b). Twelve of the eighteen needed points of intersection may now be constructed on the circle where the line segments cross.

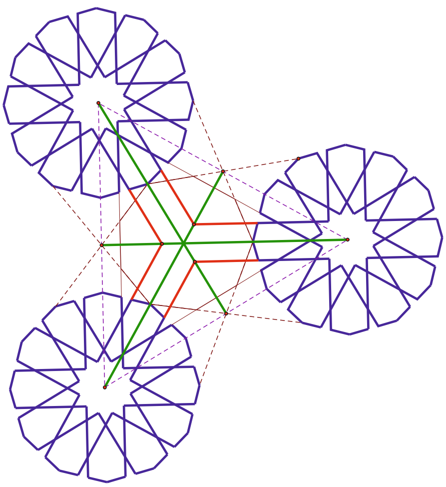

Fig. 22a

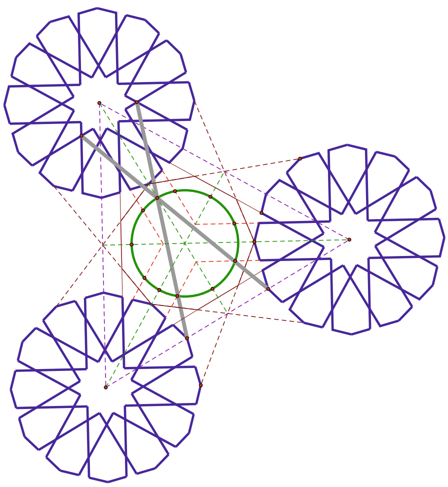

Fig. 22b

Just six additional points on the circle are needed to construct the 9-star. To obtain these, construct segments from the center of the 9-gon to the vertices of the 9-gon. These segments and the requisite eighteen points on the circle are shown in fig. 22c. Erase unneeded segments. Construct an {18/7} star now by connecting with line segments every seventh point of the eighteen along the circle, as shown in fig. 22d.

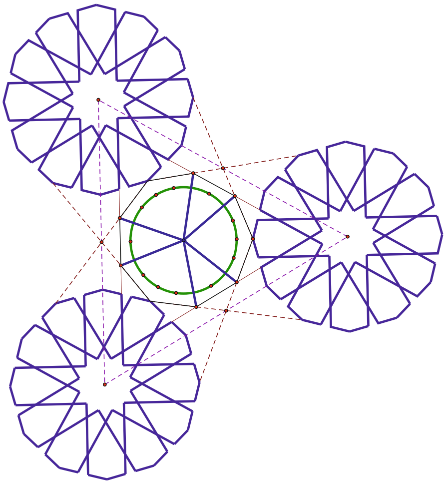

Fig. 22c

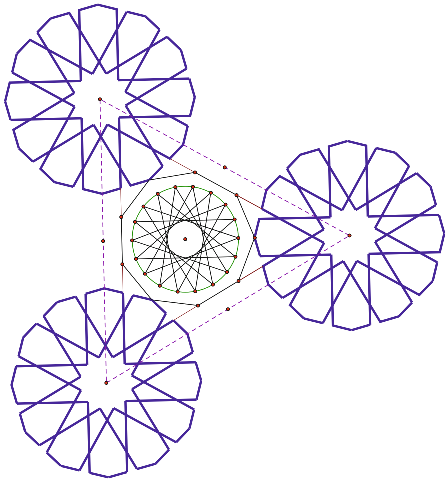

Fig. 22d

Erase the unneeded line segments (half of the segments comprising the {18/7} star) to form the 9-star shown in fig. 22e. Extend existing line segments that form the 9-star until they meet the 9-gon and highlight other existing segments to enclose the hexagonal petals of the star (fig. 22f).

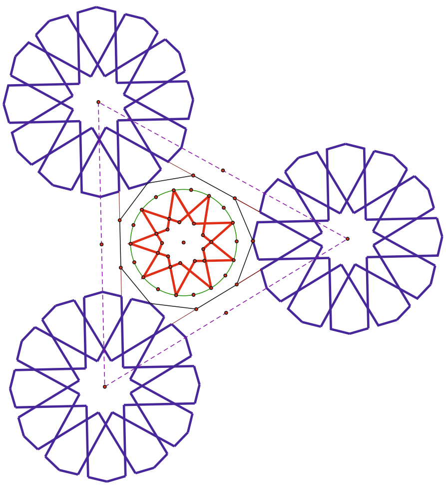

Fig. 22e

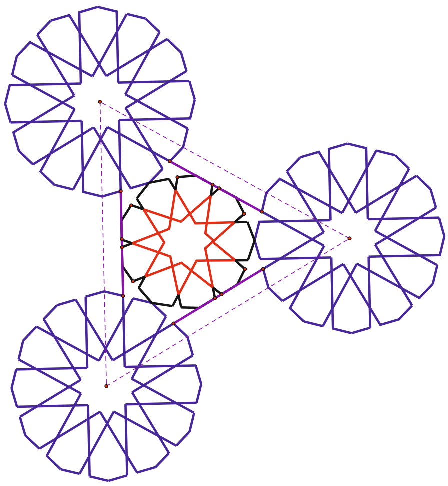

Fig. 22f

Erase unneeded line segments and the circle circumscribing the 9-star, and connect existing points to form the six pentagonal stars (fig. 22g). Erasing some line segments and highlighting others will produce more decorative pentagonal stars. Lastly, the three "arrow" shapes may be constructed by connecting existing points on the 9-gon with the midpoints of the segments joining the three 12-stars (fig. 22h). Having found a way to construct the requisite polygons in the interstice between the three 12-stars, the rest of the design (shown in fig. 3a at the beginning of this paper) can be completed by using symmetry and repetition.

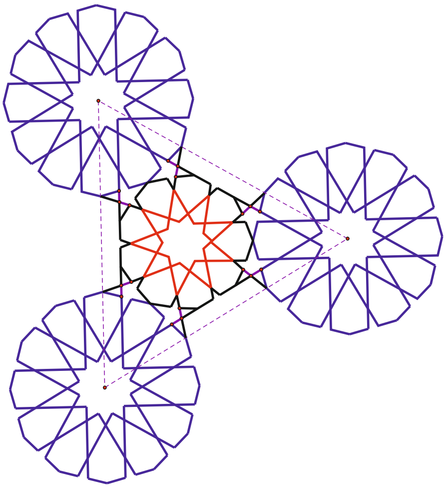

Fig. 22g

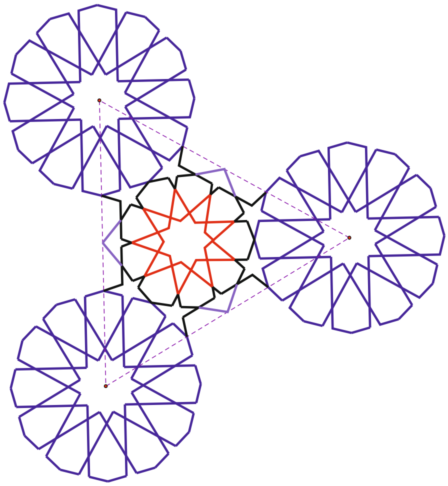

Fig. 22h

Since the Sultaniyeh example of the *B120* design (the [Wade: Ira 2701] photograph) is contained within a hexagon, we have created a copy of the reconstructed design also within a hexagonal frame for comparison purposes. Aside from the embellishments evident in the photograph, which is also slightly distorted due to the perspective of the photograph, one can see that the reconstructed pattern is identical to a skeletal version of the original, as shown in fig. 23.

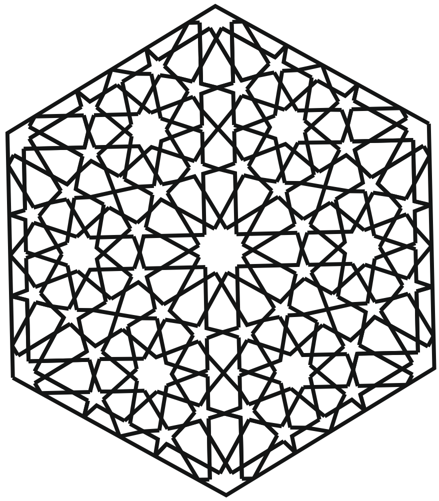

Fig. 23. left) A portion of a ceiling in the Mausoleum of Sultan Oljaytu in Sultaniyeh, Iran, dated 1313, from the Ilkhanid dynasty [Wade: Ira 2701] (photograph cropped by this author); right) the author's reconstruction

Discussion

In this paper we have described two plausible Euclidean "point-joining" compass-and-straightedge constructions for the two different nine- and twelve-pointed Islamic star polygon designs. The first design, designated as the *TS* design, was completed by recreating to the exact design proportion and placement, the polygons within a repeat unit of a sketch from the *Tashkent Scrolls*. The other design, designated as the *B120* design, was completed by first constructing three copies of the appropriate regular 12-star and then finding a way to construct a nearly regular 9-gon and then a 9-star in the interstitial space. Both methods were relatively straightforward and easy to accomplish, showing that they are plausible constructions of the designs. Although we found only one extant example of the *TS* design [Wade: Ind 0729], the *B120* design may be found rather ubiquitously in monuments across the Islamic world, from Spain to Egypt to Turkey and to India. Ironically, neither of these two nine- and twelve-pointed star polygon patterns may be found in the *Topkapi Scroll*.

The designs were very similar in overall structure; that is, the number and placement of the major stars were identical and so (in skeletal form) they may both be mathematically classified as belonging to the same crystallographic group – both are p6m designs, whereby the design elements exhibit six-fold rotational symmetry and mirror reflections, as shown for the *B120* design in fig. 24. (The analysis would be the same for the *TS* design shown in fig. 3b.) The completed pattern more clearly shows the six-fold nature of the design's rotational symmetry. Notice the six 9-stars that are equispaced at 60° angles about the center point of the central 12-star. These, in turn, are surrounded by two groups of six 9-stars and six 12-stars. A few mirror axes are also indicated on the design by the dashed line segments.

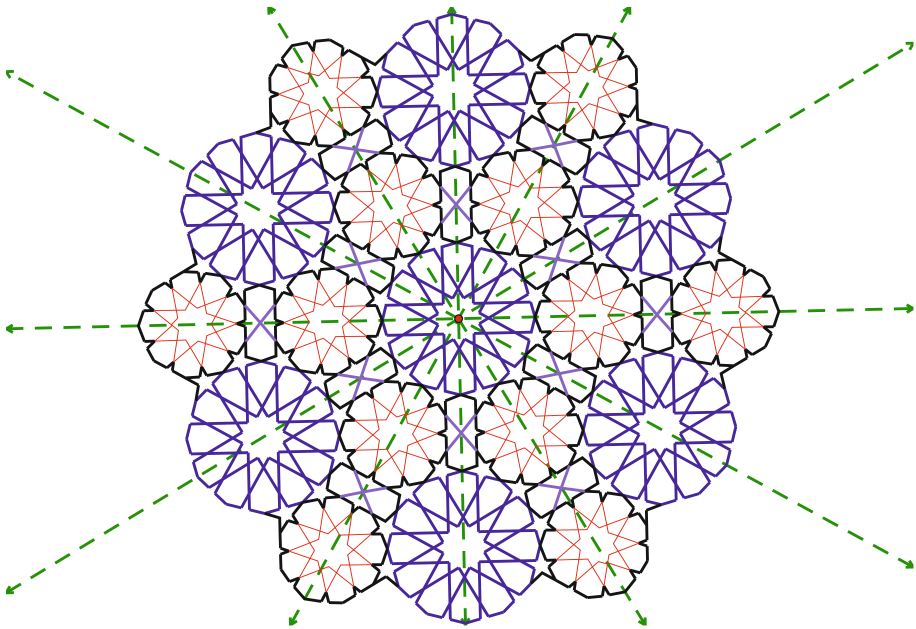

Fig. 24. The completed design, showing the p6m symmetry

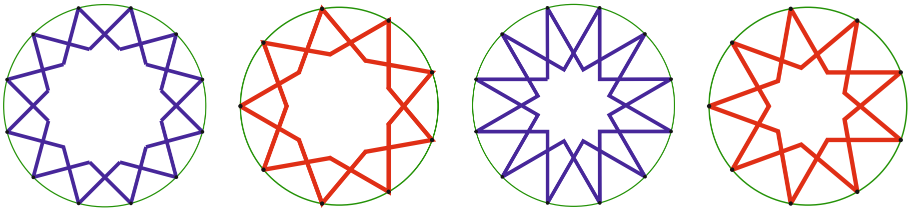

Fig. 25. The TS stars on the left ({12/4} and {9/3} star figures), and the B120 stars on the right ({12/5} and (derived from) {18/7} star polygons

It was, however, the major stars themselves that were structurally quite different despite having the same number of points, and so on, as is most evident by comparing the stars in fig. 25. Reconciling their differences was the most interesting aspect of this study.

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About the author

B. Lynn Bodner has taught a wide variety of mathematics courses to college students over the past thirty years. For the last twenty-two of these, she has especially enjoyed teaching an interdisciplinary course at her university entitled the Mathematics of Artistic Design. Of the many topics considered in that course, geometric Islamic patterns have held the most fascination and appeal for her, in addition to being a source of limitless geometric investigation.