Carol Bier

Center for Islamic Studies
Graduate Theological Union
2400 Ridge Road
Berkeley CA 94709 USA
carol.bier@gmail.com

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Research

The Decagonal Tomb Tower at Maragha and Its Architectural Context: Lines of Mathematical Thought

Abstract. Of several brick tomb towers constructed at Maragha in western Iran before the Mongol conquests, one in particular, Gonbad-e Qabud (593 H. / 1196-97 C.E.), has generated significant recent attention for its unique patterns with pentagons and decagons. Gonbad-e Qabud is also unusual in having a decagonal plan. While both plan and decoration distinguish it from earlier and later towers at Maragha and elsewhere on the Iranian plateau, the ornamental patterns follow a long line of experimentation with geometric expressions that grace many pre-Mongol buildings in Iran. This article examines in particular the overlapping polygons and radial symmetries of the tympanum of the cubic Gonbad-e Sork (542 H. / 1148 C.E.) at Maragha, and the pentagons and squares of the tympanum of the later octagonal tomb tower (486 H. / 1093 C.E.) nearby at Kharraqan. Drawing from archival sources (plans, elevations, photographs), analysis of plane patterns, and comparative architectural data, this article reevaluates the cultural significance of Gonbad-e Qabud, seeking to situate it within the histories of mathematics, architecture, and the arts.

Introduction: Decagonal intentionality

The decagonal and pentagonal symmetries present in the geometric ornamentation of the Gonbad-e Qabud at Maragha in western Iran (figs. 1, 10) have attracted widespread attention among scientists and mathematicians in recent years [Bier 2011a; Bonner 2003; Cromwell 2009; Lu and Steinhardt 2007a; Lu and Steinhardt 2007b; Makovicky 1992; Makovicky 2007; Makovicky 2008; Saltzman 2008]. But the fact that the monument itself is decagonal in plan (fig. 2) has been missing from this discussion, in which the building has been mischaracterized as octagonal, repeating mistakes of earlier publications. In 1937 M.B. Smith carefully documented the tomb towers at Maragha, producing measured scale drawings, black and white photographs, and sketches made on site, all of which are now held in the Archives of the Freer Gallery of Art [Myron Bement Smith Collection]; his annotated field plan (fig. 2) documents the decagonal exterior with an interior decagonal chamber. Daneshvari [1982; 1986] examined plans and elevations, noting the decagonal form of both interior and exterior. That it once had a conical roof composed of ten planar triangular facets, hints of which are visible in fig. 1, covered with turquoise glazed brick [Myron Bement Smith Collection], was also noted by Daneshvari [Daneshvari 1982: 289]. Other authors have documented and commented upon the metamorphic patterns [Makovicky 2008: 130; Wilber 1939; Bier 2011a], in which five-, six-, seven-, eight-, nine-, and (half) ten-pointed interlaced stars (figs. 3, 5) ornament the spandrels filling the spaces above the arches in the rectangular niched enclosures of the building's ten faces, each of which arises from the decagonal plan and is framed by a pair of engaged columns, one column placed at each exterior angle. The number ten was expressively important in the development of the plan and its

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prismatic elevation: the ten triangular facets of the roof, ten rectangular facades framed by engaged columns and framing ten arched tympanums with spandrels above. The interior chamber is decagonal, with recessed niches that complement the exterior form (fig. 2), providing a base for what was once the polyhedral conical roof. Above the niches of the interior is a twenty-sided zone of transition composed of ten blind arches and ten squinches that once supported an interior dome [Daneshvari 1982: fig. 126]. Above these interior arches and squinches is a band of inscription in Arabic, quoting verses 1-4 of chapter 67 (*al-Mulk*) of the Qur'an (discussed below), which may help to inform our understanding of this monument in its time.

The significance of the ornamental pattern that surrounds the lower exterior reaches of the building (figs. 1, 10) has achieved widespread attention because of the mathematical implications of its visual complexity that resides in combinations of five-fold and ten-fold symmetries [Lu and Steinhardt 2007a, 2007b; Makovicky 1992; Makovicky 2007; Makovicky 2008]. But decagonal intentionality extends beyond this broad band of pattern and encompasses the entire structure from the plan and its prismatic form and decagonal interior chamber to the decagonal base of its pyramidal roof; clearly, the geometry of the decagon both in theory and in practice was a primary consideration of those responsible for this building's overall design and construction [Bier 2011a].

Fig. 1. Gonbad-e Qabud, Maragha, Iran (593 H. / 1196-97 C.E.), view from WNW. Photograph by Myron Bement Smith 1937. Black and white negative (L254-11) [Myron Bement Smith Collection]

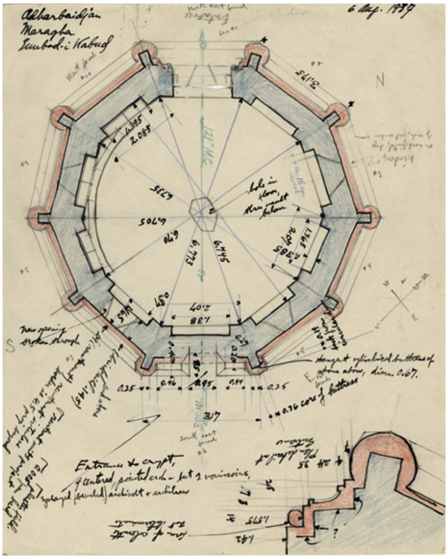

Fig. 2. Plan of Gonbad-e Qabud, Maragha, Iran. Annotated field drawing by Myron Bement Smith, 6 August 1937. Ink, pencil and color on paper [Myron Bement Smith Collection]

A reassessment of this building's cultural significance is warranted to articulate why the focus on the number ten is so vitally important for both the history of art and the history of mathematics.

Dating of Gonbad-e Qabud and its cultural environment

The dating of Gonbad-e Qabud to 593 H. / 1196-97 C.E. has been generally accepted; it rests upon Godard's historical deduction [Godard 1936: 143], placing its construction a decade after that of the tomb of Mo'mene Khatun in Nakhchevan [ArchNet], Azerbaijan, considered its model; Godard does mention reading "593" in negative impressions in the plaster where the inscription would have been [Godard 1936: 143]. Although both towers are prismatic with a decagonal plan and have engaged corner buttresses, the geometric decoration of the two decagonal buildings is not identical; that at Nakhchevan includes a pattern based on twelve-pointed stars in combination with five-pointed stars [Wilber 1939: fig. 12], and columns of ten-pointed stars in a linear arrangement. To understand the potential chronological implications of these patterns for one another, further study is needed to establish a typological development of geometric patterning in pre-Mongol ornament. But if the late twelfth century is accepted for Gonbad-e Qabud, that would place this building in a period of local rule by the Ahmadili family, who were Atabegs of Maragha [Luther 1987; Minorsky 1960], during a time of rivalry among the Sunni Ahmadili and the Shi'i Isma'ilis [Minorsky 2011], at the close of the Saljuq period shortly after the conclusion of Saljuq suzerainty in Iran and just prior to the drama and ferocity of the Mongol conquests of Iran, which began in 1219, with Maragha taken in 1221. Among the cultural highlights of the last decade of the twelfth century in Maragha was the composition in Persian by Nezami Ganjavi (1141-1209 C.E.) of his romantic epic, *Haft Paykar* ("Seven Portraits" or "Seven Beauties"), dedicated in one of two surviving manuscript traditions to Ala' al-din, who was ruling from Maragha. Related to this early literary appearance of Persian prior to the Mongol conquests and further distinguishing this building from its fraternal twin at Maragha is the incorporation of Persian verses following the foundation inscription in Arabic on the tomb tower at Nakhchevan [O'Kane 2009: 33, fig. 2.6].

Local tradition associates Gonbad-e Qabud with the mother of Hulegu Khan, Ilkhanid ruler of Iran in the thirteenth century. This association, however, is refuted by Godard [1934; 1936] on the basis of religion: neither Hulegu nor his mother, Sorghaghkani Beki, were Muslim, and this refutation has been accepted and repeated by others [Myron Bement Smith Collection; Daneshvari 1982]. Sorghaghkani Beki[1] was the mother of four sons of Mongol lineage, who inherited Chinggis Khan's empire – her youngest son, Kublai Khan, became Great Khan in 1260 and ruled over much of Asia after 1264, establishing the Yuan Dynasty from his new capital at Khanbalic, now Beijing. Hulegu was granted the southwestern territory of the Mongol Empire and he continued a policy of determined military expansion, capturing Baghdad in 1258, which brought a close to the Abbasid Empire when the seat of the Caliphate was removed to Syria. Hulegu soon returned to Maragha, and established the capital of the Ilkhanate there in 1259, where he founded an observatory (mentioned in [Makovicky 1992: 67; Chorbachi and Loeb 1992; Saltzman 2008]) and appointed Nasir al-Din Tusi chief astronomer [Saliba 2007: 244]. Born to an Isma'ili family, Nasir al-Din Tusi later became a Twelver Shiite Muslim; in the face of the Mongol conquests, he had sought protection in the Isma'ili fortress stronghold at Alamūt [Hourcade 1984], which was taken by the Mongols in 1256 and the libraries there burned. The Maragha observatory, which he oversaw, is considered to be one of the earliest monuments built under Ilkhanid patronage [Blair 2004]; recent archaeological excavation has revealed what have been identified as the remains of this observatory on a high plateau five kilometers to the south of the town [Kleiss 2002; Vardjavand 1979]. Within the annals of the history of science, Maragha was indisputably a major center for the advancement of scientific research in astronomy and mathematics [Sezgin 1998]. Kleiss [2002] mentions an extensive system of artificial caves five kilometers west of Maragha, in which he describes “catafalque-like blocks,” associated by legend as the burial place of Nasir al-Din Tusi (d. 1274).

While a pre-Mongol scientific community at Maragha has not yet been historically established, it is conceivable that the Gonbad-e Qabud, with its decagonal plan and exceptional geometric ornamentation, represents the efforts of local endeavors that engaged both mathematics and the craft of building.

Enumeration of the Gonbad-e Qabud, its inscriptions and its decoration

Based upon meticulous architectural study of the monuments at Maragha by Myron Bement Smith in 1937, through measured plans and extensive black and white photographic documentation [Myron Bement Smith Collection], one may readily recognize that the building is decagonal in plan, rising as a ten-sided prism with a round engaged column at each exterior angle (figs. 1-2), constructed of baked brick laid upon a stone foundation. Each exterior wall comprises an arched niche with a tympanum and projecting beveled archivolt that springs from above an engaged column (fig. 5), all contained within a tall rectangle, spandrels filling the area to each side of the pointed arch (fig. 1). Within each arch there is a five-lobed arch (fig. 5) composed of a shallow three-tiered muqarnas. All of these surfaces are decorated with intricate geometric patterns composed of polygonal nets executed in cut brick and turquoise glazed ceramic. Above the arches and beneath the cornice is a band of inscription in an elaborate foliated Kufic style of Arabic script (fig. 3) with several interlaced letters executed in turquoise mosaic faience (i.e., glazed bricks cut to shape), which extends around the building’s ten sides. From the top of the stone foundation to the height of the springing of the arches a broad band of geometric ornament wraps around the building (fig. 1, 10), including all ten engaged columns but only nine sides (the entrance façade is omitted).[2]

This ornamental band is executed in cut brick [Myron Bement Smith Collection] and shows no evidence of any glazed ceramic (despite publication to the contrary [Makovicky 1992: 69]). This is the band of ornament with pentagonal symmetries so carefully analyzed in the publications of Makovicky [1992; 2007; 2008] and Lu and Steinhardt [2007a; 2007b], and discussed by Bonner [2003]. The exterior decorated architectural elements, both glazed and unglazed, comprise a revetment set in plaster [Myron Bement Smith Collection]; the revetment is not integral with the structure of the tower (figs. 3, 5-6), which is among several reasons for its damaged condition.

The exterior ornamentation of the Gonbad-e Qabud is constructed of baked brick, some glazed, some unglazed, some cut, some whole. The standard brick dimension was square, as recorded by M.B. Smith, 22.5 x 22.5 cm by 5.5 cm thick [Myron Bement Smith Collection]. Except for the entry façade, each rectangular face encloses a rectangular niche comprising a panel articulated in an angular polygonal net outlined by ribs in high relief and a curvilinear interlacing in lower relief (figs. 1, 10). This broad band of visually complex geometric ornament wraps around nine faces of the building, including all ten engaged columns [Myron Bement Smith Collection; Bier 2011a]. This band of ornament is surmounted by a pointed arch in each niche, which springs from the engaged columns and has a beveled edge slightly projecting and bearing a Kufic inscription (fig. 5).[3]

Fig. 3. Gonbad-e Qabud, Maragha, Iran. Entry façade detail (upper). Photograph by Myron Bement Smith 1937. Black and white negative (L243-10) [Myron Bement Smith Collection]

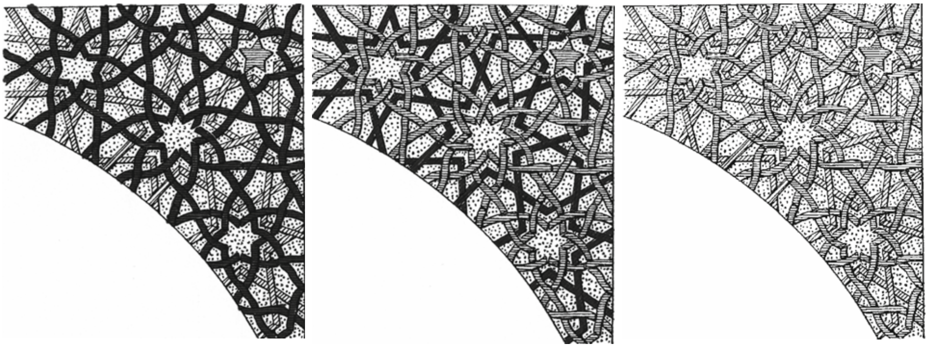

Fig. 4. Gonbad-e Qabud, Maragha, Iran. Upper spandrel, analyses by the author after Wilber [1939: fig. 17]. Three layered design of 5-6-7-6-pointed stars.
a, left) Top layer (unglazed ceramic s-curved strips) forming curvilinear polygonal net;
b, center) Lower layer (turquoise glazed ceramic strips) forming angular polygonal net;
c, right) Base layer of stucco, shown stippled, as indicated by Wilber [1939: fig. 17]) showing 6- and 7-pointed stars in negative space with 5-pointed star applied. Note that Wilber has also indicated ceramic construction breaks in polygon networks

The main arch of the niche on each facet contains a five-lobed arch that comprises a shallow three-tiered muqarnas (fig. 5); both the larger arch, and the smaller arches, and the spandrels above them, bear geometric ornament. The smaller arches bear individual designs with rotational symmetry. The upper spandrels carry a layering of two superimposed polygonal nets in which five-, six-, seven-pointed stars are interlaced (fig. 4). The topmost layer shows a curvilinear net (fig. 4a), composed of buff (unglazed) ceramic strips, with S-curves, and the lower layer is angular (fig. 4b), composed of turquoise-glazed strips that are straight (without curves). The two polygonal nets share the same center points, but the star forms of each polygonal net interlace on two separate levels. The superimposed polygonal nets are set against a ground layer of plaster [Wilber 1939: fig. 17] (fig. 4c).

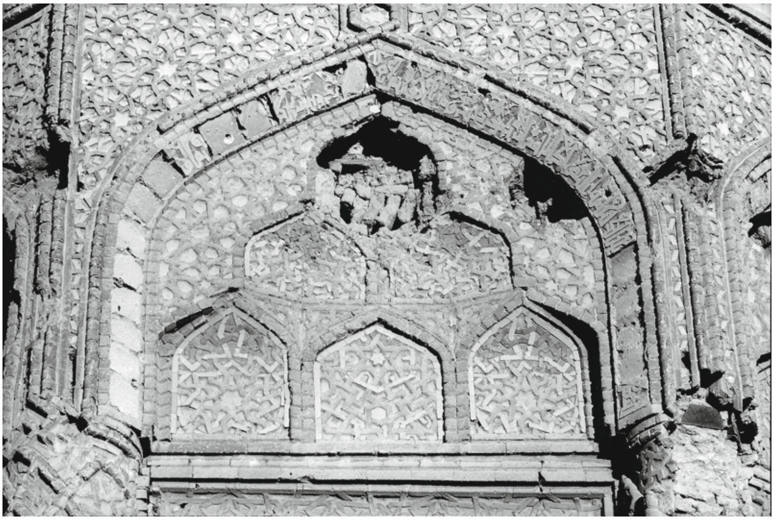

Fig. 5. Gonbad-e Qabud, Maragha, Iran. Entry façade detail: arch with tympanun and five-lobed arch composed of three-tier muqarnas. Photograph by Myron Bement Smith 1937. Black and white negative (L243-11) [Myron Bement Smith Collection]

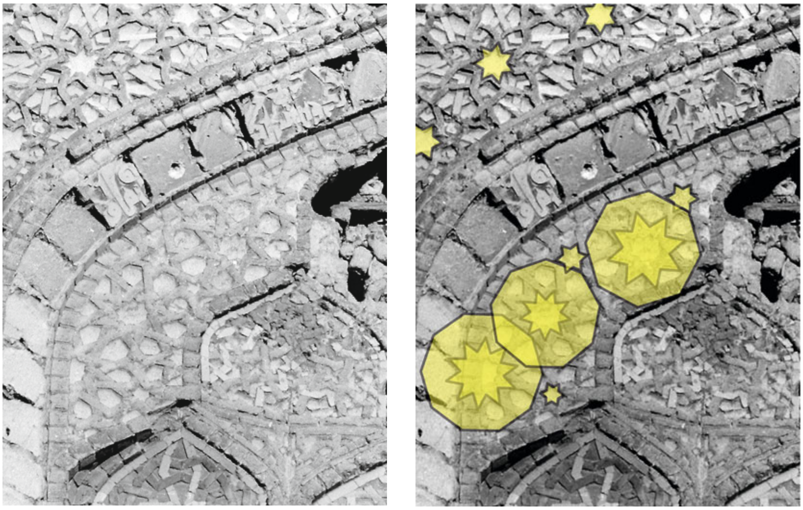

Fig. 6. a, left) Detail of a fig. 5; b, right) the same detail with stars highlighted in both upper and lower spandrels, showing progression of 8-, 9-, and 10-pointed stars in the lower spandrel

In the spandrels of the five-lobed arches (fig. 5), all of the decoration is unglazed. Here, eight-, nine-, and (half) ten-pointed stars radiate with petal-sharing (fig. 6) of the 9- and (half) 10-pointed stars [Seherr-Thoss and Seherr-Thoss 1968: pl. 36], once again lending a sense of visual complexity to the overall decorative scheme.

The visual complexity is especially evident in the spandrels, where the artisans have taken liberties in rendering the progressive sequences of 5-, 6-, 7- (fig. 4), and 8-, 9-, and (half) 10-pointed stars (figs. 5-6), adjusting lines and angles of the star polygons to allow for their diminution or expansion (fig. 4), and by "petal-sharing" of the 9- and (half) 10-

pointed stars (fig. 6). In the pattern of panels surrounding the ten-sided building, the local pentagonal and decagonal symmetries (fig. 10) are confounding to the untrained eye; that pattern also makes use of minor adjustments, as for example, in the inclusion of five-pointed stars in a lower corner [Makovicky 2007: fig. 2a].

The prismatic form of the building rises more than fourteen meters from the ground [Daneshvari 1982: 289], and rests upon a stone foundation. Limiting the visual extent of the decagonal prismatic form is a three-tiered cornice (figs. 1, 3) with muqarnas executed in glazed ceramic, which rises above an elaborate band of inscription in Arabic in Kufic style with bifurcate serifs on ascending letters. The elaborate cornice visually joins the decagonal prism and the polyhedral conical roof, each flat facet of which rises from above one of the ten sides of the building. A lower crypt is cruciform in plan with a low domical brick vault that exhibits eight-fold symmetry [Myron Bement Smith Collection; Daneshvari 1982: figs. 125-126]. The interior decagonal chamber is approached from the front (NW) by an elevated portal that would have originally had a set of stairs; the surfaces of the interior walls, niches, arches, and squinches are all plastered [Myron Bement Smith Collection]. There is evidence of an interior brick dome [Myron Bement Smith Collection; Daneshvari 1982: 289], a feature characteristic of other tomb towers on the Iranian plateau [Hillenbrand 1994: 253-330].

The varieties of geometric expressions are indeed extensive. They are all mathematically significant for their time, fitting within a substantial architectural lineage of experimentation in the expression of geometric forms from the tessellations of polygons and star polygons to the intersections of polygons in cut brick [Bier 2002], to the generation of polygonal nets and sub-grids [Bonner 2003], some of which illustrate multiple patterns superimposed upon one another and highlighted by colored glazes, the earliest example of which appears on the Gonbad-e Sork [Godard 1936; Wilber 1939] also located in Maragha (fig. 7). Developments in technologies and the use of materials are also noted at this time, ranging from the nearly exclusive use of unadorned fired brick (as at Kharraqan [Stronach and Young 1966; Bier 2002]) to the incorporation of glazed ceramic in one, two, three, or more colors, and from the use of cut mosaic faience to the application of glazed ceramic tile [Wilber 1939].

Saljuq architectural monuments exhibit extraordinary diversity in form, as well as in the incorporation of geometric ornament. This is particularly evident in tomb towers [Hillenbrand 1994: 253-330; Stronach and Young 1966; Shani 1996], although not exclusive to them [Korn 2010]. In general, an octagonal plan is far more frequently attested than other forms; similarly, for decoration, the symmetries that more easily tessellate the plane with translations to create periodic patterns with two-, four-, and eight-fold rotations, as well as those with three- and six-fold isometries, appear more often than patterns that incorporate five-fold and ten-fold symmetries [Grünbaum and Shephard 1989: chs. 1-2]. The consistent focus on the decagon in the plan, interior chamber, surface ornament, and roofing at Gonbad-e Qabud offers clear evidence of a concern that goes beyond visual impact. Although the general form of this decagonal tower is not unique, some of its geometric ornament seems to be, but many of its architectural elements find parallels elsewhere, relating it to other pre-Mongol monuments on the Iranian plateau [Hillenbrand 1994; Korn 2010].

Expressions of geometry in Iran in the eleventh and twelfth centuries

Gonbad-e Sork [Seherr-Thoss and Seherr-Thoss 1968: pls. 30-33], also referred to as Gonbad-e Qermez, is the earliest of five tomb towers still standing at Maragha [ArchNet].[4] Its construction is dated by the portal inscription in baked brick set on a base of stucco that is intricately carved above the entrance on the north side of the building [Milwright 2002]. The inscription, in a Kufic style of Arabic, follows the arcuate lines of the tympanum (fig. 7a) and gives the name of the builder and date of construction (542 H. / 1148 C.E.), as read by [Herzfeld]. The building is square in plan, with an engaged column at each corner. These architectural features are shared with the earlier tomb of the Samanids in Bukhara [Hillenbrand 1994: 287-290], the undated Gonbad-e Alaweyan in Hamadan [Shani 1996], and the later Gonbad-e Ghaffariya dated 1328 C.E. [Kleiss 2002], which is also in Maragha. The circular corner buttresses also appear in two octagonal tomb towers at Kharraqan (fig. 8), which are dated to the late eleventh century. Unusual features at Gonbad-e Sork include the incorporation of occasional glazed bricks in its ornamental revetment [ArchNet; Seherr-Thoss and Seherr-Thoss 1968: pl. 32], and an elaborately designed tympanum (fig. 7a) over the portal [Seherr-Thoss and Seherr-Thoss 1968: pl. 31] in which turquoise glazed ceramic strips articulate interlaced nonagons. This is considered to be the earliest use of glaze ceramic in Iranian architecture of the Islamic period [Wilber 1939; Necipoğlu 1995: 99 (but see note 6 below)]. The Gonbad-e Ghaffariya, although similar in form architecturally, incorporates a higher proportion of glazed bricks and more extensive use of mosaic faience with additional colors [Myron Bement Smith Collection; ArchNet]. Also unusual for its time are the double blind arches on each face of the Gonbad-e Sork, which are reflected in the interior elevation, and the patterned brickwork of the double blind niches, which uses both vertical and horizontal alignment of bricks to create the visual effect of float-patterned twill textiles. These features are also present in the Gonbad-e Alaweyan in Hamedan, where the vertical bricks project and carved stucco is affixed in the recesses [Shani 1996: figs. 6 and 32-33]; in the absence of a foundation inscription, Shani attributes this building to the late twelfth century on the basis of architectural comparisons and historical context. Each of these three square prismatic structures incorporate decorative brickwork in which the patterns reflect the laying of bricks, either vertically or horizontally.[5]

The geometric decoration in the tympanum of Gonbad-e Sork is unique among monuments on the Iranian plateau (fig. 7a-d). Within the pointed arch is an extraordinary four-layered design that includes three polygonal nets sharing the same central point.[6] The base plane is carved stucco [Wilber 1939: fig. 10]; superimposed on this are three successive layers of ceramic strips that articulate interlaced (overlapping) polygons. At the base, one interlaced layer of pattern shows six overlapping nonagons (plus nine partial nonagons), creating three elongated hexagons per nonagon, establishing a three-fold symmetry, and leaving a six-pointed star in negative space at the center (fig. 7b); this layer is highlighted by the use of turquoise-glazed strips (fig. 7a), unique for this date. This polygonal net interlaces with a second polygonal net (fig. 7c) composed of a double hexagonal grid of interlacing regular hexagons in buff (unglazed) ceramic [Seherr-Thoss and Seherr-Thoss 1968: pl. 31]. The placement of this grid allows for a centrally located regular hexagon surrounded by six partial hexagons (fig. 7c), each of which has a (partial) six-pointed star in the center created by the third, topmost layer. The uppermost layer is constructed of unglazed strips in higher relief and consists of six overlapping regular dodecagons, also yielding a centrally placed six-pointed star (fig. 7d) that has a different orientation than the six-pointed star reserved by the overlapping nonagons (fig. 7b). Three additional dodecagons are hinted at in each of three corners of the arch, indicated in the analytical drawing (fig. 7d). The overlapping dodecagons interlace with one another, but not with the lower polygonal nets. Above the arched tympanum that contains this unified geometric pattern in its arch, with the surrounding inscription band and a succession of brick borders, are the spandrels, which bear a tessellation with three-fold symmetry in cut bricks, both glazed and unglazed (fig. 7). Symmetries with overlapping polygons are also present in cut-brick ornament that decorates the exterior of the Gonbad-e Alaweyan in Hamadan [Shani 1996].

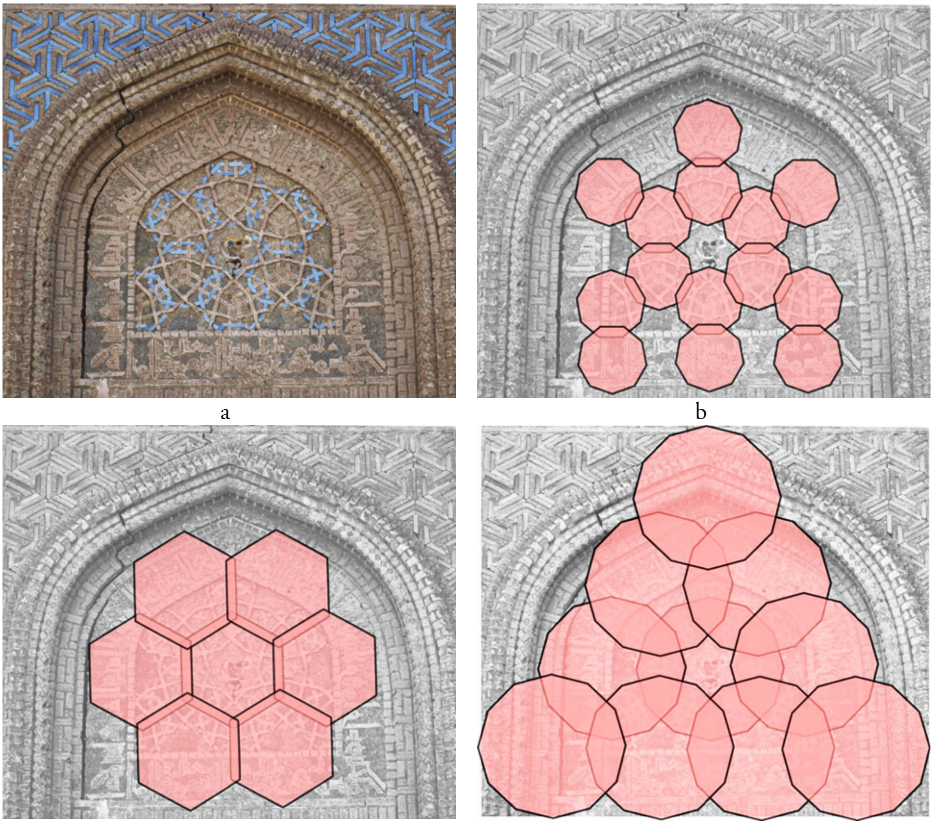
a b c d

Fig. 7. Gonbad-e Sork, Maragha, Iran (542 H. / 1148 C.E.). Tympanum above entryway, showing three interlaced polygonal nets centered around the same point (analyses by the author). a) Color transparency (C162) by Hans C. Seherr-Thoss, detail (Seherr-Thoss Photographs of Islamic Architecture, Freer Gallery of Art and Arthur M. Sackler Gallery Archives, Smithsonian Institution, Washington, D.C. Gift of Sonia P. Seherr-Thoss); b) Interlaced nonagons; c) Interlaced hexagons; d) Interlaced dodecagons. Note that traces in clay imply the extension of polygons beyond the boundary of the innermost tympanum

Gonbad-e Sork also exhibits more complicated patterns with interlocked elements executed in unglazed bricks on the engaged columns that frame the entryway [Seherr-Thoss and Seherr-Thoss 1968: pl. 30]. Like the brickwork of the sidewall niches, which visually replicates twill weaves, these patterns share visual affinities with contemporary pattern-woven textiles. The engaged columns are capped by stone capitals, another unusual feature of the period and region. The polyhedral roofing arrangement with an eight-sided pyramid is also unusual for its prominent octagonal zone of transition visible on the exterior. The transition zone is composed of bare brick and houses segmented squinches that represent an early appearance of this feature on the plateau. The clear and conscious geometry of structure is also evident in the upper panels of inscriptions, which incorporate interlaced quadrants extending from tall ascending letters.

Also relevant to the discussion of geometric patterns composed of interlocking elements and overlapping polygons are two tomb monuments at Kharraqan, located east of Maragha between Qazvin and Hamadan [Stronach and Young 1966; Bier 2002]. Constructed of brick and once surmounted by a dome, each tower displays eight niched rectangular vertical faces joined by cylindrical engaged buttresses. Built twenty-six years apart, the two buildings stand as a pair of octagonal towers rising more than twelve meters above the plain (fig. 8).[7] Each building bears an Arabic inscription that gives its date of construction. The earlier tower, to the east, was built in 460 H./1067-68 C.E. The later tower, to the west, was built in 486 H./1093 C.E. The historical inscriptions each contain, in addition to a date, the name of the architect (who seems to be the same person for both monuments), and an additional name, presumably that of the deceased, tentatively suggestive of Turkic ethnicity [Blair 1992: 135]. The exterior vertical faces of both octagonal prisms are entirely covered with geometric patterns in which extended axes of symmetry yield periodic patterns based on reflection and glide reflection. As at Gonbad-e Qabud, the buildings are clad in bricks set as revetment, the patterns visually serving to clothe the monument, leaving no exterior parts beneath the dome unadorned. The patterns are created by means of the selective cutting and placement of bricks set with mortar. The eastern tower, which is earlier, bears more than thirty patterns disposed on its eight faces and connecting buttresses. The western (later) tower has well over seventy patterns on its exterior surfaces. The play of patterns, in every instance, is based upon an algorithm consisting of a unit configuration of cut bricks that is consistently repeated to fill a designated space. Geometric relationships thus revealed are emergent structures within the pattern that is at once unitary and systemic. The presence of a third dimension, created by the relationship between bricks and mortar, seems almost incidental to the two-dimensionality of the pattern as perceived by the viewer. The play of darkness and light articulates the visible surfaces of the buildings, appearing on all eight faces of each monument as well as on the engaged columns or buttresses, which serve both to separate and join adjacent faces. The entryway of each monument, as well as the adjacent two faces, comprising five faces together, shows recessed mortar joints, whereas the back three faces, and adjoining buttresses, show flush mortar joints. This feature, combined with other aspects having to do with the distribution of patterns, suggests a clear sense of symmetry in the conception of each building.

The bricks are set in mortar and laid to form an elaborate array of symmetrical patterns that play with repeated themes of projection and recess, light and shadow, solid and void, offering numerous variations on a theme that plays continually with the passage of light without the addition of color in the earliest examples. The tombs at Kharraqan conform to the characteristics of these early examples, which pre-date the emergence of the use of color in glazed bricks and mosaic faience, both of which later appear at Maragha. The extraordinary number of patterns on the later tower at Kharraqan is unrivalled within the corpus of Saljuq brick monuments [Bier 2002].

Apart from the entry façade – which in each case is reserved for the most complicated pattern, placed in the tympanum above the entrance – there is no sense that any one pattern is more significant than any other. As for geometry, there does not seem to be any consistency or progression in the choice of patterns. The eye is not drawn to a single central focal point. Rather each panel and area designated by pattern presents multiple centers. The range of patterns displays an evident awareness, even if by way of experimentation, of the play of symmetry with its inherent ambiguities and emergent geometric relationships. Horizontal and vertical reflections and glide reflections are much in evidence, as are rotational symmetries. There is clearly play with the cut-brick units, which combine to form triangles, squares, hexagons and six-pointed stars, octagons and eight-pointed stars, and dodecagons. There is considerable attention given to illusionary interlace, visually effected by the selective cutting of bricks and their specific juxtaposition. Sometimes, as in the case of interlaced octagons and interlaced dodecagons at Kharraqan, the centers are specifically marked by a projecting dot, a feature also noted in the tympanum at Gonbad-e Sork.

Fig. 8. Tomb tower (western), Kharraqan, Iran (486 H. / 1093 C.E.). Entry façade with tympanum and inscriptions. Color transparency (C112) by Hans C. Seherr-Thoss. (Seherr-Thoss Photographs of Islamic Architecture, Freer Gallery of Art and Arthur M. Sackler Gallery Archives, Smithsonian Institution, Washington, D.C. Gift of Sonia P. Seherr-Thoss)

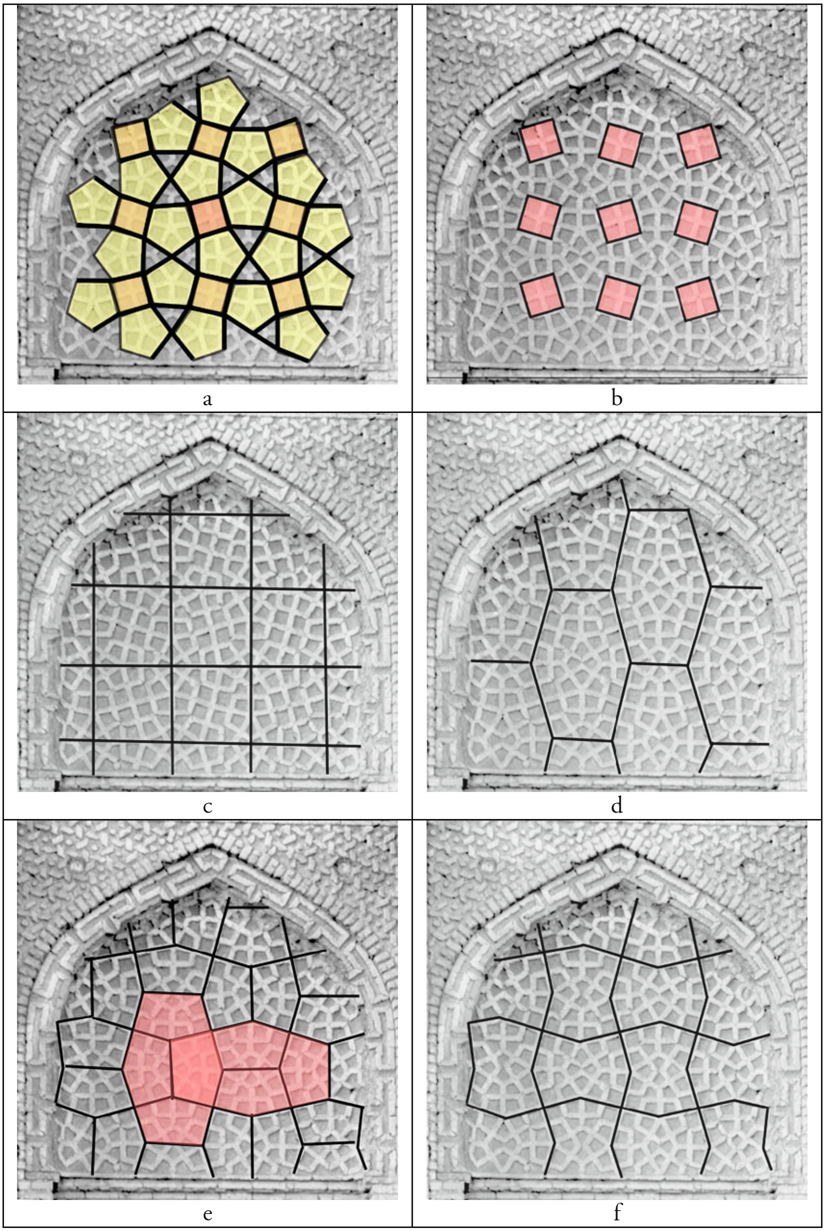

Fig. 9. Analysis of the pattern in the tympanum of the western tower at Kharraqan (shown in fig. 8) yields many possible readings (analyses by the author). a) pentagons, visible as executed in cut brick; b) squares (negative space between four pentagons); c) underlying square grid reveals a design unit that is repeated with perpendicular glide reflections forming a tessellation; d) by connecting centers of some pentagons, a tessellation of vertical hexagons is established; e) by connecting centers of other pentagons, a tessellation of horizontal hexagons is established; by overlapping these two hexagonal tessellations, a tessellation of pentagons is achieved, as illustrated; f) perpendicular lines crossing at the center of each square intersect to form a tessellation of perpendicular "bowties," each composed of two pentagons

As at Gonbad-e Sork and elsewhere, the face with the entryway (fig. 8) has received more elaborate treatment. In some areas the mortar is deeply recessed, allowing the play of light and shadow to help define the geometric patterns made by the arrangement of bricks. The entry façades, including the entry face and the faces adjacent to it on each side, show a greater proportion of recessed mortar, giving a sense of higher relief to the patterns in contrast to those faces not visible from the front of the building. Taken together, these two buildings provide early and incontrovertible evidence of a truly passionate fascination with the phenomena of patterns. In the case of the earlier tower, the tympanum above the entrance bears a geometric pattern of interlace which forms twelve-pointed stars in negative space, repeated nine times, in which the name *Allah* (God) is inscribed [Bier 2002: 73].

What is particularly relevant to the study of patterns with pentagons is the tympanum of the later (western) tower at Kharraqan (figs. 8, 9), which also shows polygonal nets. Analysis of the pattern (figs. 9a-f) shows a highly unusual design that allows for various readings – the repetition of pentagons, squares, and triangles (fig. 9a) creates a pattern in which many different tessellations may be recognized, each of the tessellations covering the plane with no gaps and no overlaps (figs. 9b-f). Symmetry analysis of the seemingly simple pattern of sectioned pentagons, squares and triangles results in the identification of an underlying square grid (fig. 9c), in which the design unit is repeated by perpendicular glide reflections to form a tessellation. The pattern exhibits both ambiguity and visual complexity, and relates this monument to Gonbad-e Qabud in unexpected ways.

Connecting the centers of selected pentagons executed in cut brick (fig. 9a) results in a tessellation of hexagons with a vertical orientation (fig. 9d), or a horizontal orientation. By superimposing these two tessellations, another tessellation is achieved (fig. 9e) of pentagons! Each hexagon, whether vertical or horizontal in orientation, is composed of four such pentagons, and each pentagon has two right angles. By dividing each pentagon in half, yet another tessellation results, this time of a quadrilateral also having two right angles. Perpendicular lines drawn through the center of each square (fig. 9f) results in the tessellation of a double pentagon, or “bowtie,” with orientations perpendicular to one another. In order to achieve this exceptional flexibility, each of the shapes has particular characteristics, the underlying geometry of which renders this high degree of pattern play and allows for great visual ambiguity. The smallest quadrilateral has two right angles, such that, reflected across the long axis, two quadrilaterals form a pentagon with four sides of equal length, two right angles, and one longer side (fig. 9e). Four such pentagons (each with a different orientation) compose a hexagon (figs. 9e). We may recognize in the cut brick elements the four-fold orientation of pentagons (fig. 9a), and the two orientations of squares (fig. 9b). The playfulness of the pattern resists easy visual analysis by the untrained eye and allows for the identification of many underlying polygons and grids. This is surely an indication of a sophisticated understanding of angles, lines, numbers and shapes in relation to two-dimensional space, and the recognition of what we call today symmetries and tessellations, described by various terms such as polygonal net, polygons in contact, and dualized grids [Saltzman 2008; Cromwell 2010; Kaplan 2005]. Below the tympanum (fig. 8) is an historical inscription in Arabic and a Qur’anic excerpt [Ali 1978: ch. 23, v. 115], which is unique to the western tower.[8]

As for ornament, there are other monuments in Iran of the Saljuq period and later, for which pentagonal or decagonal elements are paramount. The north dome of the Congregational Mosque in Isfahan (Masjid-e Jomeh) is built of brick and is constructed with a centralized pentagonal design that can be read in at least two ways (illustrated in [Grabar 1993: 147; Grabar 1990]), lending emphasis either to the concentric pentagons and their extensions, or to the rhombs and triangles created by the intersections of the extensions of the two pentagons. Alternatively, considering the three-dimensionality of interior space beneath the dome, each rib line may be seen to represent a plane that cuts a section through the dome.

Turning our attention away from the unique patterns with pentagonal shapes on the tympanum of the later tower at Kharraqan, and in the pattern surrounding the shaft of Gonbad-e Qabud in Maragha, one must return to consideration of the decagonal structure of Gonbad-e Qabud, which is very rare. The neighboring Gonbad-e Sork and Gonbad-e Ghaffariya are both square prisms, which along with octagonal prisms and cylindrical forms occur far more frequently as the architectural form for tombs on the Iranian plateau. For the Saljuq period, octagonal tomb towers are considered the norm [Hillenbrand 1994; Stronach and Young 1966; Shani 1996]. Several exceptions are worthy of note. Two hundred kilometers to the north, in what is today the Republic of Azerbaijan, stands the mausoleum tower of Mo'mene Khatun [ArchNet] in Nakhchevan, which shares with Gonbad-e Qabud an articulated decagonal plan with a correlation between interior chamber and exterior elevation. Godard [1936] pointed to the similarities in plan and elevation of these two monuments; they are both constructed of brick on a stone foundation and the tomb of Mo'mene Khatun is also likely to have had a ten-sided polyhedral roof. The exterior façade, however, is decorated with interlaced geometric patterns that show ten-, twelve- and thirteen-pointed stars, interlacing with 10-, 12-, and 13-gons [ArchNet]. In contrast to Gonbad-e Qabud, where all of the inscriptions are Arabic, the main inscription beneath the cornice of the Mo'mene Khatun monument displays an early appearance of Persian poetic verses [O'Kane 2009: 32].

The interior decagonal chamber with ten recessed niches finds a local parallel in a neighboring tomb at Maragha, the cylindrical tower ("Circular Tower" [Myron Bement Smith Collection]), dated by a foundation inscription to 563 H./1167-68.C.E. Although many other tomb towers on the Iranian plateau have a cylindrical exterior, they do not have decagonal interiors. Two other monuments have niched decagonal interior chambers, but their exterior form is quite different. One of these is the early fourteenth-century flanged tower at Bastam, around which a later shrine complex was built [Hillenbrand 1982]. Farther afield geographically but similar in date to the Gonbad-e Qabud, is the mausoleum of Kilic Arslan II (1155-1192 C.E.) in Konya, Turkey [ArchNet], which also has a decagonal plan and a ten-sided polyhedral dome that was once covered with blue tiles [Redford 1991: 55]. It is located at the center of the Alaeddin Mosque complex, founded earlier but construction of which continued until 1220; this mausoleum is associated with the burial of all but one of the Saljuq Sultans of Rum, who ruled in Anatolia. Redford describes the complex as a "curious mélange of disparate architectural elements and styles" [1991: 54].

Cultural significance of funerary architecture in Iran

There can be no doubt about the cultural significance of pre-Mongol Iranian tomb towers with respect to funerary architecture. Indeed, Hillenbrand devotes to mausolea nearly a quarter of the text of his book documenting the forms of Islamic architecture [Hillenbrand 1994: 253-330]. For the post-Mongol Ilkhanid period of the thirteenth century, Blair states that "the greatest architectural projects … are tomb complexes" [2004], which served as models for the fifteenth-century Gur-e Mir, Timur's tomb in

Samarkand, and ultimately the Taj Mahal in Agra in the seventeenth century. Ghazan (ruled 1295-1304) was the first Mongol khan to become Muslim (in 1295); he commissioned his vizier, Rashid al-Din, to write a history of the world, which was completed during the reign of Ghazan's brother and successor, Oljeitu. The cultural significance of tomb monuments, before and after the Mongol conquests, is suggested in this work, called *Jami' al-Tavarikh* (*Compendium of Chronicles*):

> Until now it has been the custom of the Mongol emperors of Genghis Khan's *urugh* [lineal descent] to be buried in unmarked places far from habitation in such a way that no one would know where they were … When the emperor became Muslim … he said, "Although such was the custom of our fathers … there is no benefit in it. Now that we have become Muslim we should conform to Islamic rites … ." Since he was in the capital Tabriz, he chose it as the site [of his tomb] and laid the foundation himself outside the city to the west … They have been working on it for several years now, and it is planned to be much more magnificent than the dome [i.e., tomb] of Sultan Sanjar the Seljuq in Merv, which is the most magnificent building in the world [Komaroff and Carboni 2002: 105].

When Yakut traveled to Merv in the mid-thirteenth century, he described the tomb of Sultan Sanjar as being beneath a great dome that was "blue and could be seen from a day's journey" [quoted in Wilber 1939: 23, n. 25]. Over the centuries, this visual aspect from afar has also been attributable to other domed monuments of the Saljuq period [Redford 1991: 55].

While architectural beauty, historical context, and commemorative significance are themes treated at length in the literature on the history of art and architecture, what often seems lacking is recognition of the cultural significance of architectural monuments with respect to the history of mathematics.[9] Parallel developments in art and mathematics in the eleventh and twelfth centuries have yet to be treated in an integrated manner, although Necipoğlu, in a chapter devoted to theory and praxis, mentions that they occur in tandem [1995:162]. Within the development of pre-Mongol brick, glazed ceramic, and stucco decoration, there seems to be a close approximation of patterns that offer visual solutions to geometric problems relating to numbers, shape, and the nature of space [Bier 2009] among new advances in algebra and geometry [Berggren 2007: 49]. But such parallels have not yet been addressed systematically, despite the admirable efforts of Necipoğlu to gather and analyze relevant documentary sources in providing context for the undated scroll in the Topkapi Palace Museum collections [Necipoğlu 1995].

Geometric forms of expression and contemporary mathematical thinking

The question of mathematical import in the patterns and form of Gonbad-e Qabud is not just of rhetorical interest, nor solely related to contemporary issues today concerning quasi-periodicity. But moving from general statements about the visual character of particular monuments and their ornament to the identification of specific relationships between patterns and mathematical thinking in Iran during the eleventh and twelfth centuries is complicated in part by the discrete histories of each of our disciplines, as well as by our different methodologies and epistemologies, to say nothing of Orientalism and historical Western attitudes towards the history of science.[10]

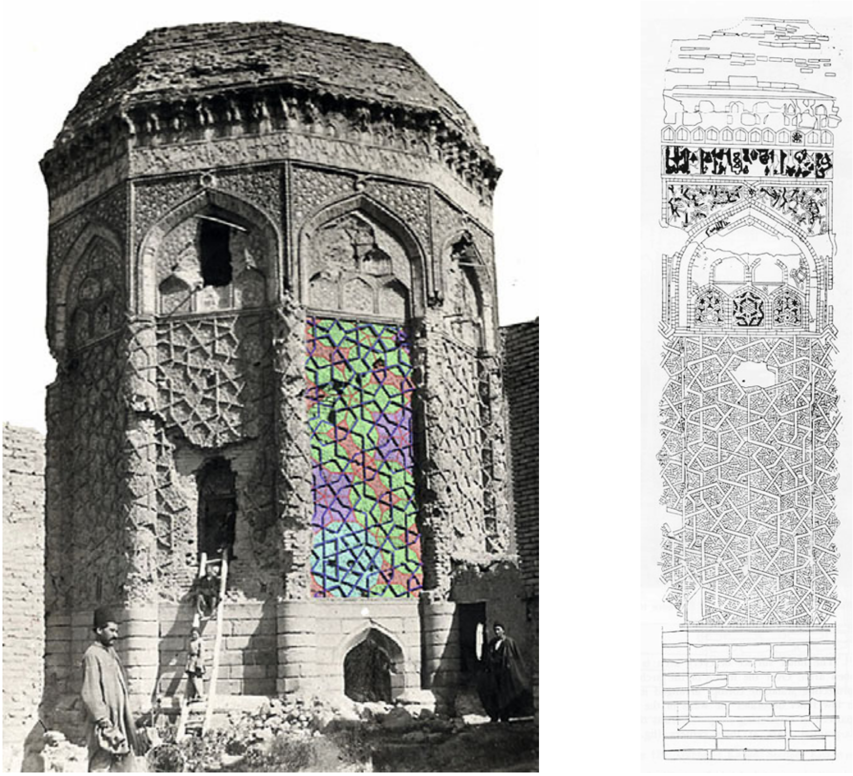

Fig. 10. Gonbad-e Qabud, Maragha, Iran. a) Pattern with pentagonal and decagonal symmetries around shaft of tomb tower, with 'girih' tiles highlighted in one panel [Lu and Steinhardt 2007a: S6 after photo by A. Sevruguin; reproduced with permission of P. Lu]; b) Drawing of one face (after Dastur Corp.) reproduced in [Daneshvari 1982: fig. 128]

Grabar correctly notes that geometry in the arts is expressive rather than representational [1993: 142], a distinction of critical importance to understanding the significance of pattern in Islamic art [Bier 2008]. For the period of the eleventh and twelfth centuries, with one hint of an exception – the association of the north dome of congregational mosque (Masjid-e Jomeh) in Isfahan with Omar Khayyam, based on an earlier supposition of Schroeder extended by Alpay Özdural [1995; 1998] – no one has yet linked specific monuments chronologically to the history of mathematics in Iran. The development of a systematic set of links will require extensive research and correlation of comparative data drawn from architecture and the arts, including textiles [Bier 2004, 2007]. For the moment, we may at least sketch a course for such a project, focusing on plane patterns and progressing from the iteration of a single shape (such as one form of brick), to the repetition of simple shapes to form a tessellation that covers the plane with no gaps and no overlaps, to the cutting, setting and laying of bricks in patterns with polygons and star polygons and the overlapping of polygons to form polygonal nets, occasionally with the addition of glazed color. Dealt with as two-dimensional patterns exhibiting symmetry of the plane, such designs are achieved through combinations of material, craft and technology, by means of which units and shapes relate to points, vectors, and angles, which together express geometric relationships and trigonometric ideas.

To consider the third dimension (spatial), one would add the differentiated forms of hemispherical domes, conical and polyhedral cones, the segmentation of squinches in zones of transition and the treatment of muqarnas, all of which are relevant to solid geometry, spherical trigonometry and conic sections. Mathematics in the eleventh and twelfth centuries in Iran comprised precisely these topics, as well as dissections and algebraic and geometric approaches to the solution of quadratic and cubic equations [Rashed 1994]. The evolutionary development of mathematical thinking that expanded upon the corpus of works of Euclid, Apollonius and others, translated from Greek into Arabic, likely found expression in the contemporary phenomena of experimentation with architectural structures and decoration, discussed above, but this remains to be demonstrated.

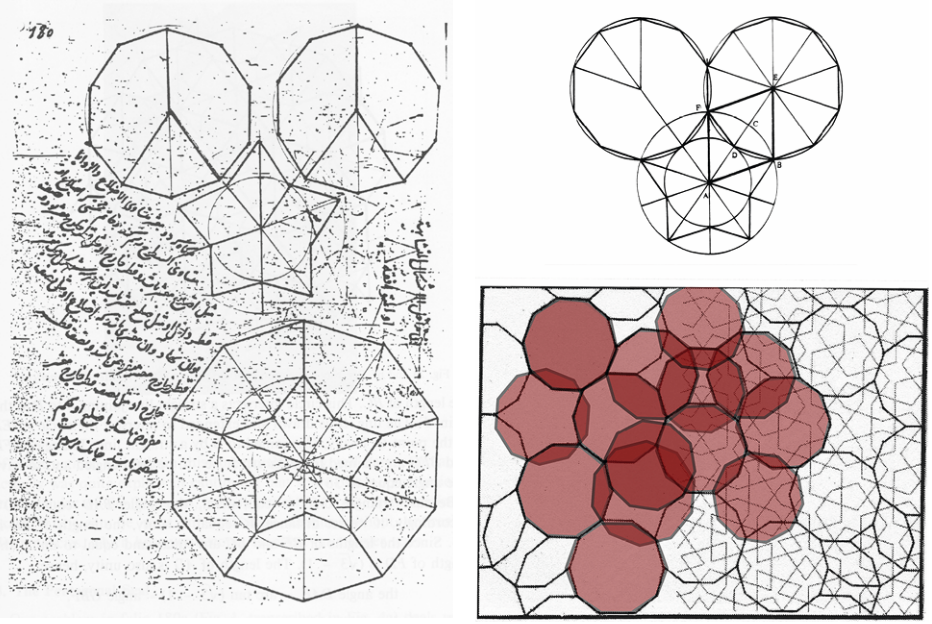

Fig. 11. a, left) *Khatam mukhammas* (pentagonal seal) from Bibliothèque national, Paris, ms persan 169, folio 180a, [Chorbachi and Loeb 1992: fig. 1];
b, upper right) Reconstruction of *khatam mukhammas* by Chorbachi [Chorbachi and Loeb 1992: fig. 4a] using overlapping decagons;
c, lower right) Reconstruction of unit cell at Gonbad-e Qabud by Saltzmann [2008: fig. 7] using grid dualization (author's highlighting of overlapping decagons)

Chorbachi is probably correct in introducing the term "tessellation" [Chorbachi and Loeb 1992: 288] to translate what Özdural [1996] and others have called "interlocking similar or corresponding" forms. She also recognized the relationship of the problems, which are set forth in the undated Paris manuscript BN ms persan 169 (fig. 11a), to what have come to be called Penrose tiles [Chorbachi and Loeb 1992: 284-286]. But she notes that the "step by step construction instructions are not given; neither is the theoretical significance of the given proportions" [Chorbachi and Loeb 1992: 290], although she reconstructed the pentagonal seal by means of overlapping decagons (fig. 11b) [Chorbachi and Loeb 1992: 291, fig. 4a]. Perhaps she and Loeb did not push this understanding far enough to recognize that the pentagonal seal (*khatam mukhammas*) is not only the self-contained design of an equilateral five-pointed star created by overlapping decagons (fig. 11b), as reconstructed from folio 180a in the undated Paris manuscript (fig. 11a), but that it may also serve as the generative form to yield a planar pattern of overlapping decagons (fig. 11c), which relates to the decagonal/pentagonal pattern on the Gonbad-e Qabud (fig. 10). Such a system of "interlocking similar" decagons also obtains from the construction methods used by Makovicky [2007] with regard to the ornament of Gonbad-e Qabud, but with neither reference to the Paris manuscript nor recognition of the implications of its use as a generative form. By linking the *khatam mukhammas* (fig. 11a-b) to the striking pattern at Maragha (fig. 11c), we may infer a relatively precise chronological connection among the history of mathematics and the history of art, for which documentary evidence in this period otherwise remains elusive.

Although the Persian word *girih* has a long history in the craft traditions of architecture and rug-weaving, present knowledge suggests that it is not textually documented in sources until the nineteenth century [Necipoğlu 1995: Blair 2004; Milwright 2001; Pugachenkova 1986]. Lu and Steinhardt [2007a] introduced the term "girih tiles" to describe the set of equilateral polygons that underlie what they consider to be a decagonal, quasi-crystalline tiling on the fifteenth-century Darb-e Emam monument in Isfahan, with roots in the decagonal symmetries of the Gonbad-e Qabud (fig. 10). But they, too, miss this aspect of the relevance of the overlapping decagons on the twelfth-century revetment of this decagonal tomb tower (figs. 1, 10, 11).

The two-dimensional construction of this visually complex pattern is variously described by different authors, none of whom seem to have made the specific connection between the textual description and construction of *khatam mukhammas* and the pattern at Gonbad-e Qabud in which overlapping decagons may readily be identified as a dual grid, following in the tradition we can now identify on the tympanum of the later (western) tower at Kharraqan. Lu and Steinhardt [2007a] ascribe "girih tiles" to a conceptual breakthrough from the "network of zigzagging lines," based on what has otherwise often been attributed to the use of compass and straightedge to form designs and periodic patterns characterized by interlacing strapwork. According to their interpretation, this breakthrough allowed for the construction of more complicated "girih patterns," with an underlying tessellation of "girih tiles." Saltzman relates the complicated visual appearance of this and other Islamic patterns to "grid dualization," which he describes as "the use of an underlying polygonal grid from which the design is derived" [Saltzman 2008: 154]. Such dual lines are often constructed using an edge mid-point rule, as in the Gonbad-e Qabud. From the perspective of both designer and practitioner, Bonner demonstrates the same results by means of a "polygonal technique" with sub-grids [Bonner 2003]. Other authors [Cromwell 2009; Lee 1987; Fleurent 1992; Kaplan 2005] ascribe similar procedures to the practical construction of such patterns using different terminology, including "polygons in contact" [Kaplan 2005] and "star and polygon compositions" [Necipoğlu 1995: 92]. The actual processes that craftsmen used in the twelfth century have not been determined definitively, but mathematicians at that time endeavored to understand relationships between geometry and algebra [Rashed 1994] and actively sought demonstrations for proofs based on Euclidean models [Kheirandish 2008]. Dissections were part of the mathematical curriculum, as were compass and straightedge constructions [Berggren 2007; Rashed 2005].

Such geometric patterns proliferate on the Iranian plateau in architectural monuments of the eleventh and twelfth centuries, including tomb towers, minarets, shrines, and mosques of the pre-Mongol period [Korn 2010]. They also appear on monuments built with Mongol patronage in the following decades [Blair 2004; Wilber 1969]. One would like to interpret their appearance in the eleventh and twelfth centuries as formative, and those of subsequent generations as normative within the confines of Islamic art and architectural ornament.

Two final points at this juncture may secure this contemporaneous interlocking of the histories of mathematics and architecture at Maragha. The first concerns the anonymous manuscript, "On Interlocking Similar or Congruent Figures" (Paris BN ms persan 169), discussed by Chorbachi [Chorbachi and Loeb 1992] and by Özdural [1996], which is appended to the text of the work by Abū'l-Wafā al-Būzjānī (d. 998), *Kitāb fīmā yahtā ju ilayhi al-sāni' min a'māl al-handasa* (*A book on those geometric constructions which are necessary for a craftsman*), in the copy held at the Bibliothèque national in Paris, indicating its relevance to issues of art and practice. The second concerns a different copy of the text of al-Buzjani at the Süleymaniye Library Istanbul [Necipoğlu 1995: 134-37, figs. 99-102], which had been copied for Ulugh Beg's royal library in Samarqand, indicating its continued relevance and usefulness as a manual of practical geometry in the Timurid capital in the fifteenth century. Ulugh Beg, grandson of Timur, was himself an astronomer and established an astronomical observatory on the model of Maragha.

Conclusion

Given the unique combination of architectural form and patterns of Gonbad-e Qabud in Maragha, that there is an emphasis on geometry is obvious (figs. 1, 3, 5, 10). The visual complexity of the dual pattern around the shaft of the monument, with its local pentagonal and decagonal symmetries, when viewed in the context of the decagonal prismatic structure with its ten-sided pyramidal roof, is potentially all the more significant in the context of an actively engaged mathematical community seeking to explore visual aspects of theoretical understanding with practical applications [Özdural 2000]. Although the presence of such a community in pre-Mongol Maragha remains speculative, it is reinforced by the very selection of Maragha in the period after the Mongol conquests as the location for the new astronomical observatory after the taking of Alamūt, where there was an earlier observatory at which Naṣīr al-Dīn Ṭūsī worked out what came to be called the "Tusi couple," which subsequently influenced the development of planetary theories in the Renaissance [Saliba 2007]. In the taking of Baghdad in 1258 the rich scientific library holdings were preserved and brought to Maragha [Saliba 2007: 244], where new instruments were developed and crafted, affecting subsequent developments in the observatory of the court of Hulegu's brother, Kublai Khan [Sezgin 1998: 266-76], and later the astronomical observatory of Ulugh Beg in Samarqand.

But even more relevant is the position Gonbad-e Qabud takes within the lineage of architectural decoration among Saljuq and pre-Mongol monuments, as evident especially in tomb towers on the Iranian plateau. Not only is there a clearly identifiable development of planar patterns based on the practical understanding of symmetry as an organizing principle (achieved through iteration of a design algorithm [Bier 2009] in cut bricks, and then with the addition of color), but there is also a noticeable evolution in experimentation with overlapping and interlacing convex polygons and star polygons, from ancient conventions of three-sided triangles and six-pointed stars, and squares and eight-pointed stars, to the appearance of secondary pentagons in polygonal nets of twelve-pointed stars, to hexagons and double hexagon grids interlacing with nonagons with a superimposed net of overlapping dodecagons, the last combination introduced at

Gonbad-e Sork also in Maragha (fig. 7). The Gonbad-e Qabud then brings to bear a conscious progression of 5-, 6-, and 7-pointed stars laid out in a dual grid (figs. 3, 4), and a geometric progression of 8-, 9- and (half) 10-pointed stars in an artistic rendering of rayed stars with petal-sharing (figs. 5, 6), situated above the visually complex pattern with a dual grid around the shaft of the monument (fig. 10). Parallel to these extraordinary and creative developments in the geometric structures and decoration of architecture are the problems set forth in texts of contemporary mathematics (fig. 11a), the understanding and practical articulation of which (figs. 10, 11c) is evident in these monuments. This direction holds great potential for future research to guide our further appreciation and comprehension of Gonbad-e Qabud as not only a nexus of architecture and visual mathematics, but a nexus also for the history of art and architecture and the history of mathematics.

Acknowledgments

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Notes

1. Recent scholarship [Jackson 2005] places the death of Sorghaghtani Beki in 1252, and refers to a subsequent burial in a Christian Church in Gansu. A Kerait princess, and the daughter-in-law of Chinggis Khan, she was herself a Nestorian Christian and is known to have retained her religious practice throughout her life. There are, however, Christian monuments at Maragha, mentioned by Kleiss [2002] in his enumeration of monuments of Azerbaijan.
2. Statements such as that in [Lu and Steinhardt 2007a: SOM-S6] and [Makovicky 2007: 1382] need to be amended to take into account the decagonal plan [Bier 2011a].
3. According to Daneshvari [1982] these badly damaged inscriptions were likely also Qur'anic passages.
4. *Gonbad* means dome, and is the term used to described a domed tomb tower; both *sork* and *qermez* mean "red," so "Red Tomb" or "Red Tomb Tower" are both reasonable translations.
5. On Gonbad-e Ghaffariya this only appears on the engaged columns [ArchNet].
6. Milwright [2002] makes note of a dark blue boss marking the center, which is just barely evident in published photographs. Also cited in [Stronach and Young 1966: 15], the spandrels of the entry façade in the later tomb tower at Kharraqan "each support a circular setting for a glazed boss" (see fig. 8). In this case the date of the earliest use of glazed ceramic in the architecture of Iran must be pushed back to the late eleventh century.
7. Much of the following section is drawn from [Bier 2002].
8. For interpretation of the Qur'anic excerpts in relation to the geometric patterns at Kharraqan, see [Bier 2002; 2008]. This is expanded on in [Bier 2011b].
9. Exceptions are the work of Chorbachi [Chorbachi 1989; Chorbachi and Loeb 1992], Özdural [1995; 1996; 1998; 2000; 2002], and Necipoğlu [1995].
10. For an outstanding discussion of such Western attitudes, see "The Notion of Western Science: 'Science as a Western Phenomenon,'" Appendix 1 in [Rashed 1994: 332-49].

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[ArchNet] ArchNet. http://archnet.org (an international web-based community based at MIT's Design Lab in the School of Architecture and Planning, with a focus on Muslim cultures and civilizations).

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About the author

Carol Bier is a Visiting Scholar (2010-12) with the Center for Islamic Studies at the Graduate Theological Union in Berkeley CA and Research Associate at The Textile Museum in Washington DC (2001-12), where she served as Curator for Eastern Hemisphere Collections (1984-2001). Her current research focuses on the historical development of Islamic patterns as intersections of geometry, art, religion, and philosophy.