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Research

Modularity and Proportions in Architecture and their Relevance to a Generative Approach to Architectural Design

Abstract. Modularity and proportions have been at stake in architecture since at least the «orders» in classical antiquity, throughout centuries, and well into our time, as highlighted by architects Rudolph M. Schindler, Hans Van der Laan, and Le Corbusier. Not all architectural modular and proportioning systems are founded on the same assumptions. Some systems, which are truly modular, are based on a unit, and deal only with integer multiples and dividers of this unit (and therefore rational ratios), others imply an irrational division of a chosen length. While both are based on proportions, music and architecture do not relate in the same way to space and time. They involve different issues regarding physics, perception and dimensions. The generative approach to architectural design means the use of transformation rules, which often involve ratios. The use of transformation rules, which relates art to the way nature generates forms, permits events like “hybridization”: how may those operations contribute to a search for “resonance”?

1 Modularity and proportions in architecture

During the twentieth century, the most important modular and/or proportional systems that have been invented are those of architects Rudolf M. Schindler (1887-1953), Le Corbusier (1887-1967), and Hans van der Laan (1904-1991). Though born very nearly at the same time, these three architects have known very different destinies and reputations, as well as their systems which are called, respectively, the *Schindler Frame*, the *Modulor* and the *Plastic number*. Though his book, *Le nombre plastique* [Van der Laan 1960] was published in 1960, van der Laan worked as early as 1930 on his system; Le Corbusier invented the *Modulor* around 1943, and Schindler first published his frame in 1945 [Schindler 1947]. We know that Schindler knew about, and was opposed to, Le Corbusier’s *Modulor*, but we don’t know if Le Corbusier knew about Schindler’s system, nor if van der Laan knew either one. Dom Hans van der Laan, being a monk as well as an architect, and much less famous (and having a less extensive body of work, consisting only of a few religious buildings for his community) was probably not known by the two other architects. The three systems will not be exhaustively examined here; to find more developed explanations one can refer to the bibliography.

1.1 Measure

When devising a modular or proportional system, architects do not deal with *space* as such, or even with *size* (or *magnitude*) as perceived, but with *measure*. Le Corbusier emphasizes this term: *L’homme mis ici en question est architecte et peintre, pratiquant depuis quarante-cinq années un art où tout est mesure* (“The man here in question is an architect and a painter, who for the past forty-five years has practised an art in which *all is measure*”; Le Corbusier refers to himself) [Le Corbusier 1951: 25; 1954: 25], and van

der Laan insists upon that too: *L'architecte, nul n'en disconviendra, est un homme sans cesse occupé de mesures et de nombres. Ses dessins sont couverts de chiffres, et la règle graduée est toujours à portée de sa main* ("The architect, no one will disagree, is a man constantly taken up with measures and numbers. His drawings are covered with numbers, and the ruler is always within his reach") [Van der Laan 1960: 1]. This latter statement leads us to question that of Le Corbusier's because most painters are surely not as obsessed with measures, or anyway with numbers, as architects are. Obviously, one does not need a ruler to draw something which has got a measure. Anything we draw has a measure, even if we don't know its value. Sculptors and painters can work without using numbers, and we can still say that in their work *all is measure*, but only in the sense that everything they do has a size, and that those different sizes, being seen together, infer relationships, and proportions.

But architecture has to be drawn before being built, and those drawings will be used by workers to do the actual building, measures will be taken by them on these documents in order to know which size they have to give to the actual elements of the building. Le Corbusier remarks that musical scales had to be invented in order to transmit, and therefore to write, music [1951: 15], and regrets that architecture has not done the same (in which he is wrong, because measuring units of length do exist, and have been used for a long time). Hans van der Laan insists a lot on the necessity to get and give measures all along the chain of transmission that starts from the drawing and finishes in the building [Van der Laan 1960: 2-5]. Architects have to think their design with numbers because they know that numbers will be necessary to build what they have conceived.

All measures are based upon a unit: measuring consists in counting how many units there are in some magnitude. What is measured in architecture is *length*, or distance, and there have been units of length for measuring from ancient times. We could even say that measuring any other phenomenon, at least in the first stages of measuring instruments, is done by referring it to a measure of length (as temperature is measured by the height of mercury in a tube). The measure of length is linked to the idea of number as such: we identify numbers with points on a straight line.

In most measuring systems, there is not only one unit, since this unit is divided, or multiplied, to furnish other units for other orders of magnitude. For instance, we measure time in seconds, minutes, hours, days, weeks, months, years. The numbers applied in these groupings are 60, 24 (or 12), 7, 30 (or 27, 28, 31), and 365 (or 366). The Imperial system uses at least three units, *inch*, *foot*, and *yard*, related by numbers 12 and 3. The metric system itself may be considered in such a way. The metre is divided into 10 decimetres and 10 metres make 1 decametre. Actually we don't use those units a lot, though centimetres, millimetres and kilometres are in common use. The great difference is that the divisions are all the same (10). The choice of 10 may be debatable, its major drawback being that it has only 2 and 5 as divisors (while 12 has got 2, 3, 4 and 6). You cannot express $1/3$ meter as an integer number of centimetres, but one must add that you cannot either express $1/5$ foot by an integer number of inches. And you cannot express $1/7$, for instance, by an integer number of sub-units in either system (while for van der Laan, that ratio is fundamental). The great advantage of the metric system is that it is coherent with our way of writing numbers, based upon 10. That makes it possible to express the multiplications and divisions of units very simply by shifting the floating point ($1\text{ cm} = 0.01\text{ m}$, for instance). Why the decimal system of writing numbers has become predominant is a question that is largely beyond the scope of this paper, but that's a fact. Though the Imperial system is at core a duodecimal system, measures must be written with the 10 symbols of the decimal system.

1.2 Module

One of the issues of a modular system is to define a *new unit* (called *module*) for measuring lengths. Our three architects reject the metric system, even if they have to deal with it (and to express their new unit in it), but don't stick to the Imperial system either (Le Corbusier intended to reconcile both systems, without much success). The metre is considered as too abstract, because it is not based upon concrete measurements like old European systems, and the Imperial system. One may object to that assumption because, if it is true that the meter was defined at first as $1/10000000$ of the distance between the North Pole and the Equator (which is actually not very easy to apprehend), it is easily and simply relied for instance to the common size of a human being (at least a male one) which can be expressed as 1.75 m (or $1\frac{3}{4}$ m if one wants to insist on the use of simple fractions). It is true that the names of units in other systems are more evocative, but when you are used to the metric system, you know as well what concretely represents 1 m, or 5 cm, or 1 km. Schindler proposed a module (new unit) of 4 ft (or 48 in). It was not maybe his preoccupation but one can remark that the common size of a (male American) human being is expressed as $1\frac{1}{2}$ module (6 ft) in that case. The average size of American males is actually 1.75 m, but it is commonly given as 6 ft, not because of any excessive pride, but only because it is easily expressed in the Imperial system. Though his system is not modular, and does not define a new unit, Le Corbusier started from the height of a human being. He settled first on 1.75 m, but then chose 1.83 m, which corresponds again to 6 ft.

Having defined a unit or module, one has to decide how to divide, and possibly multiply it. What is at stake is the issue of orders of magnitude. In the metric system, one goes from one power of ten to the next, as has been illustrated in the film *Powers of ten* (Charles and Ray Eames, 1977). Though his system is not modular, van der Laan thought a lot about this question of orders of magnitude. He concludes that the ratio 1:7 is the one that leads from one order of magnitude to another. He could have deduced from that a modular system in which a unit would be divided into 7 sub-units, and so on, going from one power of seven to the next, but he went further to build a much more complicated sequence. Schindler defined a sub-unit of 4 in, which is $1/12$ of his module; in that way he stuck to the tradition of the Imperial system.

A modular system is very close to any common measuring system; it is only based upon another unit. A modular system allows to express any magnitude as a multiple of this unit, or *module*. This leads to an arithmetical sequence of numbers. This module may be divided and multiplied to get a sequence of units: if the factor is always the same, this sequence of units is a geometrical sequence.

A modular system leads to a grid, which is a useful tool for architects. Le Corbusier's and van der Laan's systems do not allow to establish such a grid, which is maybe the major drawback of those systems.

1.3 Proportions

A measuring system, or a modular one, is an arithmetical sequence; it even can be assimilated to the sequence of integer numbers. This sequence leads to any rational number by dividing two of its elements. This allows to establish a proportion between two numbers, and even to link four numbers in a relationship ($a:b=c:d$). This is obviously very often used in the scaled drawings made by architects: from the point of view of proportions, the scaled drawing is similar to the actual plan of the building.

Weirdly, neither Le Corbusier nor van der Laan mentions the issue of scaled drawings. Their systems do not allow us to choose a scaling ratio very easily. One can hardly draw at a ratio belonging to the Modulor sequence, and if one chooses any arbitrary ratio (like 1:100) then the measurements on the drawing do not belong to the Modulor sequence... In the case of van der Laan, it is even more paradoxical, as he proposes a set of sticks for experimenting with his system, and those sticks are clearly not at an architectural scale.

Architects consider only lengths, as we said, but they look for a relationship between lengths, either in the same direction, or moreover in different directions and, more specifically, in the directions of the Cartesian coordinate system (length, width, height). Those relationships between numbers are proportions, or ratios.

Architects seem to like rational proportions, and even to prefer simple fractions. If that was understandable in old times, it is weird in an age of electronic devices. Moreover, if earlier architects (as well as painters) preferred some proportions, it is in a large part because those are easy to arrive at through a construction with compass and straightedge; that allowed them to include the irrational $\sqrt{2}$, which is very simple to construct (Palladio, for instance, proposed those proportions: 1:1, $\sqrt{2}$:1, 4:3, 3:2, 5:3, and 2:1). It is surprising that none of the three architects we consider mentions geometrical constructions. Or, to be more precise, Le Corbusier considers a geometrical construction, but he is looking for *le lieu de l'angle droit* ("the place of the right angle") where we can be certain he will not find it.

According to Lionel March [1993a], Schindler uses approximations of $\sqrt{2}$ like 7:5 ($=1.4 \approx 1.414\dots$) or of $\sqrt{3}$ like 5:3 ($=1.666\dots \approx 1.732\dots$, which is actually closer to the golden mean...), but it is more in token of classical tradition than by resorting to a geometrical construction. He definitely prefers simple ratios among those that his module, divisible into 12 sub-units, permits.

Le Corbusier's and van der Laan's systems are not modular. They may be described as *sequences* of measures which are not arithmetic (as in a modular system); they could have been geometric sequences, though actually both architects settled on sequences of integer numbers which approximate such sequences. Their two systems are very much related, the *plastic number* being an equivalent in three dimensions of the *golden number* on which the *Modulor* is founded. Hans van der Laan and Le Corbusier both were looking for a "magic" number, an ideal proportion. Le Corbusier chose a well known one, ϕ , the "golden mean" ($=1.61803\dots$). Dom van der Laan, even if he never expressed it as such, relied on another one, the "plastic number", p ($=1.32472\dots$).

Before seeing how Le Corbusier and van der Laan actually built their systems, we shall describe the characteristics of those two numbers, which are very interesting from the point of view of mathematics. While ϕ and the Fibonacci sequence are common knowledge, p and the Padovan sequence (named after Richard Padovan, who ascribed it to van der Laan [Padovan 1994, 2002]) are less known; we will present them in parallel. ϕ verifies a relationship between the length and the width of a rectangle (a , b):

$$\frac{(a+b)}{b} = \frac{b}{a}, \text{ or, for a rectangle } (1, \varphi):$$

$$\frac{1+\varphi}{\varphi} = \frac{\varphi}{1}$$

$$1 + \varphi = \varphi^2 \text{ or } \varphi^2 - \varphi - 1 = 0$$

Fig. 1. The golden rectangle

p verifies a relationship between the length, the width, and the height of a parallelepiped (a, b, c) :

$$\frac{(a+b)}{b} = \frac{c}{a} \text{ and } \frac{b}{a} = \frac{c}{b}$$

or, for a parallelepiped $(1, p, p^2)$:

$$\frac{(1+p)}{p} = \frac{p^2}{1}$$

$$1 + p = p^3$$

or

$$p^3 - p = 0$$

Fig. 2. The plastic parallelepiped

So, those numbers are zeros of equations $x^2 - x - 1 = 0$ and $x^3 - x - 1 = 0$, respectively; as such, they are called *morphic* numbers, and it has been demonstrated that they are the only ones [Aarts, Fokink and Kruijtzer 2001].

$x^2 - x - 1 = 0$ is not very difficult to solve:

$$\Delta = 1 + 4 = 5$$

and the roots of the equation are:

$$x = (1 \pm \sqrt{5})/2,$$

of which we take the positive one:

$$x = (1 + \sqrt{5})/2 = 1.61803\dots$$

$x^3 - x - 1 = 0$ is much more difficult to solve ...

In an equation of this type $(x^3 + px + q)$, we get one real solution (and two complex ones) only if

$$D = 4p^3 + 27q^2 > 0.$$

In that case $D = 23 > 0$ and then the solution is expressed by:

$$x = \sqrt[3]{\frac{-q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

$$x = 1.32472\dots$$

φ can be considered as the ratio of an irrational geometric sequence:

φ^0	φ^1	φ^2	φ^3	φ^4	φ^5	φ^6	φ^7	φ^8
1	1.61803...	2.61803...	4.23607...	6.8541...	11.0902...	17.9443...	29.0344...	46.9787...

As expected, $\varphi^2 - \varphi^1 = 1$ (and $\varphi^3 - \varphi^2 = \varphi$, and so on...).

In the same way, p can be considered as the ratio of an irrational geometric sequence:

p^0	p^1	p^2	p^3	p^4	p^5	p^6
1.0	1.32472...	1.75488...	2.32472...	3.07962...	4.07963...	5.40437...

Here, $p^3 - p^1 = 1$ (and $p^4 - p^2 = p$, p and so on...).

Those “geometric” sequences (of numbers) have their “geometrical” counterpart:

Fig. 3. Rectangle growing in a geometric progression

Fig. 4. Parallelepiped growing in a geometric progression

But φ and p are also the limits of the ratios of terms of integer sequences, called the Fibonacci sequence and the Padovan sequence, respectively.

Fibonacci sequence: $u_{n+1} = u_n + u_{n-1}$

0	1	1	2	3	5	8	13	21	34	55	89	...
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The sequence of ratios of successive terms (rational numbers), tends towards irrational number $\varphi = 1.61803\dots$

1/1	2/1	3/2	5/3	8/5	13/8	21/13	34/21	55/34	89/55	...
1	2	1.5	1.666...	1.6	1.625	1.615...	1.619...	1.617...	1.618...	...

Padovan sequence: $u_{n+2} = u_n + u_{n-1}$

0	1	1	1	2	2	3	4	5	7	9	12	16	...
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The sequence of ratios of successive terms (rational numbers), tends towards irrational number $p = 1.32472\dots$

1/1	1/1	2/1	2/2	3/2	4/3	5/4	7/5	9/7	12/9	16/12	...
1	1	2	1	1.5	1.333...	1.25	1.4	1.285...	1.333...	1.333...	...

Those sequences have also their geometrical counterpart.

According to a famous drawing of the Modulor, Le Corbusier settles on this sequence (red series, in mm):

6 9 15 24 39 63 102 165 267 432 698 1130 1829

This is almost the Fibonacci sequence issued from 6 and 9:

6 9 15 24 39 63 102 165 267 432 699 1131 1830

However, he claims to use the golden section (φ) to make it.

Van der Laan settles on this sequence (divided in three groups):

2	2.5	3.5	4.5	6	8	10.5	14
14	18.5	24.5	32.5	43	57	75.5	100
100	132.5	175.5	232.5	308	408	540.5	716

This is actually the Padovan sequence issued from 4, 5 and 9 (whose terms have been divided by 2):

4	.5	7	9	12	16	21	28
28	37	49	65	86	114	151	200
200	265	351	465	616	816	1081	1432

He claims to find constant ratios, like:

$$2.5/2 = 18.5/14 = 132.5/100 = 4/3$$

which is obviously wrong, as:

$$2.5/2 = 1.25$$

$$18.5/14 = 1.321...$$

$$132.5/100 = 1.325...$$

He also claims that his numbers are in a geometric progression [Van der Laan 1960: 26, 81], which is again obviously wrong.

The proprieties of the golden mean are well known, and its use widespread, although perhaps not to as large an extent as some authors would like. The Fibonacci sequence is common knowledge too. Those were more or less known by Le Corbusier, notwithstanding his mistakes regarding mathematics, and even geometry.

On the other hand, the so-called plastic number and the Padovan sequence are much more esoteric. Van der Laan did not know in advance that some number would have the proprieties of the plastic number. He doesn't even express this number as such. Neither is he aware that he is building a Padovan sequence. He builds his sequence intuitively, by first establishing a fundamental ratio, 7:1, chosen as being what constitutes a change of order of magnitude. Then he looks for intermediate values that have other ratios which he wants to introduce, involving the arithmetic mean and the geometric mean, which are 7:4, 7:3 and 4:3. This leads him, after a rather hard work, to insert 4/3, 7/4, 7/3, 3 and 4 between 1 and 7. As he has already settled on the number of the components of his initial sequence (eight, including first one 1 and last one 7), and even given them their names, he needs another intermediate number between 4 and 7 and settles on 21/4.

small element	great element	small piece	great piece	small part	great part	small whole	great whole
1	4/3	7/4	7/3	3	4	21/4	7

But then he reconsiders his work, and introduces some “margins”, which leads him to this new sequence:

2	2.5	3.5	4.5	6	8	10.5	14,
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which would be, divided by 2, and expressed in simple fractions:

1	5/4	7/4	9/4	3	4	21/4	7,
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where we see that the “great element” and the “great piece” have changed. Which is a problem, because in the first stage, van der Laan had very clearly stated some fundamental relationships, such as the great element relates to the small element according to the fundamental ratio which is 4:3 [Van der Laan 1960: 66].

What is very weird is that van der Laan does not use the mechanism of a Padovan sequence ($u_{n+2} = u_n + u_{n-1}$) yet arrives at one all the same.

Another issue concerning those sequences is their absolute values. One must first remark that, their rejection of the metric system notwithstanding, both architects must express their measures in some conventional system and do that in the metric system. If Le Corbusier and van der Laan had settled on a geometric sequence, they could have chosen any initial number. But they established sequences of integer numbers (or a sequence of integer numbers divided by two in the case of van der Laan) and so they are not absolutely free in their choice.

Le Corbusier chose 1829 mm as a fundamental measure, from which the entire sequence is derived. By the way, had he been more feminist, he could have chosen 1618 mm, which is close to the average height of French women (1625 mm); he would then have introduced the golden number itself in his sequence. Anyway, the measure he chose is very abstract, as it is completely dependent on the choice of the metre itself. For van der Laan, an inconspicuous choice is made – *Commençons par le petit élément du système I et attribuons-lui la valeur 100* (“Let us start from the small element of system I and let us give it the value 100”) [Van der Laan 1960: 76] – for which he never provides an explanation. One might see in this arbitrary choice an atavism of the metric system, but what is not so clear is what unit these numerical values are expressed in. He is not very interested in that point, as for him it is the ratios that matter. In his abacus, constituted of sticks, the manipulation of which is supposed to give the sensation of proportions, he must give concrete magnitudes, and those are different from the above sequence. He does not give details, but mentions 3.7 mm, 26.5 mm and 189.7 mm, the diameter of the cylindrical sticks being constant and equal to 10 mm [Van der Laan 1960: 91]. We can complete the sequences: a geometric sequence of ratio p and first element 3.7 indeed reaches 26.5. With rounded measurements (let us not forget that those are mm!), we obtain:

small element	great element	small piece	great piece	small part	great part	small whole	great whole
3.7	4.9	6.5	8.6	11.4	15.1	20	26.5
26.5	35.1	46.5	61.6	81.6	108.1	143.2	189.7

Obviously, there are only two systems in the abacus, because if we go on, we reach values which could not enter into a small box, but which begin to be interesting for architecture (rounded in mm):

190	251	333	441	584	774	1025	1358	
1358	1799	2383	3157	4182	5540	7340	9723	...

We find 1799 mm on our way, which could represent the height of a man, but in any case, those are not the same numbers than in the sequence mentioned above. It confirms that, in contrast to Le Corbusier, it is not the absolute value of measures that interests van der Laan.

2 *Relationships between architecture and music*

Much has been said about relationships between architecture and music. With few exceptions, these are the two most abstract arts, and don't have figurative aims, while for a long time painting and sculpture had to represent the real world. This lack of figurative aim, at least, relates music and architecture. This famous and often repeated quote, "Architecture is a frozen music", attributed to Goethe or von Schelling, refers to aesthetics, to a very subjective perception of music and architecture, and not to the poetics, the making of music and architecture, which is what we are more interested in here. What is at stake in music and architecture, what musicians and architects deal with, is what will be approached, especially concerning modules and proportions. The relationship of music and architecture to mathematics is the other, more positive way in which architecture and music are related.

2.1 *Spatial magnitudes and sounds*

Any physical phenomenon is perceived through our senses as a perceptive *continuum*. Henri Poincaré [1913] explains this through the example of the weight (as we feel it by hefting): suppose we make a continuous chain of weights, all of which are different, and suppose that we cannot distinguish one weight from another, though we can very easily distinguish between the first and the last ones of this chain. Poincaré then establishes, through the concept of *cut* (*coupure*) and by recursion, that such a continuum has a *dimension* (or *number of dimensions*). Weight, temperature, length, and so forth, are one-dimensional, because one point of the continuum cuts it, but colour, for instance, is three-dimensional, because we need a surface (itself two-dimensional) to cut it.

Poincaré intended to prove that space, as perceived by human beings, is three-dimensional. This is not an abstract quality of space; space, as perceived by other creatures, could be two-, three- or four-dimensional, or even higher. After all, some birds have four types of colour receptors, instead of our three, and perceive colour as four-dimensional. In his famous novel *Flatland* [2006] Edwin Abbott imagined creatures whose space was two-dimensional (and were very astounded by the visit of three-dimensional creatures).

Lengths, or distances, constitute a one-dimension space. Sound is a complex phenomenon which involves at least pitch, intensity and timbre, and most of the time it is not pure. But from this complexity, at least in a musical sound, we can extract a component that is the *pitch* and which also constitutes a one-dimension space. However, those two kinds of one-dimensional spaces are not configured in the same way. We cannot imagine a length, a distance, which would be less than nothing (less than zero, but let's remember that we are considering a phenomenon that is perceived, not measured), while, even if we cannot concretely encompass it, we can imagine a length longer than any other, or rather, we cannot imagine that there is a length which is the longest of all. Our intuitive image of the space of lengths is therefore a half-line. On the

other hand, even if our audition limits our hearing to a part of the possible pitches, we can imagine sounds that are lower-pitched or higher-pitched than those we can hear. In addition, there is a very specific event in the space of pitches: some sounds seem very alike without being at the same pitch (the same note at different octaves). The one-dimensional space of pitches can then be represented by an helix with a vertical axis, the same notes being at the same angle but at different heights.

Spatial magnitudes, i.e., lengths and distances, are perceived through our visual and kinaesthetic senses, while sounds are perceived through our hearing sense. There is a relationship between sound and length, though: a vibrating cord of some length will make a sound that we perceive as being at the same pitch, an octave lower, than a cord of half its length. The terms “low” or “high” refer to spatial qualities too; they are not only metaphoric (as with “warm” or “cold” for colours) as we actually feel that a “higher” pitch comes from a higher place in our body.

The first stage of evaluating is comparing, as we do in the experiment proposed by Poincaré. We can say that something is longer that something else when we see them near each other and in the same direction, but otherwise we shall have to use another device to which both can be compared as, for instance, a string is used to compare the nearness of two boules to the jack in the game of *pétanque*. On the other hand, comparing two pitches is a very subjective task. There is no objective way, as in comparing two lengths, to prove that such a sound is higher-pitched than another. But it seems that most people can recognize the same note at two different octaves, and that there are some chords that sound harmonious for most ears.

Any continuum needs to be discretized to be worked with. We do this through language by giving names to some stages of a continuum, which is an intuitive way of *sampling*, of *scanning* (in the first meaning of this term, which comes from Latin *scandere*, and applies to verse) it, as we do by naming colours, for instance, and as has been done in music by choosing notes and giving them names:

C	D	E	F	G	A	B	C	...
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or

do	re	mi	fa	sol	la	ti	do	...
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Modular systems are attempts to discretize lengths in architecture. Le Corbusier clearly relies his research to musical notation at the beginning of *Le Modulor*:

Le son est un événement continu, conduisant sans rupture du grave à l'aigu. ... Il fallait le représenter par des éléments saisissables, par conséquent découper le continu selon une certaine convention et en faire du gradué (“Sound is a continuous phenomenon, an uninterrupted transition from low to high. ... It was necessary to represent sound by elements which could be grasped, breaking up a continuous whole in accordance with a certain convention and making from it a series of progressions”) [Le Corbusier 1951: 15; 1954: 15].

Referring to the establishment of the different scales up to the well-tempered one, Le Corbusier wonders: *Sait-on qu'en ce qui concerne les choses visuelles, les longueurs, nos civilisations n'ont pas encore franchi l'étape accomplie par la musique?* (“How many of us know that in the visual sphere – in the matter of *lengths* – our civilizations have not yet come to the stage they have reached in music?”) [1951: 16; 1954: 16].

Defining units of measure is a way of giving names to specific lengths. Schindler defined his module (48" or 4'), and its subunit (4"), naming, or rather, qualifying them respectively as *rhythm* and *texture*. Those names, like the names of measure units, refer to orders of magnitude, the elements of each order are not named (they are expressed as numbers).

Hans van der Laan does not refer much to music, but evokes "the musical scale which Plato explains in detail in the *Timaeus*", and remarks that:

in music, however, nothing is fixed but the intervals, and every piece of music can be played at different pitches; the actual pitch of the notes must be established 'from outside' ... As soon as one tone is fixed, so too are all the others, and all intermediate tones are false (from "Architectonic space", 1989, in [Ferlenga and Verde 2001: 196]).

He however went the furthest in an analogy with music by giving names to the *elements* of his systems (each system being numbered and corresponding to an order of magnitude), in a way very similar to the musical scale:

small element	great element	small piece	great piece	small part	great part	small whole	great whole	System II
=								
System I				small element	great element	small piece	great piece	small part ...

It may be a coincidence that he uses the same number of distinct grades as in the (diatonic) scale, i.e., seven (the "great whole" being equal to the "small element" of the next system), or in the colours as defined by Newton. Hans van der Laan doesn't link his choice to either musical notes or colours. The seven notes of the diatonic scale are not completely arbitrary: fa – do – sol – re – la – mi – ti or F – C – G – D – A – E – B is a sequence of perfect fifths, considered the most consonant interval. But Newton's seven colours – red, orange, yellow, green, blue, indigo and violet – are more subjective and induced by the diatonic scale. Newton had first split the spectrum into five principal colours and inserted orange and indigo to be able to link colours and notes: violet/D – indigo/E – blue/F – green/G – yellow/A – orange/B – red/C. As we now know, any splitting of the spectrum into more than three primitive colours is completely subjective.

2.2 Modules, proportions, and musical scales

The issue of modules and proportions in architecture and their relationships to musical scales is a very crucial and controversial point. It questions the use of numbers in each discipline and the types of perception they involve.

Musical scales relate to acoustics, and therefore to frequencies. Frequencies, and the way we perceive them, lead to ratios, the most simple of them being 2:1. Two notes with fundamental frequencies in a ratio of 2:1 (or any power of two) are perceived as very similar. When frequency ratios are simple fractions (3:2, 5:4, etc.), the combination of the notes will sound *consonant*. When they are near to a simple fraction, but not exact, a physical phenomenon called "beating" is produced, and the combination is perceived as *dissonant*. So there are physical and perceptual reasons to use exact ratios in music. Those reasons may be in part cultural as well. When Europeans hear music from other parts of the world, they sometimes feel that it is not harmonious; and some contemporary works of music that explore other possibilities can hurt our ears. However, it is possible that we are simply not used to them.

Although sound is a vibration (or rather a combination of vibrations) which has some frequency, we are not aware of this when we hear it. The frequency is the scientific measure of the pitch, but what we hear in a very subjective way is the pitch. We don't *know* the ratio between two frequencies, we *feel* that the combination is consonant or dissonant. Moreover, a *geometric* sequence of frequencies (like the one that is used to tune a piano) is felt as an *arithmetic* sequence of intervals (we feel that there are equal intervals between two adjacent keys).

Concerning the perception of lengths, we could say that we can evaluate their proportions directly. When a length is $2/3$ of another one, for instance, we are able to see that, at least if the smallest one is included in the largest. But proportions are most often used for lengths in different directions (typically for the length and the width of a room). In plan it is still possible to evaluate their ratio (especially if a grid is drawn that allows us to count the units in each direction), but when we are in that room, without any measuring device, and considering that what we see is in perspective, are we still able to evaluate their ratio? Some might say that, as with the pitches, we don't have to measure or even to evaluate, that "harmonious" proportions of lengths are *felt* in the same way as consonant chords. But there are actually no rational or physical groundings for such an affirmation. Palladio wrote:

Le proporzioni delle voci sono armonia delle orecchie, cosi' quelle delle misure sono armonia degli occhi nostri, la quale secondo il suo costume sommamente diletta senza sapersi perche' fuori che da quelli che studiano di sapere le ragioni delle cose ("Just as the proportions of voices are harmony to the ears, so those of measurement are harmony to the eyes, which according to their habit delights [in them] to a great degree, without it being known why, save by those who study to know the reasons of things") [Puppi 1988: 123].

Unfortunately it is not sure that any one knows.

This objection is no reason to exclude any attempt to use modules and ratios in architecture. The architect must decide on measurements and proportions, because, anyway, they *are* in his drawing, and then in his building, even if he does not think about them. If a modular system is established, which leads to a grid, it is obvious that simple ratios will appear. Lionel March sees in ratios chosen by Schindler a relation to musical ratios [March 1993b], and he is very convincing. But must it not happen as soon as simple ratios are chosen? Schindler was not the first to use modules and simple ratios, and even the "row" [Park 2006] and its correlation to musical harmonies is present in, for instance, Bragdon's theory [2010].

Le Corbusier and van der Laan opted for irrational ratios (though they did not settle on a geometric sequence, as seen before) without relationships to musical ratios. Is this a reason to reject their choice as contrary to harmony, as we cannot establish rationally the foundations of harmony in spatial magnitudes? Supporters of the golden mean (and there are many of them) will affirm that a golden rectangle is the most harmonious of all. Perhaps some research will prove that our brain is able to perceive that in such a rectangle the length is to the width as the sum of both to the length, or that if you draw a square on the smaller edge what remains is again a golden rectangle... But we could just as well promote $\sqrt{2}:1$ which has the great advantage that such a rectangle, when cut into two equal parts, leads to two rectangles of the same proportions: which is satisfying for the mind, and very useful, and has been used to define the ISO216 standard for paper size (A0, A1, A2, and so on).

2.3 The equivalent of counterpoint (*contrappunto*) in architecture

Any transposition from one form of art to another is in a great part arbitrary. Indeed, some artists claim to do such transpositions very naturally, and even to experience a kind of synaesthesia, among them Olivier Messiaen or Wassily Kandinsky. Those synaesthaesiae mostly involve colours, sounds and graphemes (letters and numbers); there is no reference to a correspondence between musical sounds and proportions of a rectangle. Anyway, there is no need to invoke synaesthesia to look for equivalents between two forms of art. That is a legitimate concern, as any that can lead us to renew our conception.

In music, counterpoint is the relationship between two or more voices that are independent in contour and rhythm and are harmonically dependant. It involves the writing of musical “lines” that sound very different and move independently from each other but sound harmonious when played together. Through the simultaneity of two or more different lines, some harmonies may (and must) occur, though those “vertical” features are considered secondary and almost incidental in counterpoint.

Music occurs in time: a musical “line” is a sequence of notes that develops in time. One may notice that pitch and time are transcribed in musical notation into spatial positions: along a vertical axis for the pitch, and a horizontal one for the occurrence in time; this spatial representation is combined with an arbitrary code for the duration of the notes. As we saw for the continuum of pitches, time (also a one-dimensional space) is discretized into bars or “measures”, which are themselves divided into beats. As with pitches, this discretization is relative. There can be a metronome mark, but in most cases, music may be played at a slightly different tempo.

Architecture is essentially a distribution, a composition, of solids and voids in space. This distribution does not change in time, though we perceive it in time by moving inside and around it. Music can also be considered as a distribution or composition of solids and voids (sounds and silences, notes and rests), not in space (though it can also intervene), but in time. Architecture and music have therefore very different relationships to space and time.

One way to link them is to consider musical notation, and transpose time into a line in one direction of space. Some architectural features may then be considered as “lines”, equivalent to musical lines, even if those lines do not develop in time. A colonnade, a row of windows, or the vertical divisions in windows (as in the *Couvent de la Tourette*, by Le Corbusier and musician-architect Iannis Xenakis), and so on, are opportunities of linear sequences, the terms of which can be related to pitch or rhythm. In that case, we might wonder how two or more such lines may be “played” together. Hans van der Laan developed four ways in which two measures of length can interact: they may be superposed or juxtaposed, and in each case they may be in the same orientation or in the opposite. This creates the potential for an approach towards one kind of architectural equivalent to counterpoint.

But we don’t want to restrict ourselves to linear occurrences in architecture, and other possibilities have to be explored. Time may be considered in terms of perception, or as the frame of a generative process.

3 *Generative processes*

For Philip Galanter,

Generative art refers to any art practice where the artist creates a process, such as a set of natural language rules, a computer program, a machine, or

other procedural invention, which is then set into motion with some degree of autonomy contributing to or resulting in a completed work of art (from http://www.philipgalanter.com/generative_art/index.html).

More precisely, generative art is based upon transformation processes, which include cellular automata, formal grammars, L-systems, fractal processes, iterated function systems (IFS), genetic algorithms, artificial life, and so on; what is interesting is that the same types of processes may be used to create music, architecture, or any other kind of art, as participants in Generative Art conferences (<http://www.generativeart.com/>) demonstrate each year. This is a new way to consider the relationship between music and architecture, without either of them being indebted to the other, as happens when musical ratios are borrowed by architecture. IFS have been chosen to illustrate this issue. In the world of generative processes, IFS are correlated with space, since they work with geometrical transformations, but musicians have used them too, to make music [Gogins 1991] as well as analyse it [Meloan and Sprott 1997].

3.1 Attractors and time

IFS have been introduced and theorized mainly by Michael Barnsley [1998], and are a very efficient way to generate fractals, or rather, self-similar forms. A simple example of IFS is the half-way transformation: one defines a reference point, and then imagines starting from any other point and going half way towards the reference point, and going half way again from this new place, and so on. The limit of the process is the reference point, which is no surprise, and is also a kind of illustration of one of Zeno's paradoxes. But now let's take two reference points, and, starting from any other point, let's *choose at random* one of the two reference points and go half way towards it, and do the same thing from this new place and so on. What is called the *attractor* of this new process is the segment between the two reference points; which is maybe a little more surprising. If we do the same with three reference points (again choosing at random among the reference points the one towards which we are going) we obtain a famous fractal figure, a Sierpinski triangle (or gasket); if the three points are not the vertices of an equilateral triangle, it may be an irregular one. We can carry on with more reference points, the result is that we get self-similar figures, some of them fractals, others which are not fractal but are anyway self-similar (like the segment or the filled square, if the reference points are the vertices of a square).

The key point of IFS is that fractals, or self-similar forms in general, are attractors of a set of transformations, applied iteratively. The use of randomness is not mandatory, it is only one way of writing an algorithm. One can also apply systematically all the transformations of the set on the result of the previous step. Starting from any set of pixels, if you apply systematically and iteratively the set of transformations that corresponds to the Sierpinski triangle, you will obtain the Sierpinski triangle. Any process may be described by an initial stage and transformation rules, but concerning IFS and attractors, the initial stage does not matter.

Time is an important dimension of generative processes. Time is discretized, and most generative processes work step by step. Ideally, time is considered infinite, though for obvious reasons we never let a process work infinitely. Some processes converge more or less rapidly towards an attractor, while other ones evolve erratically. IFS converge towards fractal attractors, as we said, but we can also use their first steps to see how an initial form (which in this case matters) is transformed. Each step makes its mark on the form. The result is the trace of all the steps of the process. It is a way to include time inside the design itself, and not only in the perception of it; dynamics is used in the

process and the result reflects this dynamics. It is perhaps a new way to consider architecture as frozen music. The experiments shown in §3.3 below are based upon this hypothesis.

3.2 Self-similarity and proportions

Many kinds of transformations may be used in IFS, as long as they are “contracting”. Mostly the transformations are scalings (homotheties) combined with rotations and translations. It is then crucial to choose ratios for the homotheties. Those ratios may be irrational (though they will be approximated in this case) or rational numbers. Those transformations reflect the self-similarity of the attractor.

Proportions have to do with geometric similarity: two forms are similar if they have the same proportions. Self-similarity is a stronger requirement than proportions: sides of a rectangle may be in a given proportion, but a self-similar form is composed of parts that are each similar to the whole form.

Van der Laan reminds us that symmetry in ancient times meant more than what we refer to nowadays [Van der Laan 1960: 8-15]. It did not refer only to the axial symmetry, nor even to the other symmetries between parts, but to a relationship, a ratio, a proportion, between the parts and the whole. In “Disposition of the town” [1960: 177-181] he shows sketches that are strikingly self-similar. He analyses Hagia Sophia in Istanbul [1960: 115-116] and again proposes a sketch that makes evident the self-similarity of elements of the plan. Obviously, he doesn’t use the term “self-similarity”, which he didn’t probably know, but his conception of measures, and especially of orders of magnitude, had a lot to do with that geometrical concept.

Self-similar forms are also said to be scale-invariant. That could be a drawback as far as architecture is concerned. Architecture has a lot to do with scale: scale is very important in relation to its users and context. But what scale-invariance means is that at any of many scales, we find the same structure. This series of scales is not continuous. On the contrary, we are able to define ranges of scales according to the needs of our design. Besides, we are not compelled to pursue the process of diminishing scales very far.

3.3 A search for counterpoint in architecture: experiments

In order to illustrate my hypothesis on the use of IFS in a search for counterpoint in architecture, I carried out experiments on basic schemes involving the golden mean φ . Those experiments do not pretend to be architectural, though some results could be interpreted in such terms, but they do deal with a distribution of solids and voids, which is a primary concern in architecture.

The basic schemes consist in placing four squares of ratio $1/\varphi$ (and $1/\varphi^2$) either at the corners of a square, or at the centre of its edges (Fig. 7).

The use of the golden mean is relevant because as we can see for instance in the first scheme, the squares of the first ratio intersect to get those of the second one (because of the fundamental relationship: $\varphi^0 + \varphi^1 = \varphi^2$). The basic schemes provide a division of the square into squares and rectangles, among which are golden rectangles (in grey in Fig. 7). Divisions of the square were explored by Le Corbusier [1951: figs. 39-44, pp. 95-101] though most of his are not so symmetric as mine.

Fig. 7. Basic schemes

IFS 1	scaling	rotation	translation x	translation y
w1	$1/\varphi$	0°	$-(1-1/\varphi)/2$	0
w2	$1/\varphi$	0°	$(1-1/\varphi)/2$	0
w3	$1/\varphi$	0°	0	$-(1-1/\varphi)/2$
w4	$1/\varphi$	0°	0	$(1-1/\varphi)/2$

Table 1. IFS1

IFS 2	scaling	rotation	translation x	translation y
w1	$1/\varphi$	0°	$-(1-1/\varphi)/2$	$(1-1/\varphi)/2$
w2	$1/\varphi$	0°	$(1-1/\varphi)/2$	$-(1-1/\varphi)/2$
w3	$1/\varphi$	0°	$-(1-1/\varphi)/2$	$-(1-1/\varphi)/2$
w4	$1/\varphi$	0°	$(1-1/\varphi)/2$	$(1-1/\varphi)/2$

Table 2. IFS2

IFS 3	scaling	rotation	translation x	translation y
w1	$1/\varphi^2$	0°	$-(1/\varphi)/2$	0
w2	$1/\varphi^2$	0°	$(1/\varphi)/2$	0
w3	$1/\varphi^2$	0°	0	$-(1/\varphi)/2$
w4	$1/\varphi^2$	0°	0	$(1/\varphi)/2$

Table 3. IFS3

IFS 4	scaling	rotation	translation x	translation y
w1	$1/\varphi^2$	0°	$-(1/\varphi)/2$	$(1/\varphi)/2$
w2	$1/\varphi^2$	0°	$(1/\varphi)/2$	$-(1/\varphi)/2$
w3	$1/\varphi^2$	0°	$-(1/\varphi)/2$	$-(1/\varphi)/2$
w4	$1/\varphi^2$	0°	$(1/\varphi)/2$	$(1/\varphi)/2$

Table 4. IFS 4

Fig. 8. Attractors of IFS1, IFS2, IFS3, IFS4

Four IFS have been built, each constituted of four very simple transformations: scalings (homotheties) of ratio $1/\varphi$ (IFS1 and IFS2) or $1/\varphi^2$ (IFS3 and IFS4) combined with translations that put the scaled squares at the middle of the edges of the first square (IFS1 and IFS 3) or at its corners (IFS2 and IFS 4).

Attractors of those IFS (Fig. 8) are not what interests me here. The IFS are used with a systematic algorithm on a white and black bitmap, starting from a solid black square, but with a slight change: at each step, white and black pixels are switched. The bitmaps are then interpreted as 3D forms, white corresponding to solid, black to void.

I “hybridize” [Corcuff 2008] the IFS pairwise, in two ways. In the first hybridization, the algorithm chooses between the two IFS (the four transformations) at each step (Fig. 9-14); in the second one, it chooses between the transformations of each IFS at each step. In the first case, one can show all results of three steps of the process; in the second case, there are too many possibilities, so one must randomize the process to show only some of them (Fig. 15, 16).

The first hybridizations maintain the square symmetry of the basic schemes. This symmetry is counteracted by the second type of hybridization.

Fig. 9. First hybridization of IFS1 and IFS2

Fig. 10. First hybridization of IFS3 and IFS4

Fig. 11. First hybridization of IFS1 and IFS3

Fig. 12. First hybridization of IFS2 and IFS4

Fig. 13. First hybridization of IFS1 and IFS4

Fig. 14. First hybridization of IFS2 and IFS3

Fig. 15. Second hybridization of IFS1 and IFS2

Fig. 16. Second hybridization of IFS3 and IFS4

Generative processes permit us to work a little like nature itself. We can imagine individual processes, but also alter them through “mutation”, or combine two or more of them through “hybridization”, as counterpoint implies two or more musical lines. My interpretation of counterpoint in these experiments is the assimilation of the simultaneity of different lines of music to the simultaneous transformations at each step of the process, and the lines themselves to the evolution of the process. The harmonic dependence of the “lines” is attained by the use of the golden ratio. That is obviously only one way of envisaging the notion of counterpoint.

I did not try to translate a piece of music into architecture, or even ornament. There are a lot of different ways to search for an equivalent for counterpoint in architecture; my contribution is only one of them.

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