Ahmed Ali Elkhateeb

King Abdulaziz University
Department of Architecture
Faculty of Environmental Design
Jeddah, Saudia Arabia
aelkhateeb@kau.edu.sa

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Research

Domes in the Islamic Architecture of Cairo City: A Mathematical Approach

Abstract. This work analyzes mathematically and graphically the two methods used historically in the transitional zone between the circular base of the dome and the square top of the cube where the dome is supported. The time frame of this work is the distinguished historical buildings of Islamic Cairo built between the ninth and eighteenth centuries. Ten samples were chosen out of a total of thirty. A set of mathematical expressions has been derived to relate the different parts of the squinches/pendentives to the cube with a side of length l. The equations derived were validated twice, first by generating 3D graphical sequences for both squinches and pendentives for the selected domes using CAD software based on the values obtained from the driven equations, and second by executing physical models using a 3D printer for two examples of squinches and pendentives.

1 Introduction

Domes are one of the most distinguished architectural elements; their antiquity (which may date back to more than five thousand years ago) increases their ambiguity and charm, and their high flexibility in covering the very wide range of different spans from few meters to hundreds of meters increases their modernity and power. From the greatness of history and the strength of the present, domes acquired their originality and modernity, thus it deserve to be the king of all roofs. Historically there isn't any difficulty in constructing walls as boundaries for a space to fulfill the requirements of both privacy and security, as construction is one of the oldest human activities, but the real problem was how to cover or roof the space. This may be one of the reasons that limited the spans of spaces for a long time. In the various phases of construction, the dome was one of the inventions for covering large span spaces that emerged in response to the requirements of the newest functions, such as temples, palaces and the like.

Fig. 1. The oldest and the most famous domes in Islamic Architecture. a, left) The Holy Dome of the Rock, Jerusalem; b, right) The Taj Mahal, Agra

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In Islamic architecture, the subject of this present work, the dome has played a leading role since the early beginning of Islamic era. The Holy Dome of the Rock (fig. 1a), built by the Umayyad ruler Abdol Malek Ibn Mrwan between 687 and 691 A.D. in Jerusalem, may be the oldest dome in Islamic Architecture. Subsequently, Muslim architects admired the dome and used it in excellently, thus the dome applied widely with different forms as a cover for various types of buildings in the Islamic era: Masjids (Mosques), palaces, baths, mausoleums and more. With its onion dome, the mausoleum of Taj Mahal in Agra, India, built between 1632-1648 A.D. by Shah Jahan (the Mughal ruler) as a mausoleum for his favourite wife, is one of the most famous domes in Islamic architecture (fig. 1b).

1.1 Problem definition

While there are numerous research works examining the architectural features and aesthetical values of Islamic architecture, research works concerning this architecture from the analytical and technical points of view are still limited (see, for example, [Cipriani 2005; Harmsen 2006; Harmsen et al. 2007; Krömker 2010; Takahashi]). It is a well-known historical fact that the domes were supported even on squinches or pendentives [Abouseif-Behrens 1996]. This present work investigates mathematically and geometrically the types of forms that were utilized to support the domes covering the cuboid spaces (square plans). More precisely, it tries to answer two questions:

- What are the forms that were used to generate both the squinches and pendentives?
- What are the mathematical formulae that can be used to generate such forms?

1.2 Objectives

The main purpose of this work is to derive sets of mathematical formulae that express the different forms used in the transitional zone between the circular base of the dome and the square top of the cube where this dome rests. These formulae will contribute to:

- Greater understanding of the two terms *squinches* and *pendentives*, the only two methods used historically to support the domes in architecture generally and in Islamic architecture in particular;
- Differentiate and distinguish architecturally and mathematically between the formative features of the terms squinches and pendentives;
- Generate accurate drawings for both forms, which are a prerequisite for more advanced technical analysis such as acoustics and lighting that utilize simulation software and require accurate drawings for the spaces under consideration.

1.3 Methodology

For the purpose of this work, a comprehensive historical survey was conducted on the most distinguished domes of the historical buildings in Islamic Cairo in order to determine the various shapes of the historical domes, and to collect the main different forms that were utilized in both squinches and pendentives in the transitional zone.

Based on the historical survey [Elkhateeb and Soliman 2009], ten Masjids were chosen out of thirty Masjids that were built in Cairo between the *Tulunid dynasty* (868-905) and the end of *Ottoman* period in Egypt (1517-1798). Finally, the collected forms were analyzed using the mathematical principles of solid bodies and analytic geometry to derive the required formulae. The formulae thus derived were verified twice, first by producing a 3D graphical sequence for such forms based on the values obtained from the derived equations using AutoCAD (ver2008), and second by generating physical models for one example that uses squinches and one that uses pendentives, utilizing a 3D printer.

Fig. 2 summarizes the methodology applied in this work.

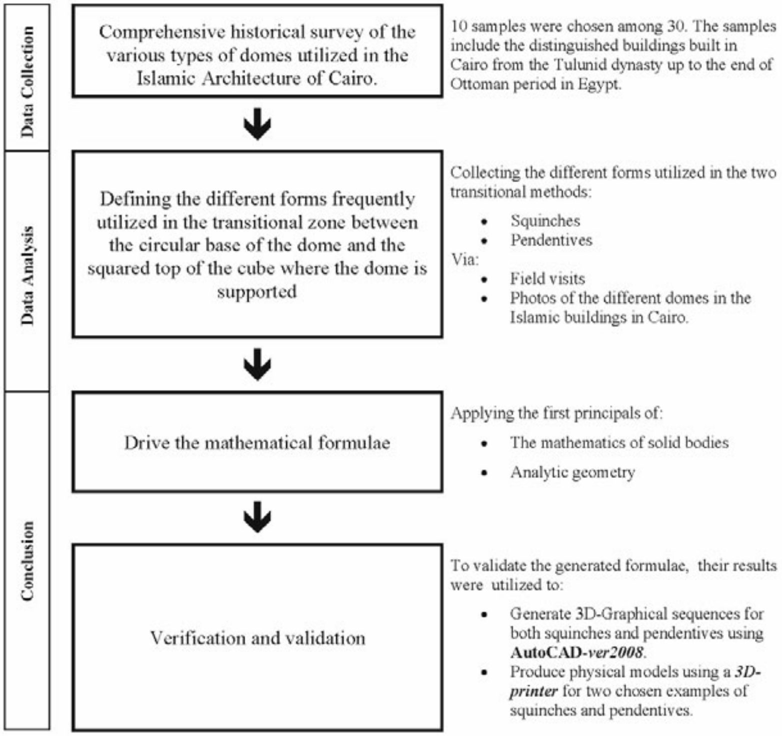


Fig. 2. Methodology

2 The origin of the domes and their types

In architecture, the dome is a vaulted roof having a circular, polygonal, or elliptical base. Since it is difficult to count the various types of domes, the next part will discuss only the most famous types of domes used in Cairo's Islamic architecture. From the architectural point of view, there are two types of domes that were used frequently almost in all of Cairo's Islamic buildings, the spherical dome (based on a perfect sphere) and the elliptical dome (based on a spheroid).

Under these two main categories, many other types of domes can be classified, such as onion and shallow (saucer) domes. Mathematically, both spherical and prolate, or vertically elliptical, domes originate from a complete rotation (360°) of a segment of an arch around its vertical axis. The type of the dome (spherical or elliptical with their different types) will be determined according to the rotated part of the arch and consequently, the position of the rotation axis Y and the horizontal axis X and the relation of both with the origin (0,0) (fig. 3a). In this context, there are two main possibilities, within which are three others, as listed as follows:

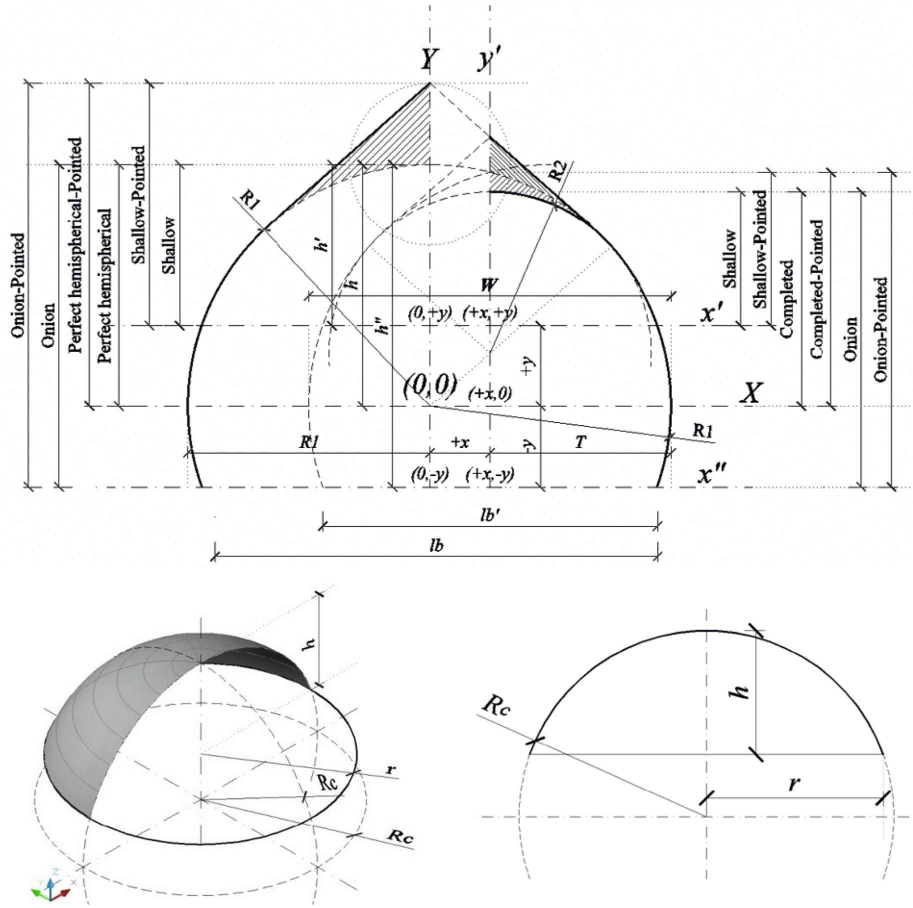


Fig. 3. The origin of the domes and their types. a, above) The origin of the most famous domes; b, lower left) Spherical dome, perspective; c, lower right) Spherical dome, mathematical values

1. The first possibility: the center of the rotated arch and the rotational axis *Y* is the origin (0,0). This is the *spherical dome*. Under this category and according to the location of the horizontal axis *X*, the other three possibilities are:
   - The horizontal axis *X* passes through the origin (0,0). This is the perfect hemispherical dome with its two types (flat or pointed);
   - The horizontal axis *X* moves to the location x' to pass through the point (0,+*y*). This is the shallow (saucer) dome with its two types (flat or pointed);
   - The horizontal axis *X* moves to the location x'' to pass through the point (0,–*y*). This is the onion dome with its two types (flat or pointed).
2. The second possibility: the center of the rotated arch is the origin (0,0) but the rotational axis *Y* is moved to the location (+*x*,0) to be at the location y'. This is the *elliptical dome*. Within this category and according to the shape of the rotating part and the location of the horizontal axis *X*, the other three possibilities are:

- The horizontal axis X passes through the location $(+x,0)$. This is the perfect half elliptical dome with its two types (flat or pointed);
- The horizontal axis X moves to the location x' to pass through the point $(+x,+y)$. This is the shallow (saucer) elliptical dome with its two types (flat or pointed);
- The horizontal axis X moves to the location x'' to pass through the point $(+x,-y)$. This is the onion elliptical dome with its two types (flat or pointed).

In shallow (saucer) domes (either spherical or elliptical), the height of the rotated part h' is less than the arch radius *R1*, whereas in the onion dome (either spherical or elliptical) the height of the rotated part h'' is greater than the arch half span *R1* or *T*. In other words, the diameter of the dome base *lb* or *lb'* is less than its width *2R1* or *W* respectively. Table 1 summarizes the previous classification.

Dome	Classification		Location of Arch center	Location of rotation Axis *Y*	Location of the *X* Axis	Remarks
spherical	Shallow	Pointed	*0,0*	*Y (0,0)*	*x' (0,+y)*	Dome height
		Flat	*0,0*	*Y (0,0)*	*x' (0,+y)*	***h'< its radius***
	Perfect	Pointed	*0,0*	*Y (0,0)*	*X (0,0)*	Dome height
		Flat	*0,0*	*Y (0,0)*	*X (0,0)*	***h= its radius***
	Onion	Pointed	*0,0*	*Y (0,0)*	*X'' (0,-y)*	Dome height
		Flat	*0,0*	*Y (0,0)*	*X'' (0,-y)*	***h'' its radius***
Elliptical	Shallow	Pointed	*0,0*	*y' (+x,0)*	*x' (+x,+y)*	Dome height
		Flat	*0,0*	*y' (+x,0)*	*x' (+x,+y)*	***h'< its radius***
	Perfect	Pointed	*0,0*	*y' (+x,0)*	*X (+x,0)*	The *X* axis passes
		Flat	*0,0*	*y' (+x,0)*	*X (+x,0)*	the arch center
	Onion	Pointed	*0,0*	*y' (+x,0)*	*X''(+x,-y)*	The diameter of
		Flat	*0,0*	*y' (+x,0)*	*X'' (+x,-y)*	dome base ***lb< its width W***

Table 1. The architectural classification of the most famous historical domes in Cairo

Historically, onion domes have a limited application in Cairo's Islamic architecture, but it is well known and widely used in the Far East, Russia, India and Turkey. Still, there are a limited number of examples in Mamluk architecture. In Cairo, the other types of domes have been used widely, because of their importance in the context of this present research, the following parts will discuss them in some details.

2.1 Spherical domes

From the mathematical point of view, the spherical domes can be completely described via its height h and radius of curvature R_c (see figs. 3a-c), where:

$$R_c = \frac{r^2 + h^2}{2h} \tag{1}$$

The volume of this type of domes can be also calculated by knowing its height $\boldsymbol{h}$ and radius of curvature R_c, or the radius of dome base r [Gieck and Gieck 2006]:

$$V_s = \frac{1}{3}\pi h^2 (3R_c - h) = \frac{1}{6}\pi h(3r^2 + h^2) \tag{2}$$

The total surface area of this dome can be calculated:

$$S_a = 2\pi h R_c = \pi (h^2 + r^2) \tag{3}$$

2.2 Elliptical domes

Elliptical domes have wide applications in architecture; they are especially useful in covering the rectangular spaces. The oblate, or horizontal elliptical, dome utilized in particular where there is a need to limit the excessive height of the space that accompanies the use of the spherical domes. The prolate, or vertical elliptical, dome, is the one most often applied from the architectural point of view, as the majority of the most famous domes in architectural history, including Islamic architecture (as will be seen later, see §4), are related to this type of elliptical dome. Usually, the mathematical description of the elliptical domes originates from the mathematical description of the spheroid itself. As the mathematical description of the elliptical domes is more complicated than that of spherical ones, this part will not be considered here, and only its applied architectural part will be considered.

3 Methods of supporting domes in Islamic architecture

Because the form of the dome initially originated from the arch, it exerts thrusts all around its perimeter, and the earliest monumental examples that utilize the dome as a cover required heavy continuous circular (or polygonal) walls to offer continuous support to withstand these forces safely. This may explain those massive walls used, for example, in the Pantheon, which allowed the engineers of the temple to create some decorative niches in the walls, aimed at decreasing its weight. To be sure, these massive walls permitted few openings (and had to be rounded or polygonal to give the required continuous support), limiting for a long time the possibilities to incorporate the domes in complex buildings that consist of many adjacent spaces required connection and continuity, especially when adjacent spaces are vaulted. And this may explain again the reason for the simplicity in the form of the early buildings that utilized domes as a cover.

The next milestone in covering a space with a dome was the transformation from the circular to the square plan (see fig. 4). By this transformation, the incorporation of domes in the building became easier, important and distinguished. It is not an exaggeration to say that with very few exceptions the most valuable historical domes in this context cover a square plans; this is clear at least in historical Islamic Cairo where most of the buildings are either square or semi-square. Historically, the main difficulty that has to be overcome by engineers is how to fill the four voids emerging between the circular base of the dome and square top of the cuboid space where the dome rests (see fig. 4). These voids were known in the different historical references as the "transitional zone" (see a perspective view in fig. 8a). For the purpose of this work, this will be called the "corner blocks".

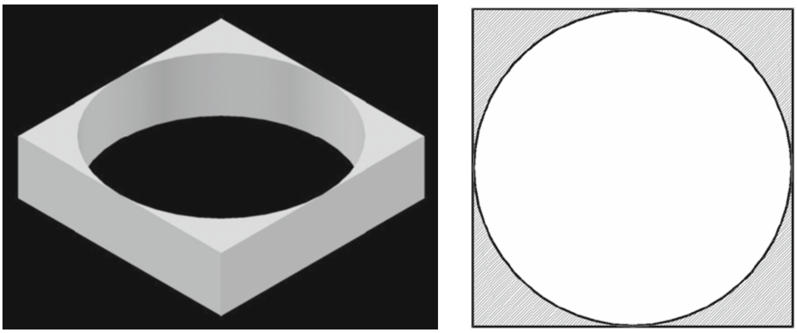


Fig. 4. The transitional zone between square and circle. a, left) perspective; b, right) plan

With their form, these blocks were not appreciated by the architects of the different eras for many reasons: they were unattractive if they were seen in such form inside the room, they were so heavy without any necessity or reason, and unsafe from the structural point of view. The transmission paths of loads coming from the dome and its weight do not require such heavy blocks. To overcome this difficulty, the architects of those days turned to *hollowing* the unwanted parts of these massive blocks. The concept of hollowing the corner blocks was a solution that was at once both aesthetic and functional, as it first makes it possible to decrease the loads of these blocks, keeping them safe to support the dome, thus it functions well structurally and second, provides unlimited possibilities to invent aesthetic methods for this hollowing, leading ultimately to beautiful and innovative forms.

Two methods were utilized to hollow those corner blocks. These methods distinguish between the ways of hollowing, and between the way the loads of the dome are transimitted to the cuboid:

1. The first method: using the squinch(es). Originally, it is a support carried across the corner of a room under a superimposed mass. In architecture, squinches can be one of any of several devices by which a square or polygonal room has its upper corners filled in to form a support for a dome. In its simplest form used in Cairo's Islamic architecture, it is a four diagonal niches, one in each corner of the room transforming the square plan of the room to an octagonal one.
2. The second method: using the pendentive(s). Originally one of the concave triangular members that support a dome over a square space, in architecture, pendentives are triangular segments of a sphere, taper to points at the bottom and spread at the top to establish the continuous circular or elliptical base needed to support a dome.

The invention of methods of supporting a dome over a cuboid space and consequently the need to hollow the four corner blocks which appear in the transitional zone is not an absolute Islamic invention. In fact, it preceded Islam itself by many centuries. However, while the the idea of hollowing itself cannot be be credited to Islamic civilization, Islamic architecture demonstates its superiority in innovating new methods for this hollowing. Successive eras of the Islamic civilization in Egypt (specially the *Burgi Mamluks*, 1382-1517) added the methods to manipulate the exterior part of the transitional zone sculpturally as well. This means that the architectural manipulation of the corner block is not limited to the interior of the space, but can be done internally and externally simultaneously. Some examples are shown in in fig. 5.

Modern structural theories indicate that a slope of 1:2 or higher in corbels (see fig. 6) transmits the loads smoothly and safely (see, for example, [American Concrete Institute 2004]. According to this rule, the height of the corner blocks has to fulfill at least the following equations:

$$r_c = \frac{l}{\sqrt{2}} \tag{4}$$

$$C_l = \frac{l}{\sqrt{2}} - \frac{l}{2} \tag{5}$$

$$C_h = \frac{2l}{\sqrt{2}} - l = 0.4142\, l \tag{6}$$

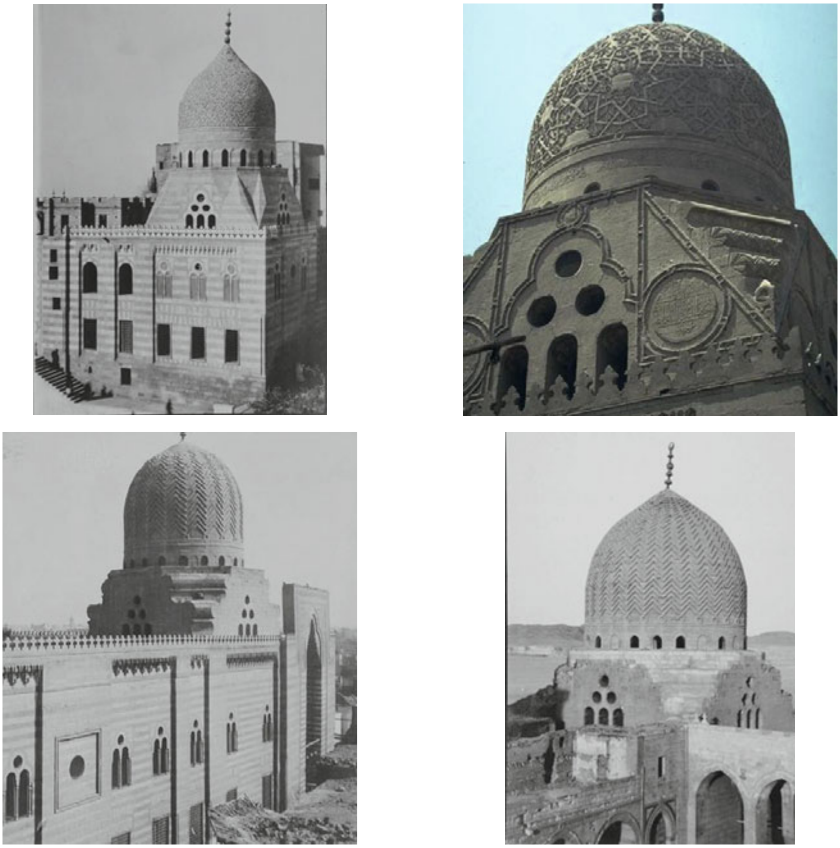


Fig. 5. Examples for the transitional zone formation in chronological order. a, upper left) Amir Qanibay al-Rammah, 1503 A.D.; b, upper right) Sultan Qaytbay, 1475 A.D.; c, lower left) Sultan al-Mu'ayyad Shaykh, 1420 A.D.; d, lower right) Sultan Farag Ibn Barquq, 1411 A.D.

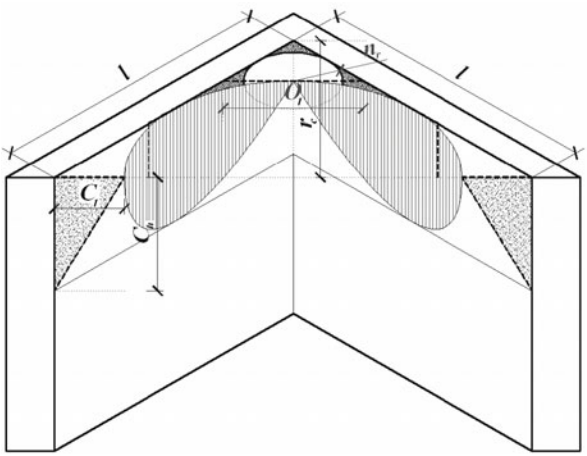


Fig. 6. The corbel length C_l, height C_h and the cube side length l

where l is the cube side length, C_l is the maximum corbel length and C_h is its height (fig. 6). Although this rule was not applied in some of the cases studied, as shown clearly in the squinches of Firouzabad Palace (the squinches of conical vault, see fig. 7), it seems to have been applied in some other examples, especially the pendentives.

3.1 The mathematical analysis of squinches

The squinch was a primitive solution to hollow the corner blocks in the transitional zone. It is believed that it was originally developed, almost simultaneously, by the Roman builders of the late Imperial period and the Sasanian in Persia, where it was used in the palace of Ardeshir, the Emperor of the Persian Empire, near Firouzabad, Iran. Later it was used by the Byzantine architects in their domed buildings, and finally by the architects of the Islamic era. It seems that Islamic architecture developed its own squinches, borrowing from the Sasanian civilization, the nearest to it in place and time.

"Squinch" is defined in the *Encyclopedia Britanica* in the following way:

> **squinch**, in architecture, any of several devices by which a square or polygonal room has its upper corners filled in to form a support for a dome: by corbelling out the courses of masonry, each course projecting slightly beyond the one below; by building one or more arches diagonally across the corner; by building in the corner a niche with a half dome at its head [see fig. 8]; or by filling the corner with a little conical vault that has an arch on its outer diagonal face and its apex in the corner [see fig. 7]

(see http://www.britannica.com/EBchecked/topic/561801/squinch).

From this definition it seems that the word squinch can be considered a category more than a terminology; this category includes every method used to fill in the transitional zone except the pendentives. Again, according to *Britannica's* definition, the main features of the squinches can be summarized as follows:

- Transformation of the square top of the cube first to a polygon (usually an octagon, where the four corner blocks, one at each corner, will appear) and then to a circle. The octagon side length *OL* can be calculated according to Eq. 7;
- The dome rests on the polygon that rests on the square;
- The lower part of the squinch must be a line, not a point, in contrast to pendentives (as will be seen later);
- The corner block is hollowed by the niches or any other means, as previously mentioned.

The length of octagon side O_L (fig. 6) can be calculated from:

$$O_L = l \tan 22.5 = 0.4142\, l \qquad (7)$$

In Cairo's Islamic architecture, the squinches started in abstracted form, only four niches with a half domes at their heads, one at each corner, as used in the Mihrab (niche) dome of Al-Hakim Masjid (1013 A.D.) from the Fatimid period. Here it is clear that the dome rests on the octagon that rests on the square (see fig. 8). Later, and over the course of many decades, squinches developed more complicated forms, for example, using the stone corbel as in the Masjid and school of Sultan Baybars Al-Jashankir (1310 A.D.), using the spherical-pointed vault as in Al-Fadawiyya Dome (1479 A.D.), and others. It seems that the conical vault (see fig. 7) used in the Sasanian architecture was not used in Islamic architecture in Cairo, as I have been unable to find any similarity, but the other types have been widely utilized in buildings of Islamic Cairo.

The squinches of a conical vault, the simplest form of the squinches, originate from the intersection between a diagonal right circular half-conical vault, its apex in the corner, and the corner block. The radius of the cone is equal to the height of the corner blocks and its height can be calculated from Eq. 5.

Fig. 7 represents in an analytical series the method for generating the kind of squinch used in Firouzabad Palace. The squinches of the niche with a half-dome originate from the intersection between the positive form of the niche with its half-dome and the corner block. Fig. 8 represents in an analytical series the method for generating the kind of squinch used in Al-Hakim Masjid. The height of the niche is determined according to Eq. 6 whereas its radius n_r (see fig. 6) can be calculated from:

$$n_r = 0.14644l \tag{8}$$

The stone corbel squinches originate by corbelling out the courses of stone: each course projects gradually and slightly beyond the one below until it reaches the dome's circular base. In Islamic architecture, stone corbels were known as "stalactite vaults" (see §3.2). In the last type of squinch where the concept of stalactites is used, the way of hollowing the corner blocks has various and different forms depending on the architect's imagination and his ability to invent new stalactite forms which, when complete, compose the stone corbel. Fig. 9 represents in an analytical series the method for generating the kind of squinch used in Masjid Sultan Baybars Al-Jashankir.

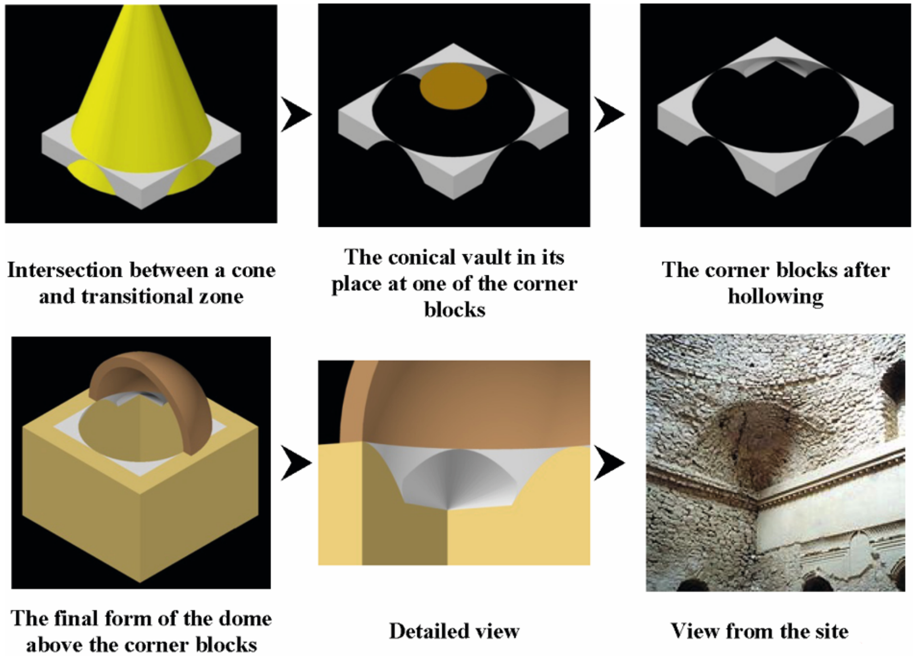


Fig. 7. The conical vault squinch used in the Palace of Ardeshir, Sasanian architecture

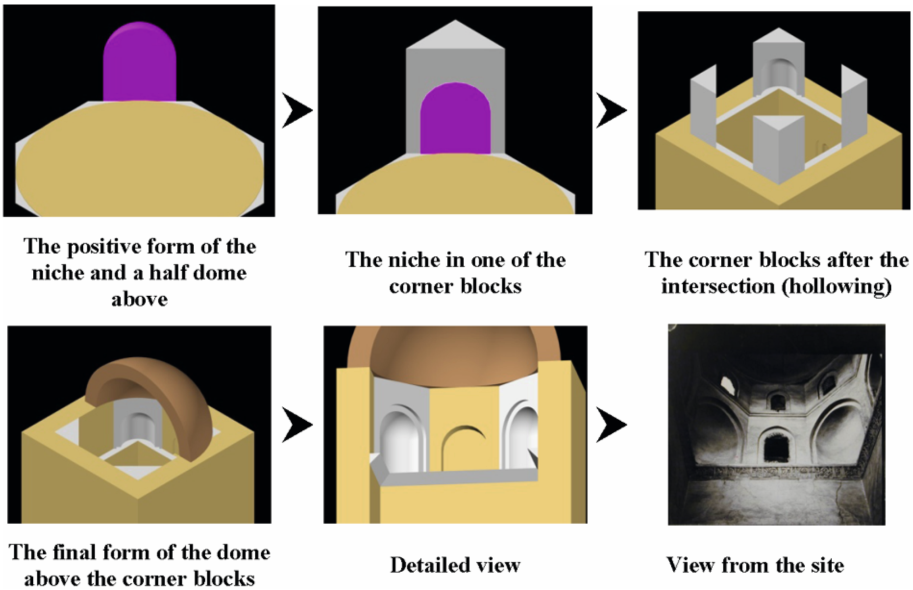


Fig. 8. The niche with a half-dome squinch used in Al-Hakim Masjid (Fatimid period)

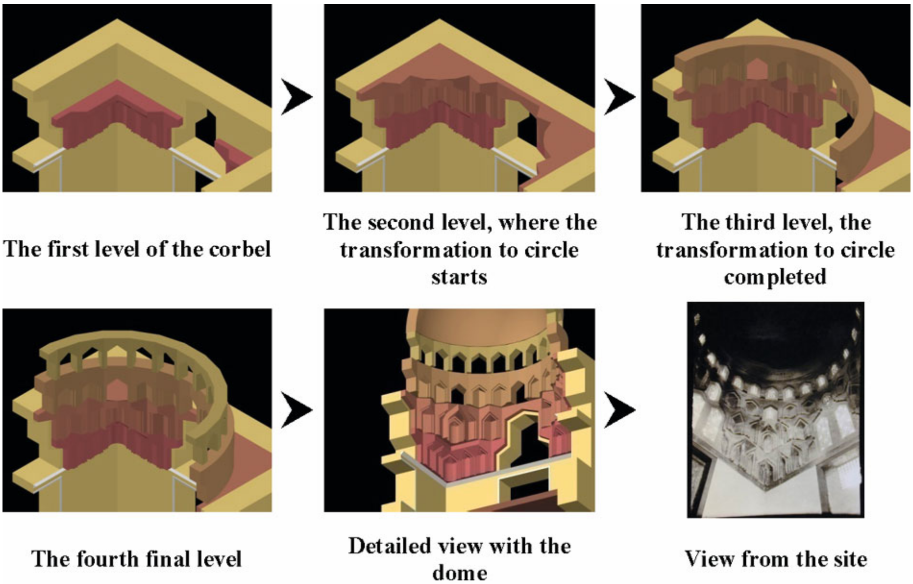


Fig. 9. The stone corbel (stalactite vault) squinch used in Sultan Baybars Al-Jashankir Masjid (Bahari Mamluks period)

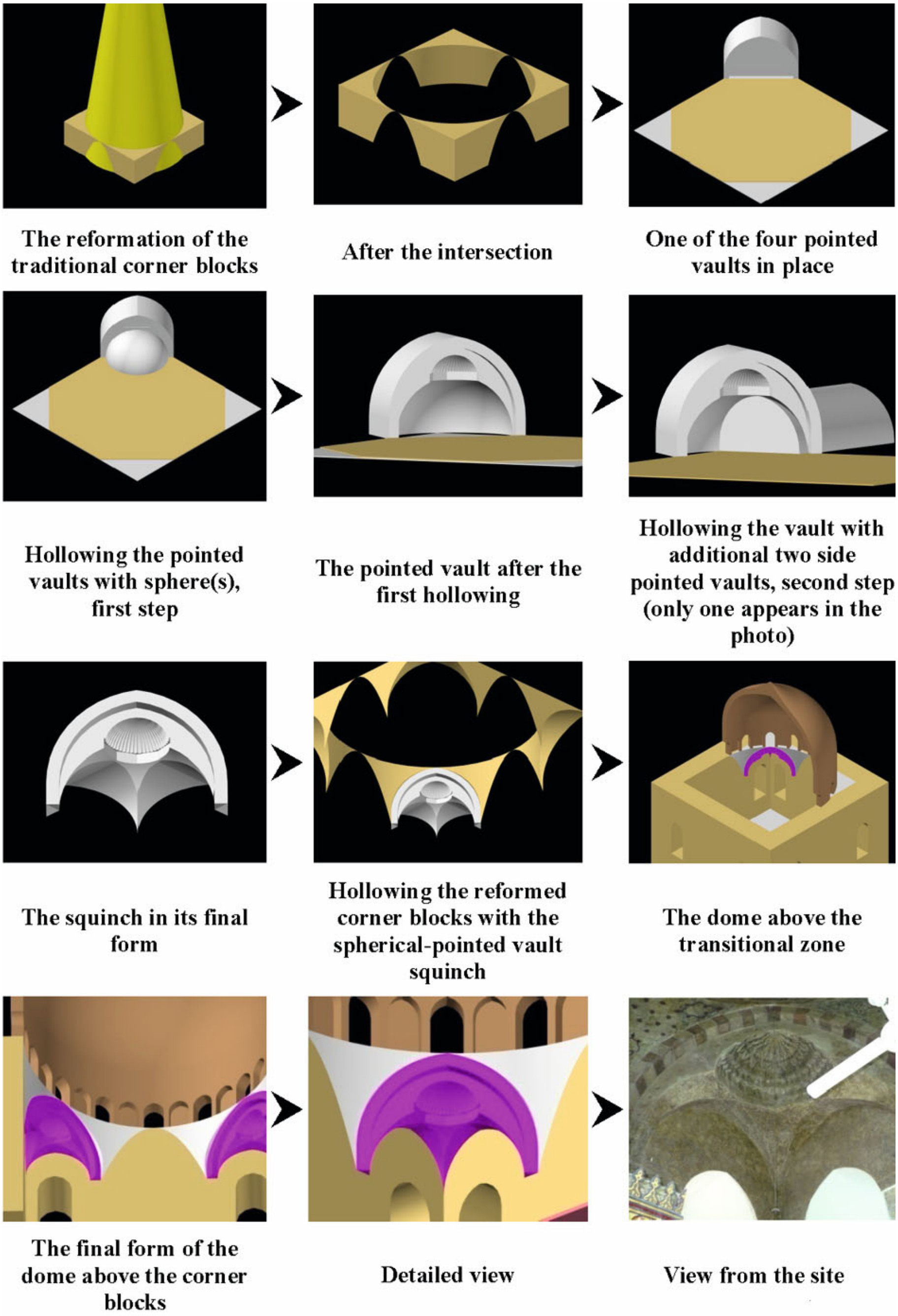


Fig. 10. The spherical-pointed vault squinch used in Al-Fadawiyya Dome Masjid (Burgi Mamluks period)

On the other hand, the most complicated type of squinch is the spherical-pointed vault. This type of squinch has been used near the end of the second Mamluks era (Burji Mamluks, 1382-1517) in Al-Fadawiyya Dome Masjid in Cairo. Later it was utilized typically in many examples up to the end of the Ottoman period in Egypt (see, for example, Masjid Sinan Pasha, 1571 A.D., and Masjid Muhammad Bey Abu al-Dhahab, 1774 A.D.). In such special type squinch, the corner blocks are firstly reformed by intersecting a frustum cone (its height is given by Eq. 6) with the traditional corner blocks. The upper and lower radii of the frustum cone (r_u, r_l respectively) can be calculated from:

$$r_l = r_u + 0.04119l \tag{9}$$

$$r_u = l/2 \tag{10}$$

$$r_l = 0.54119l \tag{11}$$

The resultant corner blocks are then hollowed in two steps:

1. The first, by using spherical forms (the maximum radius of this sphere is 0.2071*l*).
2. The second, by intersecting the resultant hollow blocks with eight pointed perpendicular vaults, two per each block.

Fig. 10 represents in an analytical series the method for generating the kind of spherical-pointed squinch used in Al-Fadawiyya Dome Masjid.

3.2 The mathematical analysis of pendentives

Byzantine architects invented a new solution – one is considered again a milestone in the long history of domes – by constructing domes on piers instead of the massive continuous cylindrical walls. The transition from the square top of the cube to the circle was achieved by four inverted spherical triangles called pendentives, which are masses of masonry curved both horizontally and vertically. Their apexes rested on the four piers, to which they conducted the forces of the dome; their sides joined to form arches over openings in four faces of the cube; their bases met in a complete circle to form the dome foundation. The pendentive dome could either rest directly on this foundation, or upon a cylindrical wall, called a drum, inserted between the two to increase height.

Two main features distinguish pendentives. The first is that their lower parts are points (not lines as in the case of the squinches). The second is that they support the dome directly on the top of the cube without using the octagon as a transitional phase between the square and the circle (as in the case of the squinches). Instead, the volume of the transitional zones here is filled with the pendentives.

Pendentives originated from the intersection of a sphere (or pendentives sphere, as it will be called later) and a virtual cube (of side length *l*), where the center of the sphere is located on the axis of the virtual cube. The resulting pendentives depend on the type of the sphere (i.e., sphere or spheroid), its diameter D and the relation between its center and the center of the virtual cube. In case of the perfect sphere, there will be four possibilities to form the pendentives:

1. The two centers (of the sphere and the cube) are identical (located in the same point), the diameter of the sphere D and the diagonal of the faces of the cube $l\sqrt{2}$ are equal. This means that:

$$D = l\sqrt{2} \qquad (12)$$

This case results in four pendentives intersecting in four points directly under the dome (see fig. 11) to form the apexes of four arches on the four faces of the cube as previously mentioned. This is the case that has been used in Hagia Sophia for example (see fig. 17b). Also, the height of the pendentives h_p resulting from such intersection equals $l/2$ (see fig. 15, Eq. 16).

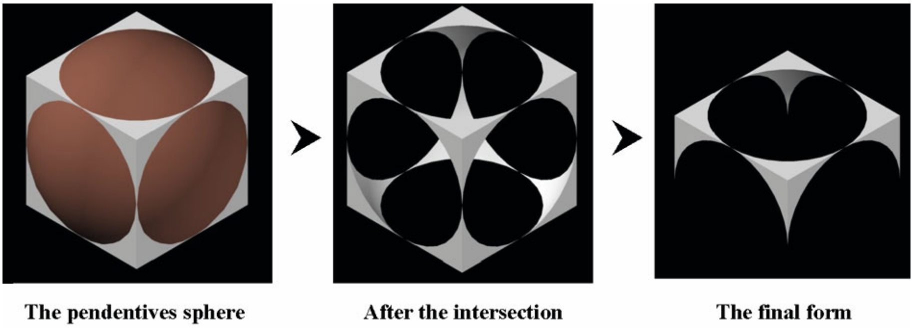


Fig. 11. Case One: the two centers are identical, and the diameter of the sphere and the diagonal of the faces of the cube are equal

It is important to mention here that in the simplest type of this form, the pendentives are part of the dome itself, and in this case the curvature of the dome R_c (see §2.1) must be equal to the vale $l\sqrt{2}$, but this is not a common case in Cairo's Islamic architecture due to the limited height of the dome.

2. The two centers are identical and the diameter of the sphere D is larger than the diagonal of the faces of the cube. Tthis means that:

$$D > l\sqrt{2} \qquad (13)$$

This case gives the common form of the pendentives in Cairo, where the pendentives bases are separated (see fig. 12) and the height of the pendentives h_p is less than $l/2$.

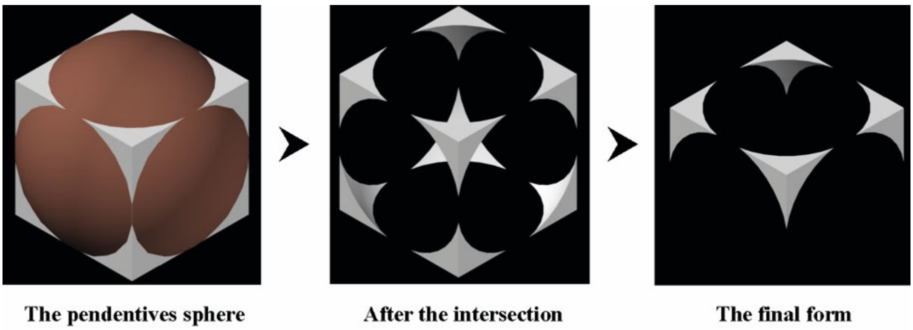


Fig. 12. Case Two: the two centers are identical and the diameter of the sphere D is larger than the diagonal of the faces of the cube

3. The center of the sphere is shifted from the center of the cube with a distance *S* (see fig. 15a), where *S* fulfills the condition:

$$l/2 + S > l/2 \tag{14}$$

In this case,

$$D = 2\sqrt{\frac{l^2}{2} + S^2 + lS} \tag{15}$$

This case results in pendentives similar to the first case where the pendentives bases all intersect in four points directly under the dome (see fig. 13). But the height of the pendentives h_p – calculated according to Eq. 16 – will be less than $l/2$.

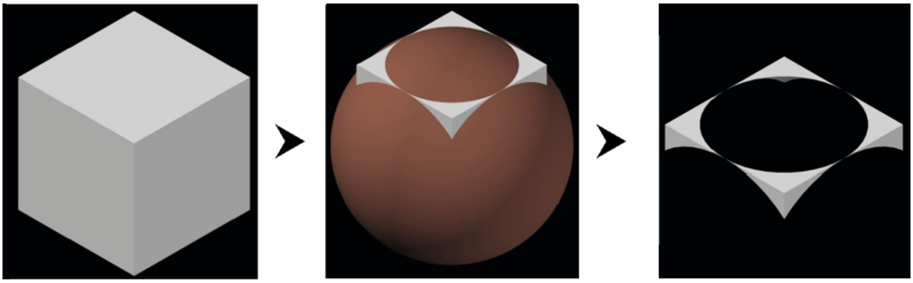


Fig. 13. Case Three: the center of the sphere is shifted from the center of the cube by a distance *S*, the diameter of the sphere is calculated according to eq. 14

4. The two centers are shifted and the diameter *D* is larger than the value resulting from Eq. 15. This case gives pendentives similar to the second case (see fig. 14) where the pendentives bases are separated and the height h_p is also less than $l/2$.

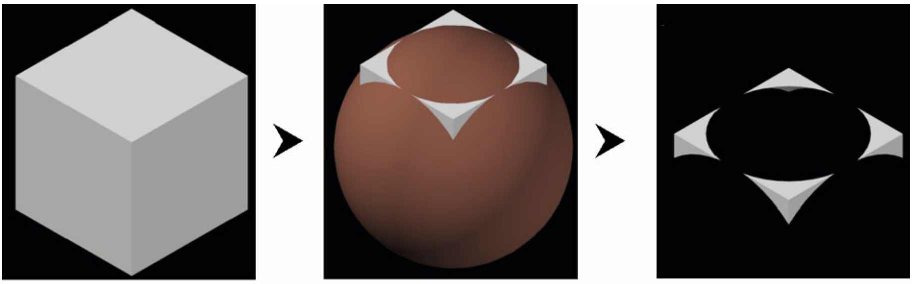


Fig. 14. Case Four: the two centers are shifted and the diameter *D* is larger than the value calculated according to eq. 14

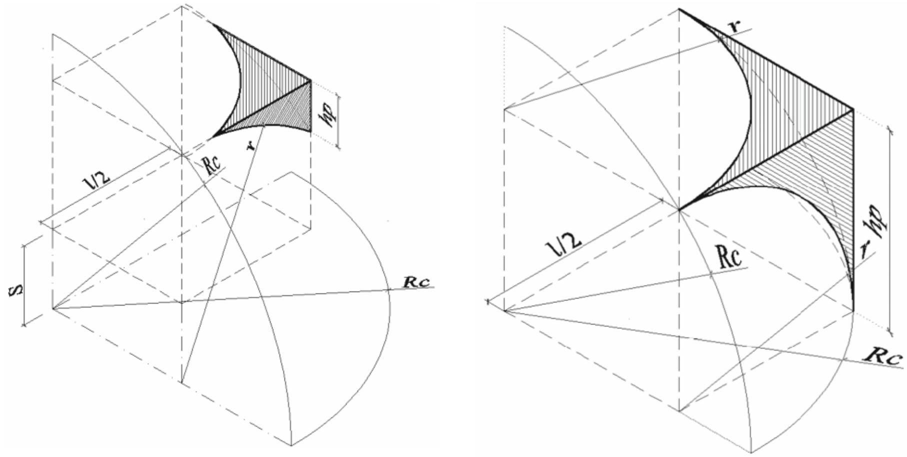


Fig. 15. The relation between pendentives height h_p , radius of curvature r_c , the virtual cube side length *l* and the radius of the intersected circle with the side faces of the cube *r* (the common case). (*r* of the side faces circles is equal to the top circle in the special case). a, left) the height h_p (the common case); the two centers are shifted with distance *S*; b, right) the height h_p (the special case); the two centers are identical and *S*=0

The height of the pendentives h_p (which must fulfill also Eq. 6) can be calculated by knowing the radius of curvature R_c (see Eq. 1), the virtual cube side length *l*, the radius of the intersected circle with the side faces of the cube *r* and the shift *S* (see fig. 15) according to the equation:

$$h_p = (l/2 + S) - \sqrt{r^2 - \frac{l^2}{4}} \qquad (16)$$

where *r* can be calculated by knowing R_c and *l* according to:

$$r = \sqrt{R_c{}^2 - \frac{l^2}{4}} \qquad (17)$$

The case of the identical centers (of sphere and cube) can be considered a special case in Eq. 16, where the shift *S* equals 0 and the radii of the intersected circles with the virtual cube faces are all equal.

It is also possible to generate the pendentives using the spheroid instead of the perfect sphere. This case generates the separated pendentives (see fig. 16) as previously mentioned in the second and fourth cases, but the intersection of the spheroid with the top face of the cube (where the dome rests) will remain a circle. The height of the pendentives h_p in this case will be also less than $l/2$. Although this is a possible method, it is beyond the scope of this present research.

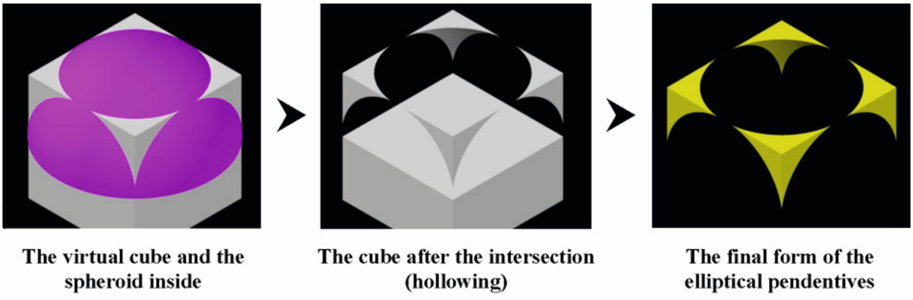


Fig. 16. Generating the pendentives using a spheroid instead of a perfect sphere

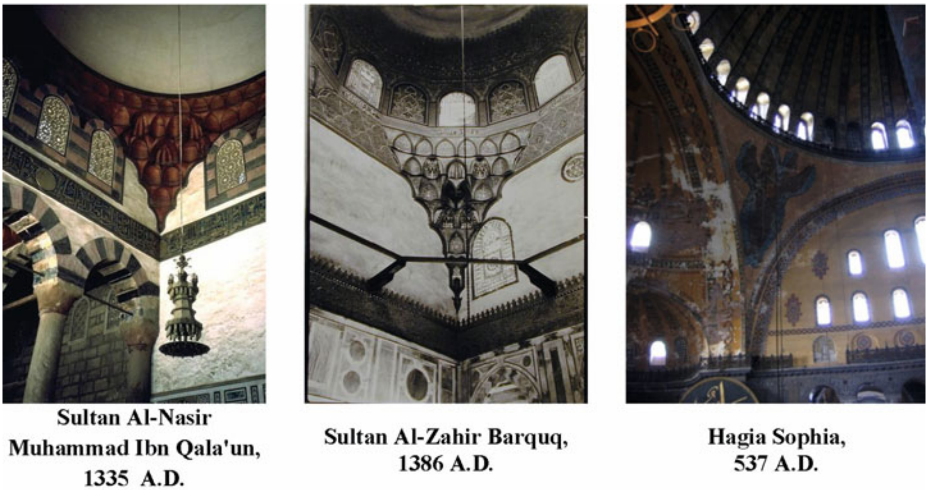


Fig. 17. Pendentives from Byzantine architecture to Islamic architecture. a, left and centre) Islamic architecture, muqarnas pendentives; b, right) Byzantine architecture, abstracted pendentives

What we see in Cairo's Islamic architecture is not the common type of pendentives (see figs. 11-14 and 16). Instead, it is a unique modified version in Islamic architecture. The architects of that era developed their own style of pendentives. They did not stop at the abstracted form of pendentives as originally invented, but they merged the concept of *muqarnas* (the Arabic word for stalactites vaults) with pendentives (see fig. 17a). A muqarnas is a special three–dimensional ornamentation consisting of tiers (layers), which themselves consist of niche-like elements. Muqarnas can be used to make a smooth transition from a rectangular basis to a vaulted ceiling. There is a great variety of these elements, yet most of them can be more or less deduced from a small set of basic elements [Dold-Samplonius and Harmsen 2004; Harmsen 2006; Hoeven and Veen 2010].

Almost all the pendentives utilized in Islamic Cairo consist of rows of muqarnas. In order to visualize how far this corbel may reach, it is sufficient to consider the dome (made of wooden beams) of the Mausoleum of Sultan Hassan (1356-1362 A.D.), which has a radius of more than 20 m [OICC 1992]. This radius results a corbel length (calculated according to Eq. 6) of more than 4 m, a long corbel that seems critical even today, with all the modern tools available.

Muqarnas pendentives – this terminology may be more expressive than the abstracted term "pendentives" – in Islamic architecture can be generated through the intersection between a *group of circles* located on the surface of the pendentives' sphere and the corner blocks as previously mentioned (see fig. 4), where the face of this block will be divided into a serious of sequential rows each of which represents one row of the muqarnas. To do that, it is important first to define the radii of the circles group. These circles must 1) be located on the surface of the virtual sphere that composes the pendentives; 2) have a radius equals to the radius of the corresponding muqarnas row.

From the mathematical point of view, all the circles that are tangential to the surface of the pendentives' sphere must be located on a diagonal circle in the sphere; its radius r_d is equal to the radius of that sphere R_c. Given the value of r_d and the height of each muqarnas row measured from the center of the sphere h_m (see fig. 18), the chords l_c can be calculated according to the equation:

$$l_c = 2\sqrt{{r_d}^2 - h_m^2} \tag{18}$$

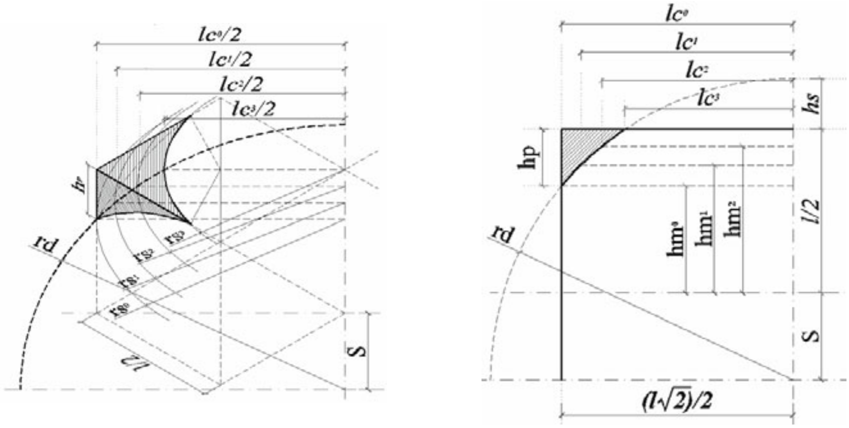


Fig. 18. The relation between the pendentives, the diagonal circle and muqarnas rows' circles.
a, left) Perspective represents the pendentives and its relation with the diagonal circle;
b, right) Diagonal section in the virtual cube

These chords represent in the third dimension the radii of the circles required to divide the pendentives into a sequential rows of muqarnas. In the special case where ($D = l\sqrt{2}$) the last formula can be rewritten to be:

$$l_c = 2\sqrt{\frac{l}{2}^2 - h_m^2} \tag{19}$$

Given the arc height h_s, the lengths l_c can be calculated according to:

$$l_c = 2\sqrt{2r_d h_s - {h_s}^2} \tag{20}$$

where h_m must fulfill the equation:

$$h_m \le l/2 \tag{21}$$

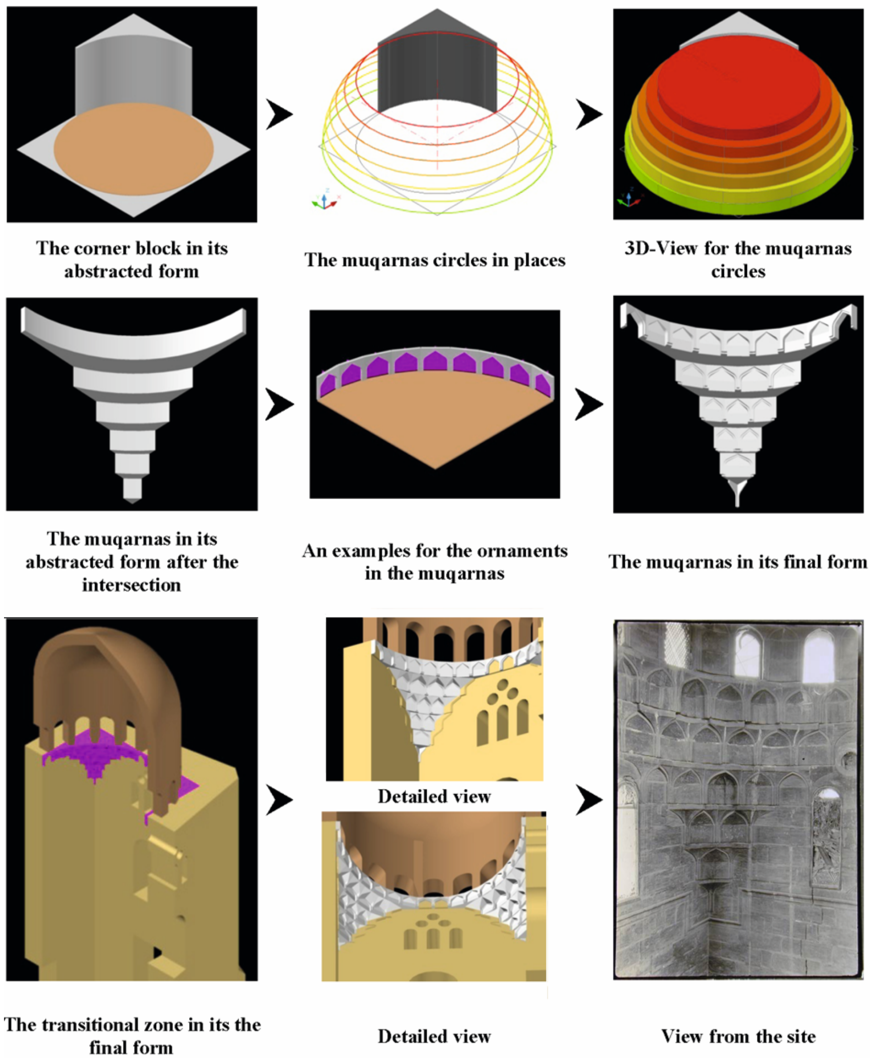


Fig. 19. Generating the muqarnas pendentives used in the dome of the Mausoleum of Sultan Al-Ashraf Barsbay (Burgi Mamluks period)

l	S	h_s	r_d=$D/2$	r	h_p	n*	R_h**
6.00	0.00	1.24	4.24	3.00	3.00	6	0.50
h_m							
0.00	0.50	1.00	1.50	2.00	2.50	3.00	—
l_c=r_s							
4.24	4.21	4.12	3.97	3.74	3.43	3.00	—

* Number of Rows

** Height of each row

Table 2. The numerical values (calculated from eqs. 12 to 21) used to generate the muqarnas pendentives in the Mausoleum of Masjid Sultan Al-Ashraf Barsbay (values in m)

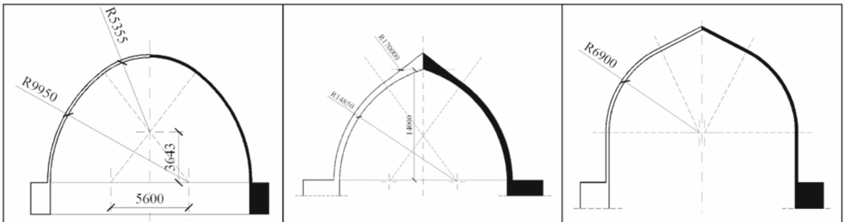
By calculating the radii of the muqarnas circles (applying the Eqs. 12 to 21), the rows of muqarnas can be generated. It is clear that the values of both h_m and l_c are determined according to the number of the muqarnas rows. Again, the design, form and arrangement of the muqarnas will depend on the designer himself, thus it may be difficult to count for all of the available forms. Fig. 19 represents in an analytical series an example for a direct application of the previous equations (values are given in Table 2) to generate the muqarnas pendentives as used in the Mausoleum of Masjid Sultan Al-Ashraf Barsbay (1425 A.D.) which, according to the historical references [OICC 1992] and the available photos for this dome, is a square room of dimensions 6 x 6 m.

4 Some common forms of the domes used in Cairo's Islamic architecture

In order to know the most common types of domes in Cairo's Islamic architecture, a sample of thrity Masjids built between 868 A.D. (the Tuluned Period) and 1798 A.D. (the end of Ottoman period in Egypt) [Elkhateeb and Soliman 2009] has been examined, of which ten were chosen. The chosen Masjids are those whose domes are the most distinguished architecturally and the most dominant visually. As a result, the following analysis does not include, for example, the domes used to cover the arcades of the Masjids.

Among the various types of domes, elliptical domes with their different forms are the most applicable in Islamic Cairo; all of the chosen samples are elliptical. Of the ten chosen, five are pointed perfect, three are flat perfect (looking from inside), one is pointed onion, and one is flat onion.

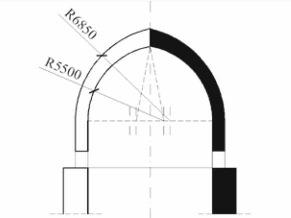
Not one of the examined samples has a spherical dome. The utilization of the squinches is also the most common; it has been used in seven of the chosen samples. The spans vary widely as well, starting from a minimum of 6 m in El-Hakim Masjid (Fatimid era) and increasing considerably, exceeded 20 m in the Masjid and school of Sultan Hassan (Bahari Mamluks era). Four of the domes chosen were used to cover a mausoleum in the Masjid, four were used to cover the space of the Masjid either partially or completely, one was placed above the niche (Mihrab) at the end of the crossed aisle and one covers the ablution fountain. Figs. 20 and 21 represent the domes that were studied.

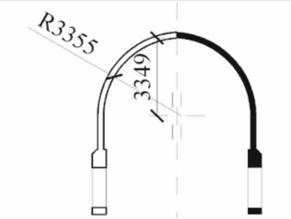


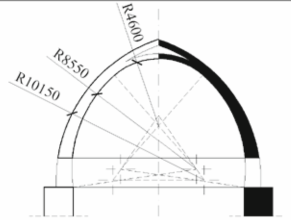
	Sultan Al-Naser Muhammad Ibn Qalaun (Bahari Mamluks) 1335 AD	**Sultan Al-Nasir Hassan (Bahari Mamluks) 1362 AD**	**Sultan Farag Ibn Barquq (Burgi Mamluks) 1411 AD**
Dome Location	The mausoleum	The mausoleum	The mausoleum
Dome Type	Flat-Perfect-Elliptical	Pointed-Perfect - Elliptical	Pointed- Perfect-Elliptical
Span (m)	14.30	21.10	14.20
C_l **(m)**§	2.96	4.37	2.94
C_h **(m)**¤	5.92	8.74	5.88
Transitional Zone	Stalactite Pendentives	Stalactite Pendentives	Stalactite Pendentives

§,¤ Calculated according to Eqs. 5 and 6

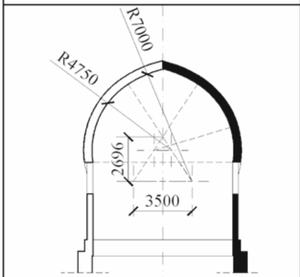
Fig. 20. Domes supported on muqarnas (stalactite) pendentives in a chronological order

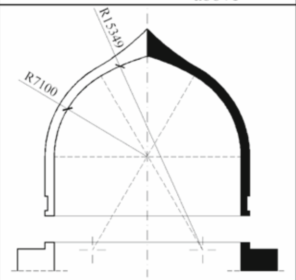


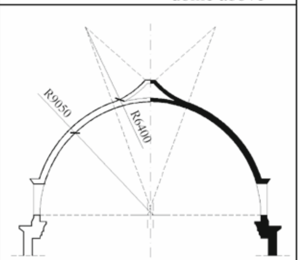




	Ahmed Ibn Tulun (Tulunid) 879 A.D.	El-Hakim (Fatimid) 1013 A.D.	Sutan Qalaun (Bahari Mamluks) 1285 A.D.
Dome Location	Ablution Fountain	Above the niche (Mihrab)	The mausoleum
Dome Type	Pointed-Perfect-Elliptical	Pointed-Perfect - Elliptical	Flat-Onion-Elliptical (from inside)
Span (m)	10.80	6.00	11.72
C_l (m)[§]	2.235	1.24	2.425
C_h (m)[¤]	4.47	2.485	4.85
Transitional Zone	Squinches, Corbel	Squinches, niches with half dome above	Squinches, niches with half dome above







	Amir Sorgotmosh (Bahari Mamluks) 1362 A.D.	Al-Fadawiyya Dome (Burgi Mamluks) 1479 A.D.	Sinan Pasha (Ottoman) 1571 A.D. Identical in Muhammad Bey Abu al-Dahab (Ottoman, 1774), span 15.50m
Dome Location	The Qibla Iwan	The Masjid	The Masjid
Dome Type	Pointed-Onion-Elliptical	Pointed-Perfect - Elliptical	Flat-Perfect-Elliptical (from inside)
Span (m)	8.55	14.20	17.65
C_l (m)[§]	1.77	2.94	3.655
C_h (m)[¤]	3.54	5.88	7.31
Transitional Zone	Squinches, Corbel	Squinches, Spherical-Pointed Vaults	Squinches, Spherical-Pointed Vaults

Fig. 21. Domes supported on squinches in a chronological order

5 Validation

Based on the derived mathematical expressions stated previously, AutoCAD 3D drawings were generated. From among the drawings generated, two drawings (one for squinches and the other for the pendentives) were chosen for building physical models using a 3D printer. Both domes belong to the Burgi Mamluks period. They are:

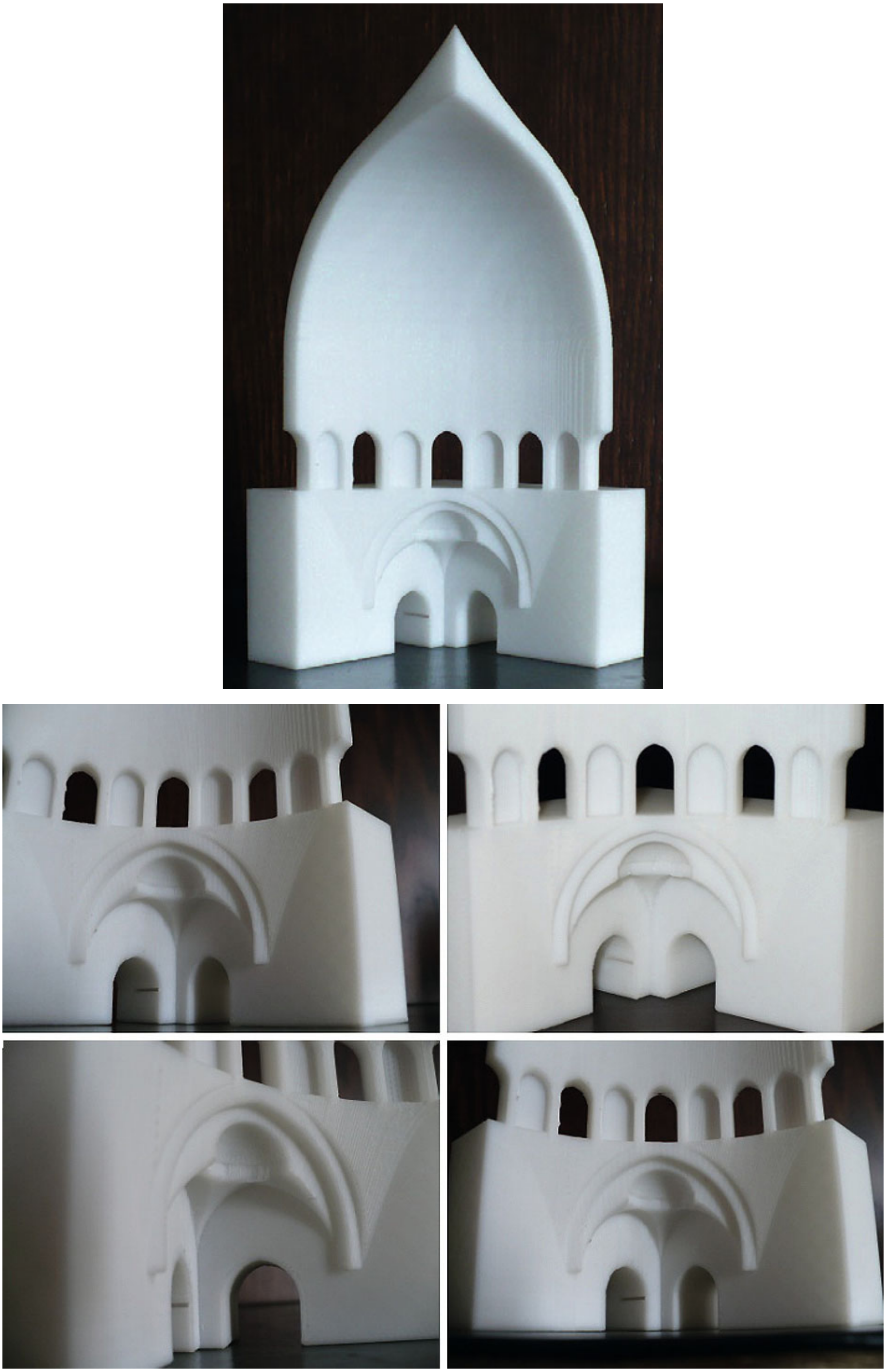


Fig. 22. The spherical pointed vault squinches in Al-Fadawiyya Dome Masjid, photos from the model. a) The dome above the squinch; b) detailed views

– Squinches model (see fig. 22): Al-Fadawiyya Dome Masjid (1479 A.D.), a square room of dimensions (14.20 x 14.20 m). The dome is used to cover the Masjid. The total room height under dome is 24.56 m [OICC 1992].
– Pendentives model (see fig. 23): the Mausoleum of Sultan Al-Ashraf Barsbay Masjid (1425 A.D.), a square room of dimensions (6.00x6.00m). The total room height under dome is about 28 m [OICC 1992].

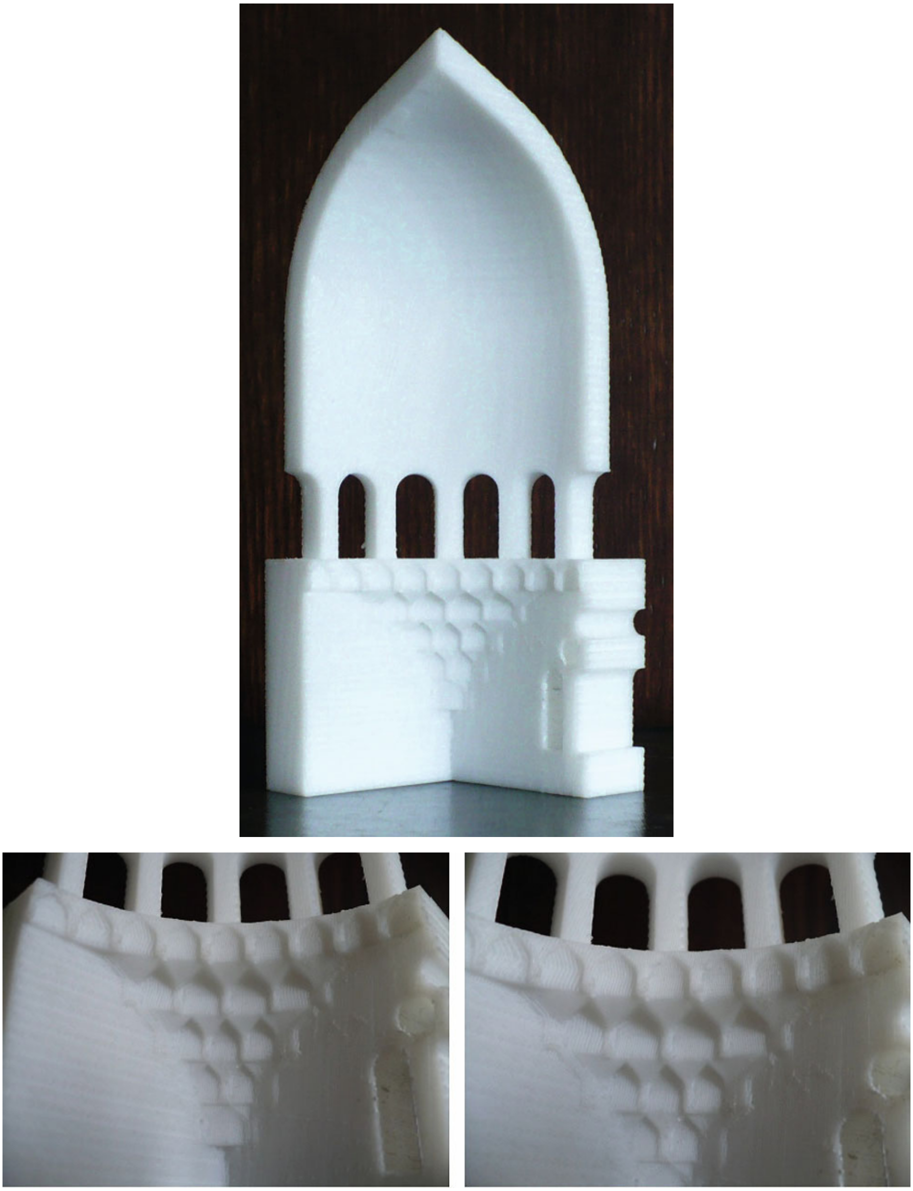


Fig. 23. The muqarnas pendentives in the Mausoleum of Sultan Al-Ashraf Barsbay, photos from the model. a) The dome above the pendentive; b) detailed views

6 Summary

Among the different methods to cover a space, the dome is the oldest and most distinguished. Although its development began some sixty centuries ago, it continues to develop more and more. This development enables it to cover a diversity of spaces and spans from few to hundreds of meters. One of the most important problems that faced the architects of the old days was how to transform the square top of the cube to a circle where the dome rested, which is known as the transitional zone. For structural and aesthetical reasons, the architects hollowed the transitional zone, applying one of two techniques: squinches and pendentives. The first is a support carried across the corner of a room under a superimposed mass. In architecture, squinches can be one of any of several devices by which a square or polygonal room has its upper corners filled in to form a support for a dome. In Cairo's Islamic architecture, they first appeared as four diagonal niches (with a half dome at their heads), one in each corner of the room to transform its square plan to an octagonal one where the dome was supported. The pendentive is originally one of the concave triangular members that support a dome over a square space. In architecture, the pendentives are triangular segments of a sphere, taper to points at the bottom and spread at the top to establish the continuous circular or elliptical base needed to support a dome.

Through the previous work a set of mathematical formulae has been derived to express the most famous forms for both squinches and pendentives. These formulae relate the different parts of the squinch or pendentive to the cube side length l where the dome will ultimately rest. Figs. 24 and 25 are graphical representations of the formulae stated previously.

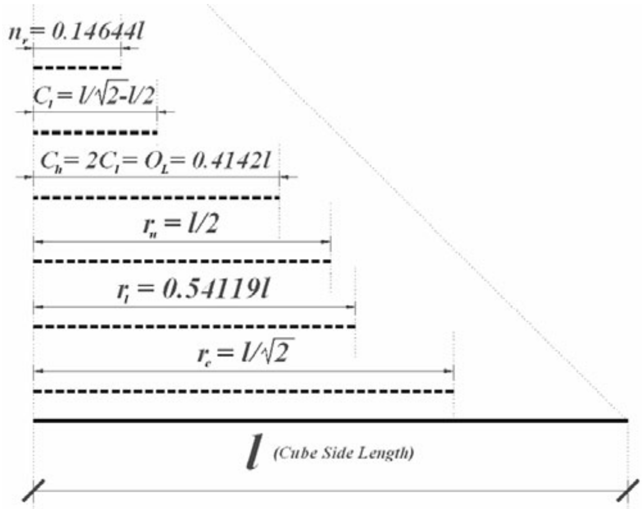


Fig. 24. The relation between the cube side length l and the different parts of the squinch

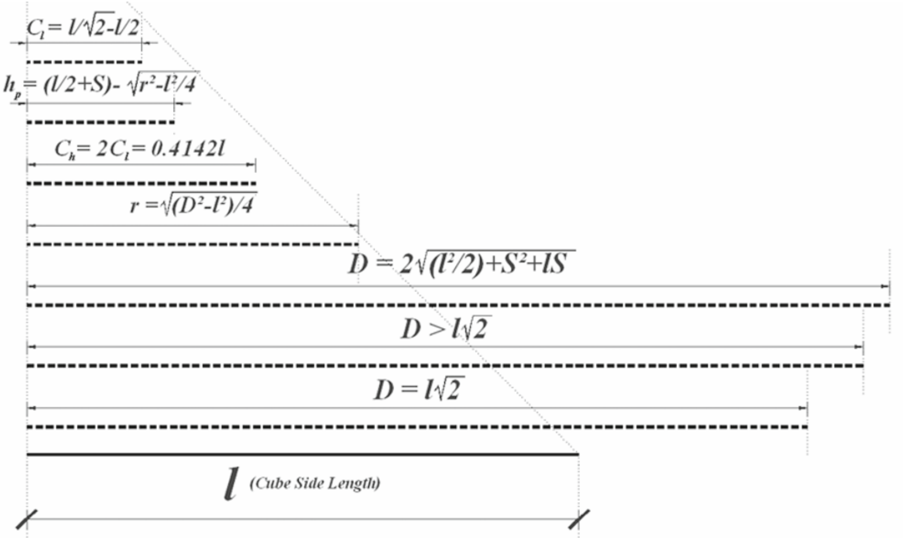


Fig. 25. The relation between the cube side length *l* and the different parts of the pendentive

Although modern materials and construction systems can accommodate various methods for supporting a dome over a square room, many architects in the Islamic world still prefer and appreciate the traditional methods (squinches and pendentives) in the transitional zone even though it may be constructed with new materials (i.e., reinforced concrete). Fig. 26 shows two examples that utilize squinches and pendentives in contemporary Masjids.

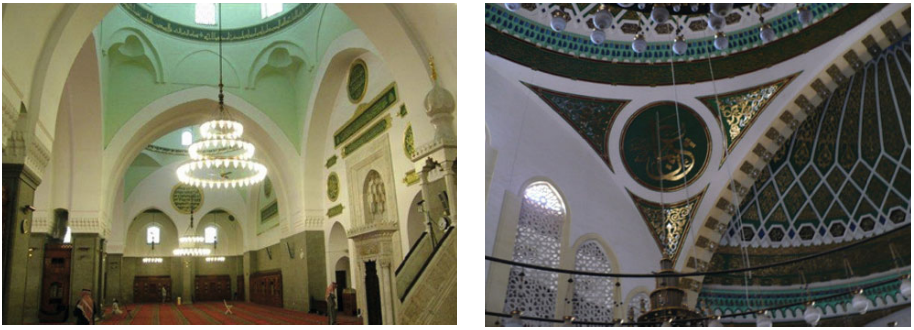


Fig. 26. Although the various methods available now to support a dome, the squinches and pendentives are still favorable for the architects of the Islamic world. a, left) Spherical-pointed vault squinches in Qeba'a Masjid, Almadena Almonawara, Saudi Arabia; b, right) pendentives in Elhosaree Masjid, 6th of October City, Egypt

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About the author

Ahmed Ali Elkhateeb graduated in 1990 from the Department of Architecture, Faculty of Engineering of Ain Shams University (Cairo, Egypt). He worked in the same department until summer 2011. He completed his M.Sc. and Ph.D. in the field of architectural acoustics, which is his primary area of interest, in addition to mathematics and its relation with architecture. As of fall 2011, he is a Professor (A) of architecture and building science in the Department of Architecture in Faculty of Environmental Design of King Abdul Aziz University, Jeddah, Kingdom of Saudi Arabia.