Snežana Lawrence

School of Education
Bath Spa University
Newton Park
Bath, BA2 9BD, United Kingdom
snezana@mathsisgoodforyou.com

Keywords: developable surfaces, Gaspard Monge, descriptive geometry, ruled surfaces, Frank Gehry, Leonhard Euler

Research

Developable Surfaces: Their History and Application

Presented at Nexus 2010: Relationships Between Architecture and Mathematics, Porto, 13-15 June 2010.

Abstract. Developable surfaces form a very small subset of all possible surfaces and were for centuries studied only in passing, but the discovery of differential calculus in the seventeenth century meant that their properties could be studied in greater depth. Here we show that the generating principles of developable surfaces were also at the core of their study by Monge. In a historical context, from the beginning of the study of developable surfaces, to the contributions Monge made to the field, it can be seen that the nature of developable surfaces is closely related to the spatial intuition and treatment of space as defined by Monge through his descriptive geometry, which played a major role in developing an international language of geometrical communication for architecture and engineering. The use of developable surfaces in the architecture of Frank Gehry is mentioned, in particular in relation to his fascination with 'movement' and its role in architectural design.

Introduction

Contemporary use of developable surfaces and their geometric manipulation with or without the use of computer software has been increasingly well documented (see in particular [Liu et al. 2006] and the bibliography therein). The application of developable surfaces is wide ranging – from ship-bulding to manufacturing of clothing – as they are suitable to the modelling of surfaces which can be made out of leather, paper, fibre, and sheet metal. Some of the most beautiful works of modern architecture by architects such as Hans Hollein, Frank O. Gehry and Santiago Calatrava use the properties of developable surfaces, yet the history of this type of surface is not well known; this is the task the present paper is set to achieve.

Developable surfaces form a very small subset of all possible surfaces: for centuries cylinders and cones were believed to be the only ones, until studies in the eighteenth century proved that the tangent surfaces belong to the same mathematical family (see in particular [Euler 1772] and [Monge 1780, 1785]).

There are therefore three types of developable surfaces (excluding a fourth type, the planar surface):

1. surfaces in which generating lines are tangents of a space curve: this type of surface is spanned by a set of straight lines tangential to a space curve, which is called the edge of regression (fig. 1a);
2. surfaces which can be described as a generalised 'cone' where all generating lines run through a fixed point, the apex or vertex of the surface (fig. 1b);
3. surfaces which, in the same manner, can be described as a generalised cylinder, where all generating lines are parallel, swept by a set of mutually parallel lines (fig. 1c).

DOI 10.1007/s00004-011-0087-z; *published online* 22 November 2011

It should be mentioned that the developable surfaces are also those which contain elements of any of the above mentioned general cases of developable surfaces, as long as they can be flattened onto a planar surface, without creasing, tearing or stretching (fig. 1).

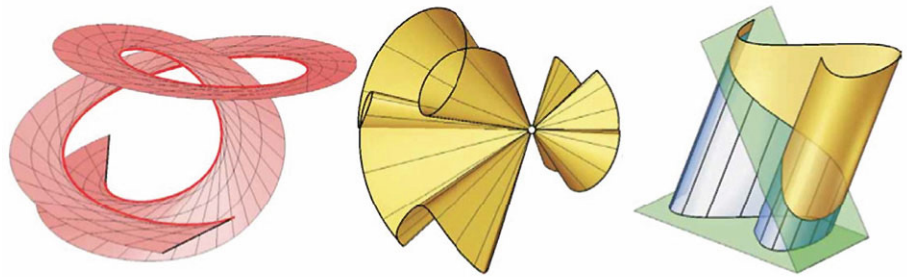

Fig. 1. The three kinds of developable surfaces: a, left) tangential; b, centre) conical; c, right) cylindrical. Curves in bold are directrix or base cruves; straight lines in bold are directors or generating lines (curves)

Developable surfaces are a special kind of ruled surfaces: they have a Gaussian curvature equal to 0,[1] and can be mapped onto the plane surface without distortion of curves: any curve from such a surface drawn onto the flat plane remains the same. In this context, it is important to remember the following property of Gaussian curvature: if the surface is subjected to an isometric transformation (or more plainly bending), the Gaussian curvature at any point of the surface will remain invariant.[2] The Gaussian curvature is in fact determined by the inner metric of a surface, therefore all the lengths and angles on the surface remain invariant under bending, a property immensely important in using developable surfaces in manufacturing. As a consequence, developable surfaces, having the Gaussian curvature equal to zero, the same as plane surfaces, can be obtained from unstretchable materials without fear of extending or tearing, but by transforming a plane through folding, bending, gluing or rolling.

The use of developable surfaces in contemporary architecture, for example in the work of Gehry,[3] to which we will return later on in this paper, have been, more recently, made possible by the development of computer software which, given user-specified three-dimensional boundary curves, generates a smooth developable surface.

In our usual world of three dimensions, the one in which we make architectural objects, all developable surfaces are ruled.

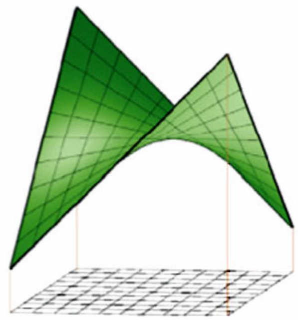

Fig. 2. A hyperbolic paraboloid is a ruled, but not developable, surface

The ruled surface is such that it contains at least one family of straight lines. Ruled surfaces are generated by a *directrix* or *base curve* – the curve along which the ruling, director, or generating curve moves. The *rulings* or *directors* of ruled surfaces are straight lines. Although all developable surfaces (in three dimensions) are ruled, not all ruled surfaces are developable. For example, the hyperbolic paraboloid is not developable but it is a ruled surface (fig. 2).

To summarise: the developable surfaces are cylinders, cones, and tangent developable surfaces, or a composition of these. The tangential developable surfaces can be best described and visualised as surfaces formed by moving a straight line in space (the director as explained above) along a directrix. If you imagine a generating line describing a tangential surface, then all points on that generating line share a common tangent plane.

The history of developable surfaces

The history of developable surfaces can be traced as far back as Aristotle (384-322 B.C.): in *De Anima* (I, 4) he states that 'a line by its motion produces a surface' [Aristotle 2004: 409], although this does not mean that it was Aristotle who defined or named the ruled or developable surfaces. Nevertheless, the statement had profound influence on perceiving the generation of surfaces through movement. More than twenty centuries later, Monge (1746-1818) used the principle of generating surfaces in a task he was given while he worked at the Mézières[4] as a draftsmen in the 1760s [Taton 1951; Sakarovitch 1989, 1995, 1997], which led to his discovery of a technique which later gained the name of 'descriptive geometry'. We will return to his invention, but first let us look at how similar treatment of motion was used in practical geometry before him.[5]

A first example is William Hawney, a minor eighteenth-century author on surveying, who in 1717 described the cylinder as a surface 'rolled over a plane so that all its points are brought into coincidence with the plane'.[6] In 1737 Amédée François Frézier (1682-1773) also considered the rolling of the plane to form a circular cylinder and cone (see [Frézier 1737-39]), but he did not generalise on the mathematical properties of this process, nor did he distinguish between developable and ruled surfaces.

Almost half a century later, Euler (1707-1783) and Monge became interested in developable surfaces and used differential calculus to study their properties. Only in 1886 however, was the term "differential geometry" coined. It was used for the first time by Italian mathematician Luigi Bianchi in his textbook *Lezioni di geometria differenziale* (Pisa, 1886). The investigations of Euler and Monge therefore preceded the beginning of a study of differential geometry, and initiated investigations in the field of developable surfaces.

Before we get there, a reminder needs to be made about the nature of the study of change through differential calculus. In the seventeenth century, independently of each other,[7] Isaac Newton (1643-1727) and Gottfried Leibniz (1646-1716) discovered calculus, which deals with the study of change. Newton used calculus to determine an expression for the curvature of plane curves. As the study of tangential surfaces is mentioned throughout this paper, it is worth noting the comparison of the study of curves in two dimensions through differential calculus. For example, finding a tangent to a curve at any point involves seeking the first derivative of that curve and using this result to find an equation of a tangent at a particular point to the curve.

Euler, who was by that time blind, wrote a paper entitled “De solidis quorum superficiem in planum explicare licet” (On solids whose surface can be unfolded onto a plane, E419) [1772] in which his perception of space is clearly shown to be that of identifying and describing surfaces as boundaries of solids, not as collection of solids.[8]

Euler’s approach includes differential treatment combined with geometrical. First he looks at any surface and assumes that a limiting value from this surface will be its derivative: this he obtains through selecting an infinitesimally small right triangle on the surface and seeking its relationship to a congruent triangle in a plane. At this point Euler is generalising a problem via a system of differential equations, but he then proceeds to seek its solution via geometrical treatment. Euler draws three lines on a sheet and describes them simply as Aa, Bb, Cc (fig. 3).

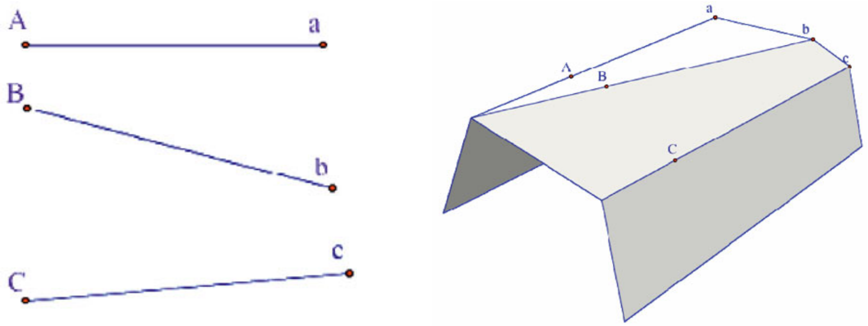

Fig. 3. Euler’s generalisation begins from three straight lines on a sheet of paper, and the bending of the paper along those lines

He then proceeds to investigate what happens when this sheet is bent along a straight line; however that happens, it is always possible to conceive of a solid which fits that bent sheet. From this it follows, Euler concludes that, besides prismatic and pyramidal bodies, there are any number of other kinds of bodies which may be covered in this manner by such a sheet, and whose surfaces may accordingly be unfolded upon a plane:

> Let us now increase to infinity those lines Aa, Bb, Cc, etc. so that our solid acquires a surface everywhere curved, as our problem demands according to the law of continuity. And now it appears at once, that the surface of such bodies should be so constituted that from any point in it at least one straight line may be drawn which lies wholly on this surface; although this condition alone does not exhaust the requirements of our problem, for it is necessary also that any two proximate straight lines lie in the same plane and therefore meet unless they are parallel (translation by Florian Cajori [1929]).[9]

The enormity of Euler’s contribution in this respect is simple to establish. He extended the study of surfaces to developable surfaces by posing a simple question about ***which kind of surfaces can be developed into a plane***:

Notissima est proprietas cylindri et coni, qua neorum superficiem in planum explicare licet atque adeo haec proprietas ad omnia corpora cylindrical et conica extenditur, quorum bases figuram habeant quamcunque; contra vero sphaera hac proprietate destituitur, quum eius superficies nullo modo in planum explicari neque superficie plana obduci queat; ex quo nascitur quaestio aeque curiosa ac *notatu digna, vtrum praeter conos et cylindros alia quoque corporum genera existant, quorum superficiem itidem in planum explicare liceat nec ne? quam ob rem in hac differtatione sequens considerare constitui Problema:*

Inuenire aequationem generalem pro omnibus solidis, quorum superficiem in planum explicare licet, *cuius solutionem variis modis sum agressurus.*

[Euler 1772: 3].

Euler proceeds to look at the intersections of the pairs of lines: if connected these intersections themselves would form a twisted surface of double curvature, a surface, as it were, of a 'higher' degree than the one from which we started (of single curvature, a developable surface). Euler then established the analytical relationship between this twisted curve and the points on the developable surface. His exposition was based on three aspects of the study of such surfaces: the study of developable surfaces by the means of analytical principles; their study by the means of geometrical principles; and finally the study in which the second is applied to the study of the first. It is interesting to note that, although Euler's opus includes some 832 original works,[10] nevertheless this paper of Eulier is considered by some to be his very best mathematical piece.[11]

Monge's study of developable surfaces and the invention of Descriptive Geometry

Independently from Euler, and at almost the same time, a young draughtsman glimpsed a new way of imagining spatial relations through principles of generation of geometrical entities by movement. Monge therefore studied developable surfaces, having started from an entirely different viewpoint. In, or around, 1764, and while working at the l'École Royale du Génie de Mézières, Monge was given the task of determining the necessary height of an outer wall in a design of fortification (fig. 4).

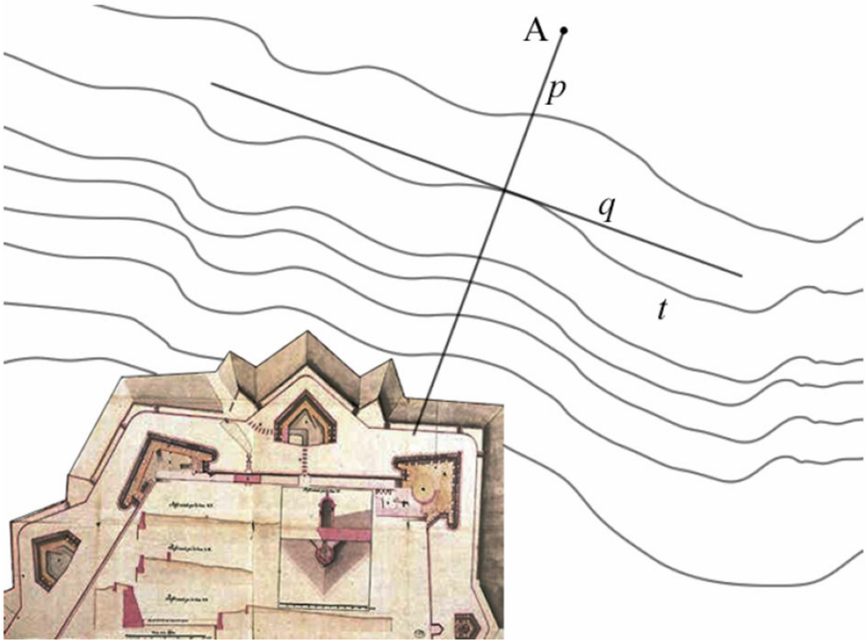

Fig. 4. Monge's work on determining the height of the fortification wall included construction of developable surfaces

Up to that time, there were two methods most widely used for this problem. One involved determining numerous view points from the terrain; the triangles thus determined by the view point, a point of the edge of the fortification, and the height of the wall sufficient to offer effective protection, were all necessary measurements; a lengthy empirical process was involved. The other method, which was employed in the school in Mézières, was based on long calculations with the heights of each crucial point being measured directly on the terrain and noted on a plan. Instead of using these methods, which would have taken a week to yield the final result, Monge found a way to finish his work in two days. Monge conceived of a new technique, which he called *Géometrie Descriptive* and which was naturally related to the methods widely used at the time. The method was examined with the highly critical eye of his superintendents who had their suspicions about Monge's speed in resolving the problem; when properly understood, it was ruled a military secret.

To solve the problem he was faced with, Monge determined a plane tangent to the terrain. This plane is determined by a point and a line: point A lies in the ground plane of the plan of fortification (fortification designs were usually predetermined and their plans were strictly geometrical), line q is drawn as a perpendicular to line p (p is a tangent to t), which is the contour line of the terrain).

Monge further considered conical surfaces which used lines such as p as generatrixes and points such as A as centres to determine the height of the fortification (the thickened line perpendicular to the plane in which the fortification plan rests).

Once it became a military secret, Descriptive Geometry saw little light until the reform of the educational system of France, which took place during the Revolution. However, Monge's first publications were on developable surfaces: these came years before his technique of descriptive geometry could be made public. His paper, *Mémoire sur les développées, les rayons de courbure et les différents genres d'inflexions des courbes à double courbure* [Monge 1785; written in 1771, but published 1785] gave his theory of developable surfaces in an abstract and purely mathematical manner. The paper "contains a broad exposition of the whole differential geometry of space curves" [Struik 1933: 105], introducing the rectifying developable, and describes such crucial terms in the study of developable surfaces as normal plane, radius of first curvature, and the osculating sphere [Reich 2007].

It is now known that Monge read Euler's paper (E419) only after this, his own (and first) publication on developable surfaces, which made him more interested in the subject and upon which he wrote his second paper *Mémoire sur les propriétés de plusieurs genres de surfaces courbes, particulièrtement sur celles des surfaces d´eveloppables, avec une application à la theorie des ombres et des pénombres* [Monge 1780]. In this paper Monge establishes his simplification of Euler's findings

> … a memoir of Mr Euler … on developable surfaces … in which that illustrious Geometer gave the formulas for recognizing whether or not a given curved surface has the property of being able to be mapped to a plane, … I arrived at some results which seem much simpler to me and easier to use for the same purpose.[12]

These 'simpler' results are summarized in his definition of a developable surface as one which is "flexible and inextensible, one may conceive of mapping it onto a plane, … so that the way in which it rests on the plane is without duplication or disruption of continuity" [Monge 1780: 383].

In this, his second paper on the topic, Monge begins with a curve of double curvature (a twisted curve – for example, a helix) and defining a point on it. Through this point he draws a plane which is perpendicular to a line tangent to the curve at this point; he does the same with another point, tangent and plane that go perpendicularly through this tangent. The two planes intersect in a line. If one imagines a curve which goes around the original surface (of double curvature), takes consecutive points on it and does the same as above (i.e., draws through each point a tangent, and through each tangent a normal plane), then finds the intersections of these planes, a developable surface will have thus been constructed. The generalisation to which Monge arrived at was that, in fact, any curvature of double surface can be enveloped by a space curve; this tangent curve can be used, as explained, to generate a developable surface. Monge's spatial and Euler's analytical insights, proved that the tangent surfaces belong to the same family as conical and cylindrical ones.

Further, developable surfaces are, as will be shown, an integral part of the Mongean treatment of space. His descriptive geometry was born out of an insight into a possibility of constructing an imaginary tangential plane to a terrain in order to solve the real problem of fortification design. But it was not only that his description of the technique itself is a study in developable surfaces; it was that in fact, all objects imagined through the use of descriptive geometry as given by Monge are ruled surfaces (and most are developable). Monge describes all geometrical objects through generation: a point is a generatrix of a line; similarly any plane is generated by two lines. In descriptive geometry the position of any element is determined by its position with respect to the projection planes, of which there are two (for simplicity of explanation, the horizontal and vertical). The generation of a plane surface could be described by the lines in which the plane in question intersects the projection planes. These two lines determine the plane in full and are called the traces of the plane.

In order to arrive at a fuller understanding of Monge's conception of space we will bring one example as follows. The traces of the plane on the projection planes may also be regarded as the lines that generate the plane. It should be now easy to see that those two traces meet on the straight line in which the planes of projection meet each other. In Monge's drawing (fig. 5) this means that in the plane, which has been named from its traces as BAb, those traces, namely the line AB and Ab, meet on the line AC=LM, which represents the intersecting line of two projection planes.

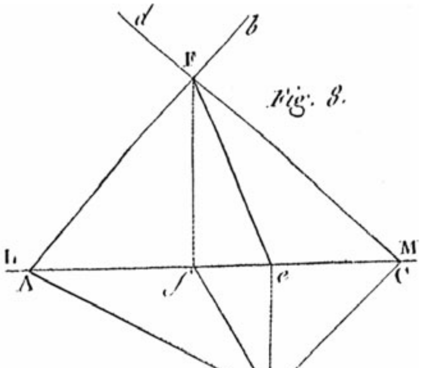

Fig. 5. Plate III from the 1811 Hachette édition of Monge's *Géométrie Descriptive* showing the primary operations with two planes. The system, although determined by two planes of projection, is actually represented only by their line of intersections, AC=LM on this diagram

If we were to consider a second plane (this plane is labelled DCd) we would be able to see how this plane meets the first plane, and their intersection line is EF, the first projection of which is given as Ef and second as eF. We may bear in mind that this is the process that is used in the construction of any solid body with plane surfaces for its sides.

Two important things that explain, in basic terms, the principles of the technique, are to be learnt from this example. First, every three-dimensional geometrical body can be described as a sum of planes which are generated by some mode of motion and intersect each other. This means that the solid is not regarded as an independent entity in itself, but as a product of motion and intersection of its primary elements. Had we further elaborated upon the above example, and said that, for example, the two planes which intersect should be orthogonal to each other, and if we had continued by adding four further planes in a similar manner so that each of them is orthogonal to the one it intersects, and that they should be on the same distance to the parallel ones on all sides, we would generate a cube. In this way the perception of all geometrical entities is changed from a collection of various forms, to a collection of various methods and processes by which those forms are being generated.

Monge realised that, although descriptive geometry had a very practical application in stone-cutting, engineering, and architectural drawing, this treatment of space had also implications for the study of the properties of curves. To Monge, developable surfaces were not just mathematical curiosities which arose out of his spatial insight underpinned by the technique of descriptive geometry which he perfected. In fact, the study of surfaces was an integral part of his interest in stonecutting; in the first edition of *Descriptive Geometry* he describes, for example, how the surfaces of the voussoir joints must be developable

> …This is a practical consideration: if such a surface were not a ruled surface, it would not be executable sufficiently rapidly to be economically viable or with sufficient precision to ensure proper contact between the voussoirs. The fact that the surface is also developable allows greater precision with the use of panels for tracing [Monge 1795b, quoted in Sakarovitch 2009: 1295].


FEUILLES D'ANALYSE
APPLIQUÉE
A LA GÉOMÉTRIE,
A L'USAGE
DE L'ÉCOLE POLYTECHNIQUE,
PARIS

Fig. 6. One of Monge's main works on developable surfaces, *Feuille d'Analyse* [1795a]

But being practical, however useful, was not the main achievement of Monge. The visual insight and ability to manipulate objects in an imaginary space with his technique, meant that his results in three-dimensional differential geometry were superior to those of Euler. The two papers mentioned earlier summarising his results were followed by his *Feuilles d'analyse appliquée à la géométrie* [1795a] (fig. 6).

This book offered a systematic treatment and translated properties of curves and surfaces into analytical language which could be manipulated through partial differential equations. In the words of Morris Kline, "Monge recognised that a family of surfaces having a common geometric property or derived by the same method of generation should satisfy a partial differential equation" [1990: II, 566].

Monge's treatment of developable surfaces is, as can be glimpsed from these short insights into his work on the topic, entirely different from that of Euler. While Euler's treatment was 'of a profoundly analytic spirit' [Taton 1951: 21], Monge had keen geometrical intuition which manifested itself most profoundly in his conception of descriptive geometry, and which enabled him to apply analysis to geometry, rather than consider the two disciplines as separate ways of inquiry.

The collaborative fraternity of Monge and Gehry

Separated by centuries and different professions, it may seem that Monge and Gehry may not have anything in common apart from an interest in developable surfaces. And perhaps one would be right to disregard any further analysis of their common interest. Let us however, entertain an idea that both, in their own times and in their own ways, were interested in these surfaces not by chance, but because they embodied certain ideas about motion that had a particular slant on creative processes that resonated with both Monge and Gehry. In Monge's case this manifested itself in his involvement in all social, political, and educational aspects of his involvement in building the new Republic. Monge built on the tradition and knowledge of stonemasons, resulting in his conception of a new, all-encompassing technique which would serve as a language of graphical communication throughout the territory in which French educational system would extend its influence. This proved to be a large territory, with influences felt up to modern times, and with descriptive geometry still surviving in many national educational systems.[13] Monge's wholehearted involvement with the ideals of the Enlightenment and the founding of the revolutionary educational institutions of the new Republic, gained him the title of the Father of École Polytechnique (see [Sakarovitch 2009]).

The teaching of descriptive geometry in this institution was certainly not a continuation of the educational tradition of the Ancien Régime; it was a revolutionary subject taught in a revolutionary way, in the first of revolutionary schools:

> A scholastic discipline which was born in a school, by a school and for a school (but maybe one should say in the École Polytechnique, by the École Polytechnique, and for the École Polytechnique), descriptive geometry allows the passage from one process of training by apprenticeship in little groups which was characteristic of the schools of the Ancien Régime, to an education in amphitheatres, with lectures, and practical exercises, which are no longer addressed to 20 students, but to 400 students. Descriptive geometry also stems from revolutionary methods. A means to teach space in an accelerated way in relation to the former way of teaching stereotomy, an abstract language, minimal, rapid in the order of stenography, descriptive geometry permits a response to the urgent situation as for the education of an elite, which was the case of France at the moment of the creation of École Polytechnique [Sakarovitch 1995: 211].

It is here, in the context of such sweeping changes, that we move our focus back to the architecture of Gehry (fig. 7). In particular, his architecture is habitually defined by the use of developable surfaces constructed from sheet metal. Gehry himself lists several things that led him to the use of developable surfaces in his later designs, the most important of which he lists as collaboration, movement, and context. All of these in their own way correspond to some of the sentiments that descriptive geometry inspires by its method. Movement is, for Gehry, the leading force:

Fig. 7. Frank O. Gehry, the Guggenheim Museum Bilbao, completed 1997. Photo: iStockphoto

> I became fascinated with the idea of building in a sense of movement with static materials. Historically there are a lot of references. At the Parthenon, Phidias played with that idea – not in the building, but in the sculptures. And the Indian Siva figures are extraordinary, if you look at the best ones: when you look away, you are sure they moved. That has always fascinated me [Gehry 1996: 38].

His interest in collaboration – the building on the tradition of unions – is somewhat similar to Monge's sentiment regarding the stonemasons;[14] in Gehry's case this manifested itself in his involvement with the metal workers' union. He highly values their input – and testifies to a bond between the designer and the executor in terms of their regard for the material to which both are dedicated and from which new structures are built:

> The [Sheet Metal Workers Union of America] is connected to metal workers around the world, and they have agreed to provide me with technical assistance wherever I go. They have lived up to that agreement, most recently on a project in Bilbao, Spain. They have helped me work within my budget and still achieve the forms I want. This collaboration has certainly made possible some of my successes in the use of metal materials [Gehry 1996: 40].

Finally it is movement, in all its symbolic and contextual form, that Gehry perhaps embodies best through his architecture: the sweeping movements of lines generating developable surfaces 'that is the sort of thing that occurs in a fast-moving city; as one tries to respond to the context, it changes, always more rapidly' [Gehry 1996: 42].

Conclusion

In the nineteenth century the mathematics of developable surfaces temporarily left the realm of the real world. This happened for very many reasons but suffice it to say that, for example, the freer and more common use of the imaginaries of all kinds: talking about 'circular points of the line at infinity' and being able not only to describe but manipulate imaginary numbers with increasing ease, meant that the study of surfaces increasingly involved interest in the imaginaries linked to such entities. This in turn led to the discovery and study of minimal surfaces.[15] In another development, Henry Lebesgue (1875-1941), best known for his theory of integration, made an astonishing discovery of surfaces which, although resembling developable surfaces, are neither developable nor ruled, and can be flattened into a plane without stretching or tearing (but with creasing) [Picard 1922].[16] Lebesgue's own description of such surfaces is somewhat convoluted, but nevertheless its very short version is given in a footnote to a simple explanation given to Lebesgue's discovery by Émile Picard (1856-1941):

> According to general practice, we suppose in the preceding analysis, as in all infinitesimal geometry of curves and surfaces, the existence of derivatives which we need in the calculus. It may seem premature to entertain a theory of surfaces in which one does not make such hypotheses. However, a curious result has been pointed out by Mr Lebesgue (*Comptes Rendus*, 1899 and *thesis*); according to which one may, by the aid of continuous functions, obtain surfaces corresponding to a plane, of such sort that every rectifiable line of the plane has a corresponding rectifiable line of the same length of the surface, nevertheless the surfaces obtained are no longer ruled. If one takes a sheet of paper, and crumples it by hand, one obtains a surface applicable to the plane and made up of a finite number of pieces of developable surfaces, joined two and two by lines, along which they form a certain angle. If one imagines that the pieces become infinitely small, the crumpling being pushed everywhere to the limit, one may arrive at the conception of surfaces applicable to the plane and yet not developable [in the sense there is no envelope of a family of planes of one parameter] and not ruled [Picard 1922], quoted in [Cajori 1921: 436].

An apparent departure from the 'real world' into the abstract construct of imaginaries permitted a new understanding of developable surfaces. But their physical embodiment in the architecture of Frank Gehry perhaps provides the most fitting monument to the study of developable surfaces in three dimensions, and to Mongean treament of space.

Notes

1. Gausian curvature is equal to zero when the product of minimal and maximal curvatures of a curve is equal to zero, i.e., $k = k_{\min} \times k_{\max}$. Carl Friedrich Gauss (1777-1855), called sometimes "the Prince of Mathematicians' contributed to a huge array of fields in the study of mathematics, including analysis, differential geometry, number theory, astronomy and optics.
2. Beyond the scope of this paper, this property of Gaussian curvature is none the less fascinating. It was expounded upon in Gauss's *Theorema Egregium* (Remarkable Theorem), which states that the Gaussian curvature of a surface can be determined by measuring distances and angles of a surface without reference to the way in which the surface is positioned in three-dimensional space.
3. For further details see [Amar and Pomeau 1997].

4. The Royal School of Engineering at Mézières was founded in 1748 and was closed in 1794 when it transferred to the School of Engineering at Metz.
5. Omitted from this paper for the reasons of length and simplicity are descriptions of the work of Jean Baptiste Marie Charles Meusiner (1754-1793) and Piere Charles François Dupin (1674-1873). Dupin's main contribution was the invention of a method for describing the local shape of a surface by, what is now called *Dupin indicatrix*. The indicatrix is the end product of a limiting process by which the plane approaches a tangent plane of the surface studied. Meusiner, on the other hand studied minimal surfaces.
6. See [Cajori 1929: 431]; the description used in Hawney is 'in form of a Rolling stone used in Gardens' [1717: 154].
7. Although this fact was vehemently denied by Newton who pursued the claims of plagiarism until Leibniz's death; see [Hall 1980].
8. In fact this approach was followed as late as the nineteenth century in relation to constructive geometry: for a different treatment of space in particular in context of generating surfaces (Gaspard Monge, France) and manipulating objects (Peter Nicholson, England) see [Lawrence 2002; 2010].
9. Meaning that if there is a surface, there lie in it at least two lines which are parallel [Euler 1772].
10. Euler's papers were catalogued by Gustaf Eneström (hence the E numbers). Most of them can be found at The Euler Archive online: http://www.math.dartmouth.edu/~euler/.
11. According to Andreas Speiser (1885-1970), a Swiss mathematician and philosopher of science, quoted in [Reich 2007: 483, note 7].
12. Full text states: *Ayant repris cette matière, à l'occasion d'un Mémoire que M. Euler a donné dans le Volume de 1771, de l'Académie de Pétersbourg, sur les surfaces développables, et dans lequel cet illustre Géomètre donne des formules pour reconnoître si une surfache courbe proposée, jouit ou non de la propriété de pouvoir être appliquée sur un plan, je suis parvenu à des résultats qui me semblent beaucoup plus simples, et d'un usage bien plus facile pour le même objet* [Monge 1780; quoted in Reich 2007: 490, note 15].
13. Currently a research project by the author, to be presented at the Second International Conference on the History in Mathematics Education, Portugal, October 2011.
14. Without wishing to imply any parallels with Gehry's own sentiments in this regard, it can, however, be noted that the ideals that defined Monge's political and social life were also those of the French Revolution – *liberté*, *égalité*, *fraternité* (without the terror) – and which were also manifest in Monge's admission to the French Freemasonic order, the *Grand Orient*; see [Lawrence 2002: 118].
15. As already mentioned, Meusiner's study of surfaces via a calculus of variations led him to discover that a surface is minimal if and only if its mean curvature vanishes. The most common 'real-life' explanation of minimal surfaces is their comparison with the soap film formed around a wire boundary.
16. Lebesgue's own explanation of the surfaces that are not ruled yet can be 'flattened' into a plane goes like this:

> To obtain surfaces that are not ruled I take an analytic developable one and upon it an analytic curve C, not geodesic [shortest path between points on a curved surface]. One knows that there exists another developable surface passing through C, such that one can make the two developables applicable to a plane in such a manner that to each point of C, whether considered as belonging to the one or to the other of the two surfaces, there corresponds one and the same point in the plane. The curve C divides the first developable in two pieces, A, B, also the second into two pieces A′, B′.
>
> Two of the four surfaces (A, A′), (A, B′), (B, B′) are applicable to the plane without tearing or duplication and are indeed such that one can detach from them a finite piece enjoying the above property and containing an arc of C. This C is then a singular line. As before, one may pass from this one singularity to an infinite number of singularities and one thus obtains surfaces applicable to the plane, yet not containing any segment of a straight line [Lebesgue 1899].

References

ARISTOTLE. 2004. *De Anima*. H. Lawson Tancred, ed. London: Penguin.

AMAR, Ben M. and Y. POMEAU. 1997. Crumpled paper. *Proceedings of the Royal Society* 453: 719-755.

CAJORI, F. 1929. Generalisations in Geometry as Seen in the History of Developable Surfaces. *The American Mathematical Monthly* 36, 8: 431-437.

CAYLEY, Arthur. 1850. On the developable surfaces which arise from two surfaces of the second order. *Cambridge and Dublin Mathematical Journal* V: 46-57.

———. 1862. On Certain Developable Surfaces. *Proceedings of the Royal Society of London* 12: 279-280.

CLAIRAUT, Alexis-Claude. 1731. *Recherches sur les courbes à double courbure*. Paris.

COLLIDGE, Julian Lowell. 1940. *A history of geometrical methods*. Oxford: Oxford University Press.

DARBOUX, Gaston. 1904. The Development of Geometrical Methods. *The Mathematical Gazette*, (3:48) 100-106.

DELCOURT, Jean. 2011. Analyse et géométrie, histoire des courbes gauches De Clairaut à Darboux. *Archive of the History of Exact Sciences* 65: 229-293.

DOMINGUES, João Caramalho. 2008. *Lacroix and the Calculus*. Berlin: Birkhaüser.

DUPIN, Charles. 1819. *Essai historique sur les services et les travaux scientifiques de Gaspard Monge*. Paris: Bachelier.

EULER, Leonhard. 1772. *De solidis quorum superficiem in planum explicare licet* (E419). *Novi commentarii academiae scientiarvm imperiatis Petropolitanae* XVI (1771): 3-34. Rpt. *Leonhardi Euleri Opera Omnia*: Series 1, vol. 28, pp. 161-186.

FREZIER, Amédée-François. 1737-1739. *La théorie et la pratique de la coupe des pierres et des bois … ou traité de stéréotomie a l'usage de l'architecture*. Strasbourg-Paris: Jean Daniel Doulsseker-L. H. Guerin.

GEHRY, Frank O. 1996. Current and Recently Completed Work. *Bulletin of the American Academy of Arts and Sciences* 49, 5: 36-55.

GLAESER, G. and F. GRUBER. 2007. Developable surfaces in contemporary architecture. *Journal of Mathematics and the Arts* 1, 1: 59-71.

HALL, A H. 1980. *Philosophers at War: The Quarrel between Newton and Leibniz*. New York: Cambridge University Press.

HAWNEY, W. 1717. *The Complete Measurer*. London.

KLINE, Morris. 1990. *Mathematical Thought from Ancient to Modern Times*, 3 vols. Oxford: Oxford University Press.

LAWRENCE, Snežana. 2002. Geometry of Architecture and Freemasonry in 19th century England. Ph.D Thesis, Open University UK.

———. 2010. Alternative ways of teaching space: a French geometrical technique in nineteenth-century Britain. Pp. 215-224 in *Echanges entre savants français et britanniques depuis le XVIIe siècle*, Robert Fox and Bernard Joly, eds. Oxford: College Publications.

LEBESGUE, H. 1899. Sur quelques surfaces non réglées applicables sur le plan. *Comptes Rendus de l'Académie des Sciences* CXXVIII: 1502-1505.

LIU, Y., H. POTTMAN, J. WALLNER, Y-L. YANG, and W. WANG, W. 2006. Geometric Modeling with Conical Meshes and Developable Surfaces. *ACM Transactions on Graphics* 25, 3: 681-689.

MONGE, Gaspard. 1769. Sur les développées des courbes à double courbure et leurs inflexions. *Journal encyclopédique* (June 1769): 284-287.

———. 1780. Mémoire sur les propriétés de plusieurs genres de surfaces courbes, particulièrtement sur celles des surfaces d´eveloppables, avec une application à la theorie des ombres et des pénombres. *Mémoires de divers sçavans* 9: 593-624 (written 1775).

———. 1785. Mémoire sur les développées, les rayons de courbure, et les différents genres d'inflexions des courbes à double courbure. *Mémoires de divers sçavans* 10: 511-50 (written 1771).

———. 1795a. *Feuilles d'analyse appliquée à la géométrie*. Paris. Second ed. with the title *Applications de l'analyse à la géométrie*. Paris : Baudouin, 1811.
———. 1795b. *Géométrie descriptive*. First ed. in *Les Séances des écoles normales recueillies par des sténographes et revues par des professeurs*, Paris. Re-edition pp. 267-459 in *L'Ecole normale de l'an III, Leçons de mathématiques, Laplace, Lagrange, Monge*, Jean Dhombres, ed. Paris: Dunod, 1992.
———. 1850. *Application de l'analyse a la géométrie,* 5th ed. Paris.
PICARD, E. 1922. *Traite d'analyse*, 3rd ed. Paris: Gauthier-Villars et fils.
REICH, Karin. 2007. Euler's Contribution to Differential Geometry and its Reception. Pp. 479-502 in *Leonhard Euler: Life, Work and Legacy*, Robert E. Bradley and C. Edward Sandifer, eds. Amsterdam: Elsevier.
SAKAROVITCH, Joël. 1989. Theorisation d'une pratique, pratique d'une theorie: Des traits de coupe des pierres à la géométrie descriptive. Ph.D. thesis, Ecole d'Architecture de Paris La Villette.
———. 1995. The Teaching of Stereotomy in Engineering Schools in France in the XVIIIth and XIXth centuries: An Application of Geometry, an 'Applied Geometry', or a Construction Technique? Pp. 205-218 in *Between Mechanics and Architecture*, Patricia Radelet-de Grave and Edoardo Benvenuto, eds. Basel: Birkhäuser.
———. 1997. *Epures D'architecture*. Basel: Birkhäuser.
———. 2009. Gaspard Monge Founder of 'Constructive Geometry'. Pp. 1293-1300 in *The Proceedings of the Third International Congress on Construction History*, Cottbus. http://www.formpig.com/pdf/formpig_gaspar monge founder of constructive geometry _sakarovitch.pdf (last accessed 30 June 2011).
STRUIK, Dirk J. 1933a. Outline of a History of Differential Geometry I. *Isis* 19, 1: 92-120.
———. 1933b. Outline of a History of Differential Geometry II. *Isis* 20, 1: 161-191.
TATON, Rene. 1951. *L'Œuvre scientifique de Monge*. Paris: Presses Universitaires de France.
———. 1966. La première note mathématique de Gaspard Monge. *Revue d'histoire des sciences et de leurs applications* 19, 2: 143-149.
VELIMIROVIC, L. and M. CVETKOVIC. 2008. Developable surfaces and their applications. Pp. 394-403 in *Mongeometrija 2008*, Proceedings of the 24th International Scientific Conference on Descriptive Geometry, Serbia, September 25-27, 2008.

About the author

Snežana Lawrence is a Senior Lecturer in Mathematics Education at the Bath Spa University, in Bath, England. She has been involved with a number of UK national initiatives to promote the use of the history of mathematics in mathematics education. Her website (www.mathsisgoodforyou.com) is a popular resource for secondary school children and teachers, attracting more than 60,000 page views a month. Snezana is interested in the historical issues relating to mathematics education in professional setting and as such has written and researched on the mathematical education deemed necessary for architects and engineers from the eighteenth century until today, in the countries of Western Europe and the United States. Snezana is on the editorial boards of *Mathematics Today* (a journal of the Institute of Mathematics and Its Applications, UK), and the *British Society for the History of Mathematics Bulletin*, and is the first Education Officer of the same society. Her chapter on the history of mathematics in the Balkans has appeared in the recent *Oxford Handbook of the History of Mathematics* (Jackie Stedall and Eleanor Robson, eds., 2008). As a member of the Advisory Board of the History and Pedagogy of Mathematics group (an affiliate of the International Commission on Mathematics Education) she is a regular contributor to their publications.