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Geometer's Angle

Squaring the Circle: Marriage of Heaven and Earth

Abstract. It is impossible to construct circles and squares of equal areas or perimeters precisely, for circles are measured by the incommensurable value *pi* (π) and squares by rational whole numbers. But from early times, geometers have attempted to reconcile these two orders of geometry. "Squaring the circle" can represent the union of opposing eternal and finite qualities, symbolizing the fusion of matter and spirit and the marriage of heaven and earth. In this column, we consider various methods for squaring the circle and related geometric constructions.

I Introduction

From the domed Pantheon of ancient Rome, if not before, architects have fashioned sacred dwellings after conceptions of the universe, utilizing circle and square geometries to depict spirit and matter united. Circular domes evoke the spherical cosmos and the descent of heavenly spirit to the material plane. Squares and cubes delineate the spatial directions of our physical world and portray the lifting up of material perfection to the divine.

Constructing these basic figures is elementary. The circle results when a cord is made to revolve around a post. The right angle of a square appears in a 3:4:5 triangle, easily made from a string of twelve equally spaced knots.[1] But "squaring the circle"—drawing circles and squares of equal areas or perimeters by means of a compass or rule—has eluded geometers from early times.[2] The problem cannot be solved with absolute precision, for circles are measured by the incommensurable value *pi* (π = 3.1415927…), which cannot be accurately expressed in finite whole numbers by which we measure squares.[3] At the symbolic level, however, the quest to obtain circles and squares of equal measure is equivalent to seeking the union of transcendent and finite qualities, or the marriage of heaven and earth. Various pursuits draw from the properties of music, geometry and even astronomical measures and distances. Each attempt offers new insight into the wonder of mathematical order. In this column, we consider methods for achieving circles and squares of equal perimeters, focusing on geometric approaches conducive to design applications and setting aside for now the problem of achieving circles and squares of equal areas.

Definitions:

The **circle** is the set of points in a plane that are equally distant from a fixed point in the plane.

The fixed point is called the center. The given distance is called the radius. The totality of points on the circle is called the circumference.

"Circle" is from the Latin *circulus*, which means "small ring" and is the diminutive of the Latin *circus* and the Greek *kuklos*, which mean "a round" or "a ring" [Liddell 1940, Simpson 1989].

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The **circumference** is the line that forms the encompassing boundary of a circle or other rounded figure. The circumference (*c*) of a circle is $2\pi r$, where (*r*) is the length of the radius, or πd, where (*d*) is the length of the diameter. The area (*a*) of a circle is πr^2.

$$c = 2\pi\, r = \pi\, d$$
$$a = \pi\, r^2$$

The Latin for "circumference" is *circumferentia* (from *circum* "round, about" + *ferre* "to bear"), which is a late literal translation of the Greek *periphereia*, which means "the line around a circular body" or "periphery" [Liddell 1940, Simpson 1989].

The **square** is a closed plane figure of four equal sides and four 90° angles. "Square" is an adaptation of the Old French *esquare* (based on the Latin *ex-* "out, utterly" + *quadra* "square," which is from *quattuor* "four") [Harper 2001, Simpson 1989].

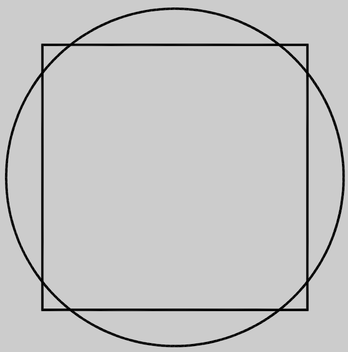
Perimeter is the term for the continuous line or lines that bound a closed geometrical figure, either curved or rectilinear, or of any area or surface. The perimeter (*p*) of a square is equal to four times the length of one of its sides (*s*):

$$p = 4s$$

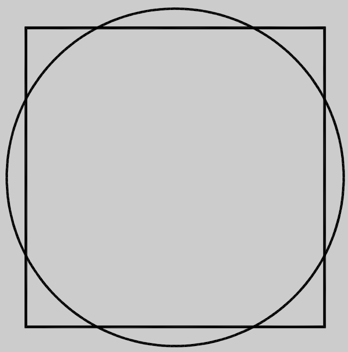
The Latin for "perimeter" is *perimetros,* which means "circumference or perimeter," from the Greek *perimetros* (from *peri* "around" + *metron* "measure") [Lewis 1879, Simpson 1989].[4]

A circle of radius 1 is **equal in perimeter** to a square of side of $\pi/2$. (Each perimeter equals 2π.)

A circle of radius 1 is **equal in area** to a square of side $\sqrt{\pi}$. (Each area equals π.) (fig. 1)



Circle of radius 1
Square of side $\pi/2$
Perimeters of circle and square = 2π



Circle of radius 1
Square of side $\sqrt{\pi}$
Areas of circle and square = π

Fig. 1

II Vesica Piscis

A vesica piscis initiates our first technique for drawing a circle and square of equal perimeter, and is offered by John Michell.[5]

- Draw an indefinite horizontal line. Locate the approximate midpoint, at point O.
- Place the compass point at O. Draw arcs of equal radius that cross the horizontal line on the left and right, at points A and B.
- Set the compass at an opening that is slightly larger than before. Place the compass point at A. Draw an arc above and below, as shown.
- With the compass at the same opening, place the compass point at B. Draw an arc above and below, as shown.
- Locate points C and D where the two arcs intersect.
- Draw an indefinite vertical line through points C, O, and D.

Point O locates the intersection of the horizontal and vertical lines (fig. 2).

- Place the compass point at O. Draw a circle of indefinite radius, as shown.
- Locate point E where the circle intersects the vertical line, above.
- Place the compass point at E. Draw a circle of radius EO.

The horizontal line is perpendicular to the radius EO and tangent to its circle (fig. 3).

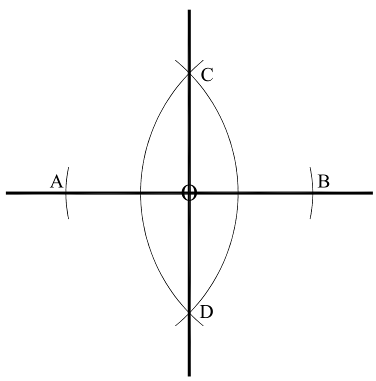


Fig. 2

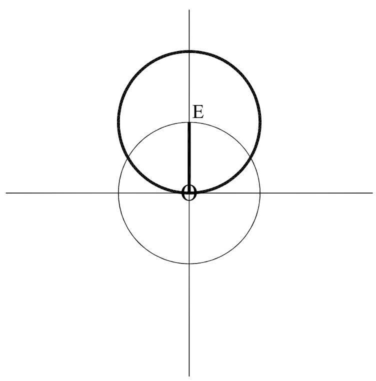


Fig. 3

Definition:

The **tangent** to a circle is any straight line in the plane of the circle that has but one point in common with the circle. The **point of contact** is the common point shared by the circle and the tangent. A straight line that is tangent to the circle is perpendicular to the radius drawn to the point of contact. A tangent exists at each point along the circumference.

"Tangent" is from the Latin *tango* or *tactus* ("to touch") and from the Greek *tetagôn* ("having seized"). The Greek for "tangent" is *epaphê* ("touch, touching, handling"), in geometry meaning "point of contact" [Lewis 1879, Simpson 1989].

- Locate point E, then remove the circle whose center is point O.
- Locate point F where the remaining circle intersects the vertical line, as shown.
- Place the compass point at F. Draw a circle of radius FE.
- Extend the vertical line to the circumference of the circle (point G), as shown (fig. 4).

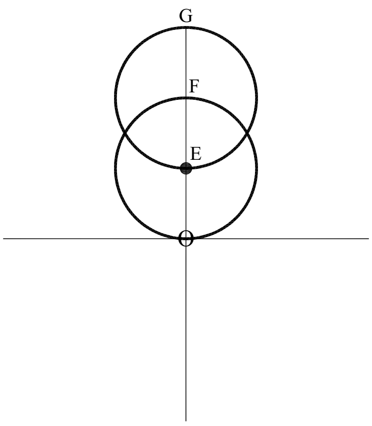


Fig. 4

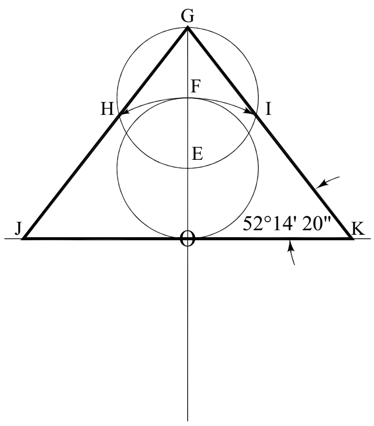


Fig. 5

- Place the compass point at O. Draw an arc of radius OF that intersects the upper circle at points H and I, as shown.
- From point G, draw a line through point H that intersects the horizontal line (JK) at point J.
- From point G, draw a line through point I that intersects the horizontal line (JK) at point K.
- Connect points G, K and J.

The result is an isosceles triangle whose base angles measure 52°14′20″ (52.2388…) (fig. 5).

- Place the compass point at O. Draw a circle of radius OJ.
- Place the compass point at J. Draw a half-circle of radius JO through the center of the circle (point O), as shown.
- Place the compass point at K. Draw a half-circle of radius KO through the center of the circle (point O), as shown.
- Locate points L and M, where the circle of radius OJ intersects the indefinite vertical line.
- Place the compass point at L. Draw a half-circle of radius LO through the center of the circle (point O), as shown.
- Place the compass point at M. Draw a half-circle of radius MO through the center of the circle (point O), as shown.

The four half-circles are of equal radius and intersect at points N, P, Q and R.

- Connect points N, P, Q and R.

The result is a square (fig. 6).

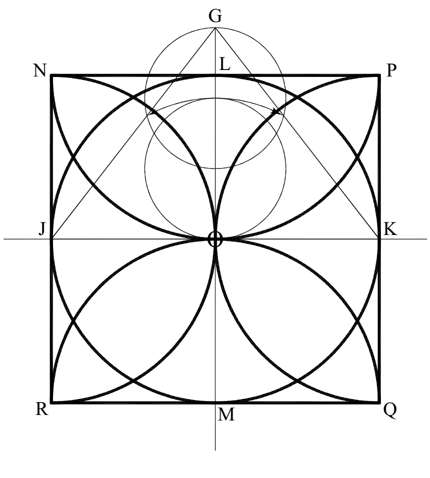


Fig. 6

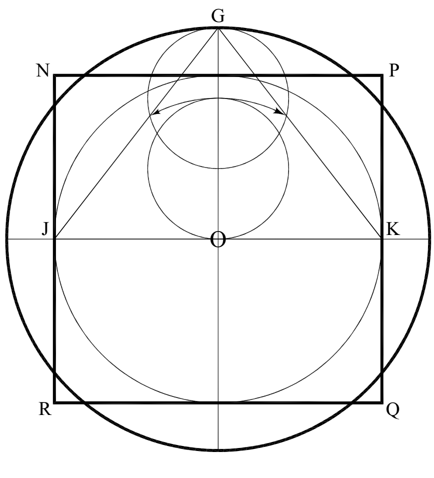


Fig. 7

- Remove the four half-circles.
- Place the compass point at O. Draw a circle of radius OG.

If the radius OJ equals 1, then the radius (OG) of the large circle equals 1.29103….

If the value of π equals 3.14159, the circumference of the large circle equals 8.1117….

The side (NP) of the square equals 2 and the perimeter equals 8.0.

The circle and square are equal in perimeter within 1.4% (fig. 7).

III Double Vesica Piscis

Another method, based on a double vesica piscis, has been observed in traditional temple plans in India [Critchlow 1982, 30-31; Michell 1988, 40-42, 70-72].

- Repeat figure 2, as shown. Extend the horizontal and vertical lines in both directions.

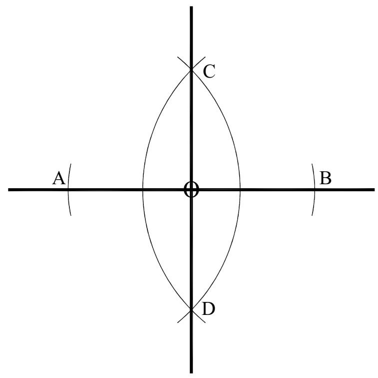


Fig. 2

- Place the compass point at O. Draw a circle of indefinite radius.
- Locate points E and F where the circle intersects the horizontal line.
- Locate points G and H where the circle intersects the vertical line (fig. 8).

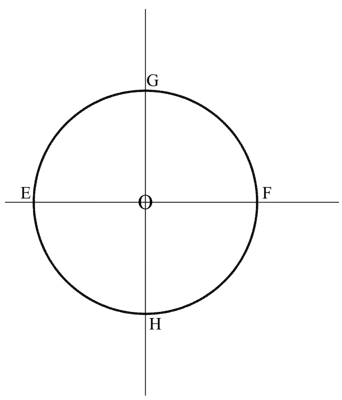


Fig. 8

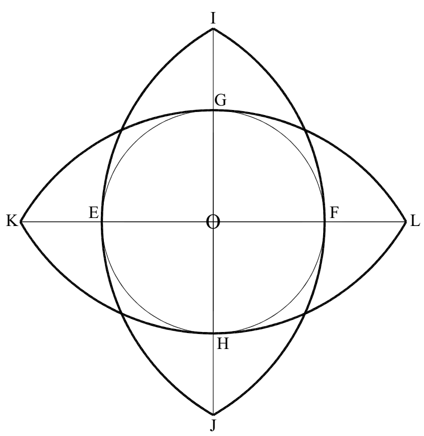


Fig. 9

- Place the compass point at E. Draw an arc of radius EF that intersects the vertical line at points I and J.
- Place the compass point at F. Draw an arc of radius FE that intersects the vertical line at points I and J.
- Place the compass point at G. Draw an arc of radius GH that intersects the horizontal line at points K and L.
- Place the compass point at H. Draw an arc of radius HG that intersects the horizontal line at points K and L (fig. 9).
- Locate points M, N, P and Q where the four arcs intersect.
- Connect points M, N, P and Q.

The result is a square.

- Locate the circle of radius OE that is contained within the double vesica piscis.

If the radius (OE) of the circle equals 1 and the value of π equals 3.14159, the circumference of the circle equals 6.28318.

The side (MN) of the square equals 1.64575… and the perimeter equals 6.58300….

The circle and square are equal in perimeter within 4.8 % (fig. 10).

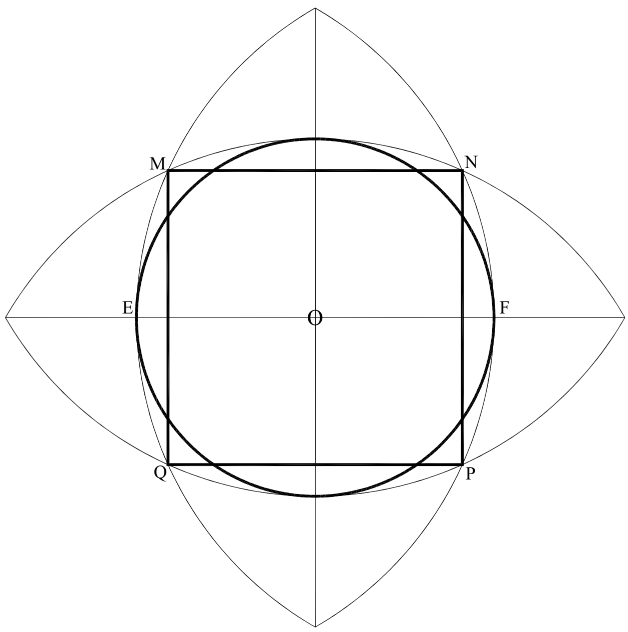


Fig. 10

IV Golden Section

This technique, offered by Robert Lawlor, utilizes the Golden Section, or Golden Ratio of 1 : *phi* or 1 : ϕ ($\phi = \sqrt{5}/2 + 1/2$), which translates numerically to the incommensurable ratio 1 : 1.618034….[6]

- Repeat figure 2, as shown.
- Place the compass point at O. Draw a circle of indefinite radius.
- Locate point E where the circle intersects the horizontal line, on the left.
- Place the compass point at E. Draw a circle of radius EO.
- Locate point F where the circle intersects the horizontal line, on the right.
- Place the compass point at F. Draw a circle of radius FO (fig. 11).

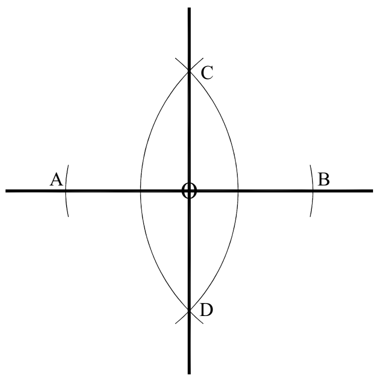


Fig. 2

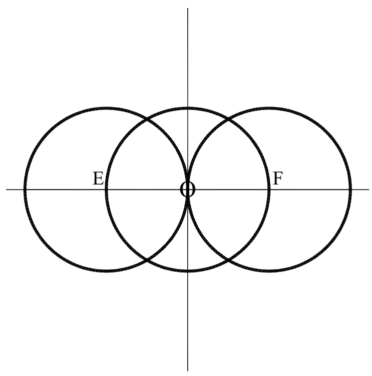
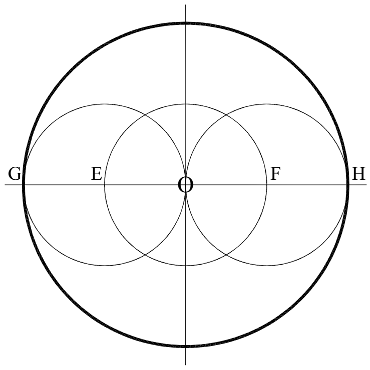
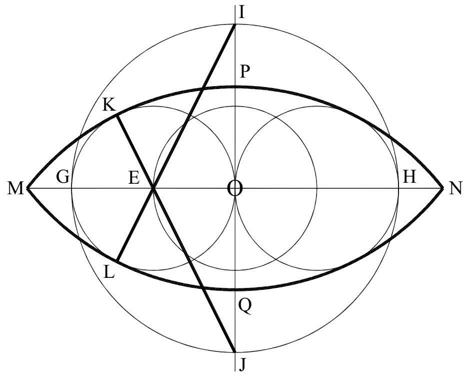


Fig. 11

- Locate points G and H along the horizontal line, as shown.
- Place the compass point at O. Draw a circle of radius OH that encloses the three circles (fig. 12).





IP : PO :: PO : OH

$\frac{1}{\Phi^2} : \frac{1}{\Phi} :: \frac{1}{\Phi} : 1$

Fig. 12

Fig. 13

- Locate points I and J where the large circle intersects the vertical line.
- Draw a line from point I to point E. Extend the line IE to the circumference of the left circle (point L).
- Place the compass point at I. Draw an arc of radius IL, which intersects the extension of the horizontal diameter (GH) at points M and N.
- Draw a line from point J to point E. Extend the line JE to the circumference of the left circle (point K).
- Place the compass point at J. Draw an arc of radius JK, which intersects the extension of the horizontal diameter (GH) at points M and N.

If the radius (OH) of the large circle is 1, the radius (IL) of the arc (MN) equals *phi* ($\phi = \sqrt{5}/2 + 1/2$ or 1.618034...), and half of the arc's long axis (ON) equals $\sqrt{\phi}$ (1.272019…).

If the short axis (PQ) of the arc equals 1, the diameters (GH and IJ) of the large circle equal ϕ^7 (fig. 13).

- Locate point I at the top of the vertical diameter (IJ) of the large circle.
- Place the compass point at I. Draw a half-circle of radius IO through the center of the circle (point O), as shown.
- Locate point H at the right end of the horizontal diameter (GH) of the large circle.
- Place the compass point at H. Draw a half-circle of radius HO through the center of the circle (point O), as shown.

- Locate point J at the bottom of the vertical diameter (IJ) of the large circle.
- Place the compass point at J. Draw a half-circle of radius JO through the center of the circle (point O), as shown.
- Locate point G at the left end of the horizontal diameter (GH) of the large circle.
- Place the compass point at G. Draw a half-circle of radius GO through the center of the circle (point O), as shown.

The four circles are of equal radius and intersect at points R, S, T and U.

- Connect points R, S, T and U.

The result is a square (fig. 14).

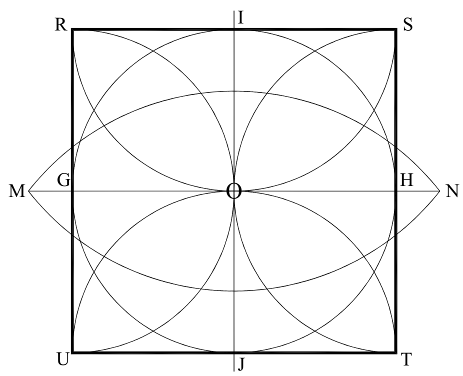


Fig. 14

- Remove the four half-circles.
- Place the compass point at O. Draw a circle of radius ON.
- Locate the radius (OH) of the smaller circle, as shown.

If the radius (OH) of the small circle equals 1, the radius (ON) of the large circle equals $\sqrt{\phi}$ (1.272019…).

If the radius (ON) of the large circle equals $\sqrt{\phi}$ and the value of π equals 3.14159, then the circumference of the large circle equals 7.99232….

The side (RS) of the square equals 2 and the perimeter equals 8.0.

The circle (of radius ON) and the square are equal in perimeter within 0.1% (fig. 15).[8]

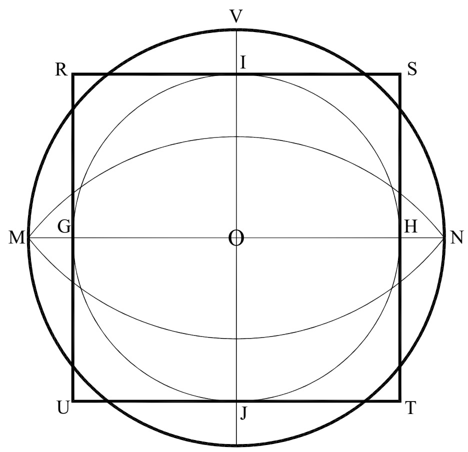


Fig. 15

V The Great Pyramid

- Connect points O, H, and V.

The result is a right triangle of sides 1 (OH) and $\sqrt{\phi}$ (OV). The hypotenuse (HV) is ϕ.

Angle VHO equals 51°49′38″ (51.827…) (fig. 16).

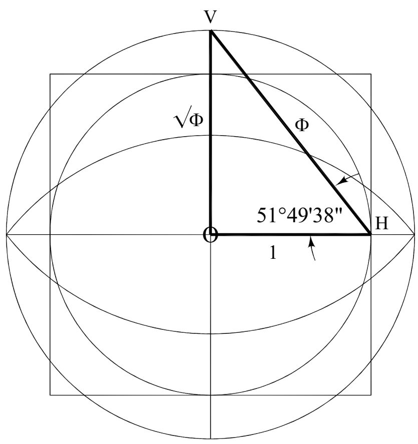


Fig. 16

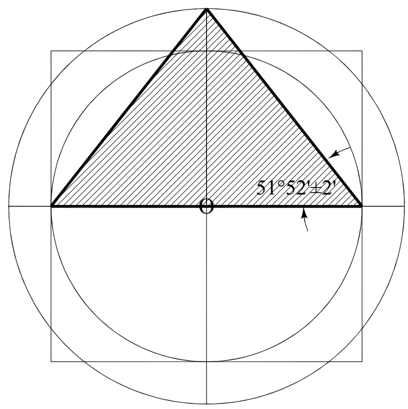


Fig. 17

The Great Pyramid of Khufu, second king of the Fourth Dynasty (2613–2494 B.C.) and known to the Greeks as Cheops, is the largest of three pyramids at Gizeh, approximately eight miles from modern Cairo. The pyramids are built largely of limestone blocks with some granite, and erected near the edge of the limestone desert that borders the west side of the Nile valley. The approximate mean face angle of the Great Pyramid, based on calculations by Flinders Petrie, is 51°52′± 2′ (51.866...) [Petrie 1990, xi, 12-13] (fig. 17).

Another method for achieving the proportions of the Great Pyramid, offered by John Michell, derives from a rhombus inscribed within a vesica piscis [1983, 158]. Fig. 18A presents an isosceles triangle whose base angle of 51°36′38″ (51.61055…) approximates the face angle of the Great Pyramid.[9] Fig. 18B presents a square whose side equals the base of the triangle and a circle whose radius equals the height of the triangle. The circle and square are equal in perimeter within 0.8%.

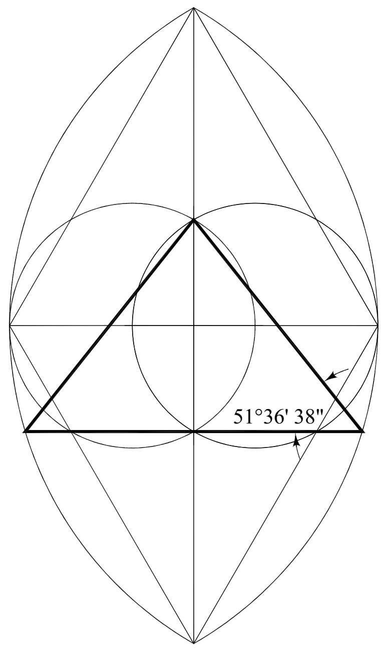


Fig. 18a

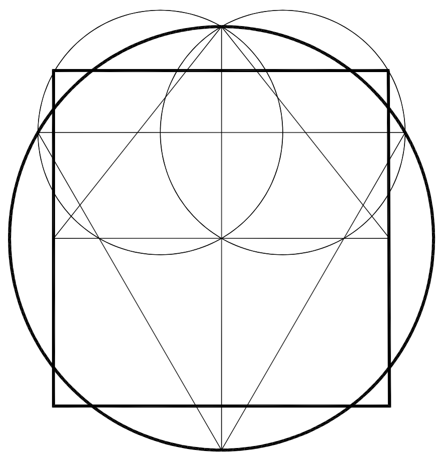


Fig. 18b

Kurt Mendelssohn proposes a practical method for achieving the Pyramid's proportions that utilizes the diameter and circumference, or revolution, of a rolling drum. The technique is not a true squaring of the circle, because it employs an instrument other than a compass and rule. But the relationship between circles and squares of equal perimeters is expressed in precise terms [Mendelssohn 1974, 73].

- Let the height (VO) of an isosceles triangle (VHG) equal the length of four drums stacked tangent to one another.
- Let half the base (OH) of the triangle equal the length of the circumference, or one revolution of one drum.
- Place the compass point at O. Draw a circle of radius OV.
- Place the compass point at O. Draw a circle of radius OH.

- Draw a square (RSTU) about the circle of radius OH.

If the diameter of each drum equals 1, the radius (OV) of the large circle equals 4, and the radius (OH) of the smaller circle equals π.

The circumference of the large circle (radius OV) and the perimeter of the square (RSTU) each equal 8π precisely.

If π equals 3.14159, angle VHO equals 51°51′14″ (51.854…), which is approximately the mean face angle of the Great Pyramid (fig. 19).

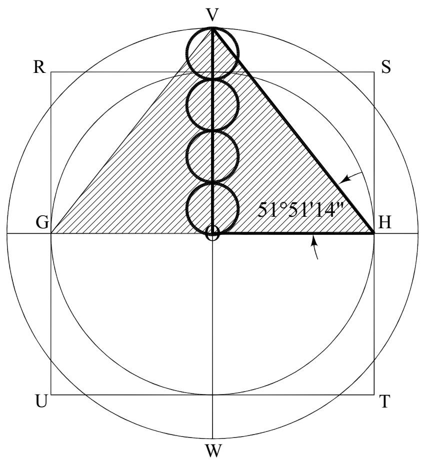


Fig. 19

VI Relative Measures of Earth and Moon

One solution for squaring the circle, offered by John Michell, derives from actual astronomical measures. The construction is based on a circle of radius 5040, representing in miles the combined mean radii of the circles of the earth (3960) and the moon (1080).[10]

- Draw a circle representing the earth (mean radius 3960 miles) and a circle representing the moon (mean radius 1080 miles) tangent to one another, as shown (fig. 20).
- Draw a square about the circle of radius 3960 (earth).
- Draw a circle about the combined radii of 3960 (earth) and 1080 (moon), or 5040.

If π equals the Archimedean value of 22/7, the circle of radius 5040 and the square drawn about the "earth" circle of radius 3960 are exact (31,680) (fig. 21).

The measures in Michell's construction express added meaning when converted to different scales and units of measure, suggesting that the different measuring systems are interrelated. For example, 31,680 in miles is both the circumference of the circle drawn on the combined radii of the earth and moon and the perimeter the square containing the circle of the earth alone. But the number 31,680 in furlongs is the mean radius of the earth (3960 miles) and in inches is half a mile [Michell 1988, 33, 173].

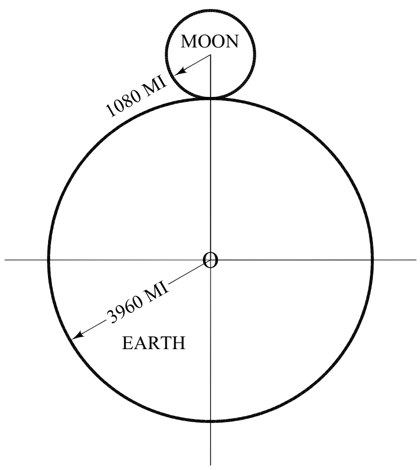


Fig. 20

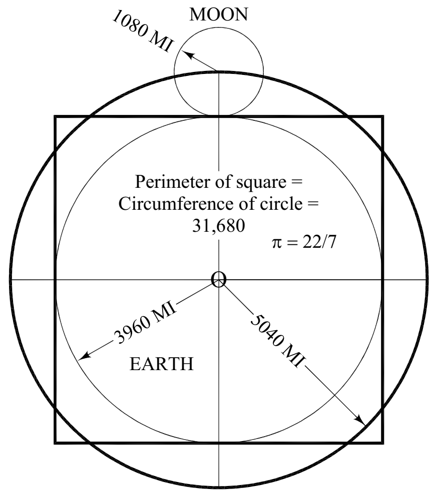


Fig. 21

In miles, the radius of the moon is 1080. One hundred and eight (108) is the atomic weight of silver, the metal we traditionally associate with the moon, whose silvery surface reflects the light of other bodies. The mean distance between the earth and sun is four times 10,800 diameters of the moon. The Roman half-pace of 1.216512 feet divides 108,000,000 times into earth's mean circumference. The Hebrew calendar divides the hour into 1080 units, called *chalaki*, based on the number of breaths one is presumed to take in one hour. The number 108 appears in religious symbolism, such as the 108 beads in the Hindu or Buddhist rosary, 10,800 stanzas in the *Rigveda*, and 10,800 bricks in the Indian fire altar [Michell 1988, 180-181].[11]

The ideal city-state Magnesia, envisioned by Plato in the *Laws*, consists of 5040 individual allotments of land to be distributed among 5040 citizens [Plato 1961: *Laws* V, 737e, 1323]. In miles, 5040 is the combined radii of the earth and moon. The number 5040 is the product of 1 x 2 x 3 x 4 x 5 x 6 x 7 and contains sixty individual divisions. The number 7920, which is the mean diameter of the earth in miles, is the product of 8 x 9 x 10 x 11. Thus, the product of 5040 and 7920 is the product of the numbers 1 through 11 [Michell 1988, 109-110].

Michell observes this arrangement of astronomical measures in temple plans throughout history. St John's New Jerusalem, the celestial city described at the beginning of the Christian Era in the New Testament book of *Revelation*, is based on a square of 12 x 12 furlongs, containing a circle of circumference 14,400 cubits. This translates to a circle of 7920 feet in diameter and 24,883.2 feet in circumference, compared to the earth's diameter of 7920 miles and circumference of 24,883.2 miles. The perimeter of a square circumscribing the circle is 31,680 feet [Michell 1988, 24-25].

The New Jerusalem plan is an idealized vision of heaven and earth, but it can also be a house, temple, village, city, or entire world-order. Glastonbury, England, where some believe Druidic mysteries yielded to Christianity in the west, is associated with Arthurian legend. The original width of St. Mary's Chapel, built at Glastonbury Abbey, is 39.6 feet. Its footprint, a 1 x √3 rectangle, would be circumscribed by a circle of diameter 79.2 feet. The perimeter of the circumscribing square would be 316.8 feet [Michell 1988, 28-29].

Stonehenge, the megalithic monument in Salisbury, England, also reproduces the dimensions of St. John's city on a reduced scale of 1:100, when expressed in feet. Thus, the mean circumference of the outer circle of sarsen stones is 316.8 feet. The diameter of the inner ring of bluestones is 79.2 feet [Michell 1988, 31, 173].

Michell's geometric symbol contains additional layers of meaning, which may be accessed through "gematria," a term from medieval Kabbalah adopted from the Greek *geômetria* or "geometry" that associates the letters of Greek, Hebrew and other ancient alphabets with numerical values, musical tones and vibrations, colors, and geometric images [Liddell 1940, Simpson 1989]. In this way, numbers and measures convey musical, astronomical and mythological content. For example, by gematria, the Greek το αγιον πνευμα or *to hagion pneuma*, which means "the Earth Spirit," and το γαιον πνευμα or *to gaiôn pneuma,* which means "the Holy Spirit," each yield the number 1080.[12]

VII The Heptagon

Definitions:

The **regular polygon** is a plane figure in which all sides are equal and all interior angles are equal.

In a regular polygon with (*n*) sides, the **interior angle** is (180-360/*n*) degrees. The sum of the polygon's interior angles is (180*n*-360) degrees.

"**Polygon**" is via late Latin from the Greek *polugônos* (from *polu* "many" + *gônia* "corner, angle") and *polugonos*, which means "producing many at a birth, prolific" [Harper 2001, Liddell 1889, Liddell 1940].

"**Heptagon**" is from the Greek *heptagônos* (from *hepta* "seven" + *gônia* "corner, angle") [Harper 2001, Liddell 1940, Simpson 1989]. A regular heptagon contains seven equal sides that meet at seven equal interior angles of 128°34′17″ (128.57142…).

A regular heptagon cannot be constructed precisely with a compass and rule, but one approximate construction relates to the squaring the circle. Let us begin with the method for squaring the circle that is based on the Golden Section.

- Repeat figure 16, as shown.
- Locate points G and H where the small circle intersects the horizontal diameter.
- Locate points V and W where the large circle intersects the vertical diameter.
- Connect points V, H, W and G.

The result is a rhombus (fig. 22).

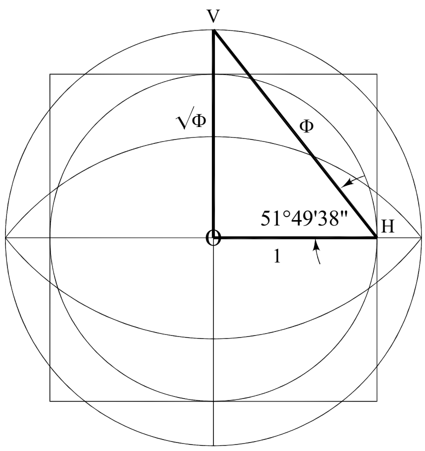


Fig. 16

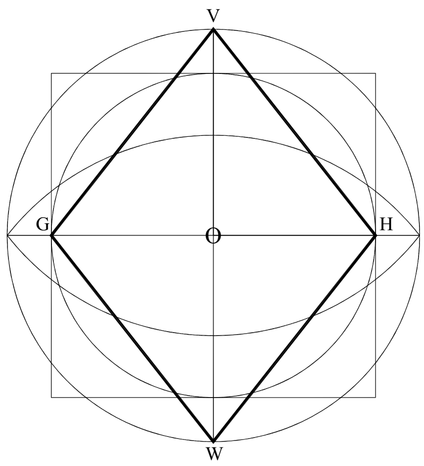


Fig. 22

Definition:

The **rhombus** is a four-sided figure whose side lengths are equal and whose opposite angles are equal. "Rhombus" is via the late Latin *rhombus* from the Greek *rhomboeidês* ("rhomboidal") and *rhombos*, which means "a spinning-top or wheel" [Liddell 1889, Liddell 1940].

- Place the compass point at G. Draw a circle of radius GO.
- Place the compass point at H. Draw a circle of radius HO (fig. 23).

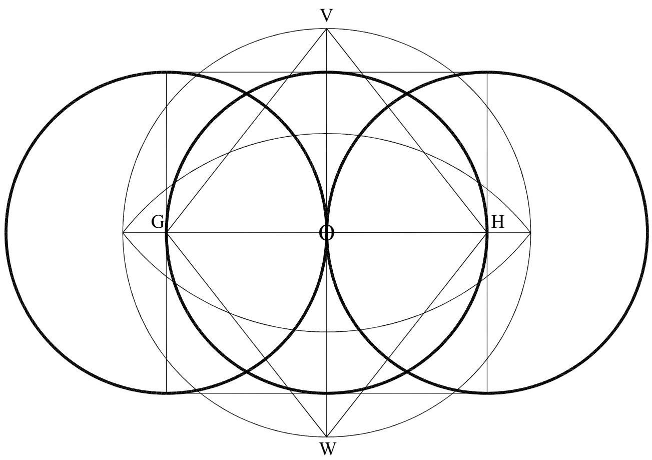


Fig. 23

- Locate point A where the line VH intersects the right circle, as shown.
- Locate point B where the line WH intersects the right circle, as shown.
- Connect points OA and OB (fig. 24).

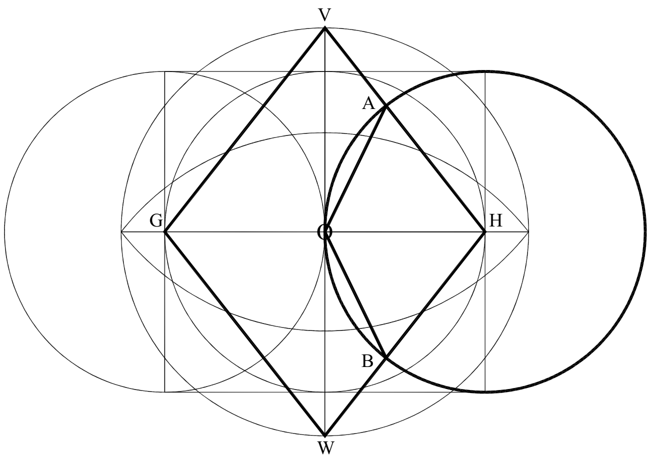


Fig. 24

Lines OA and OB locate two approximate sides of a regular heptagon inscribed within the right circle.

- Place the compass point at A. Draw an arc of radius AO that intersects the right circle at point C, as shown.
- Place the compass point at B. Draw an arc of radius BO that intersects the right circle at point D, as shown.
- Place the compass point at C. Draw an arc of radius CA that intersects the right circle at point E, as shown.
- Place the compass point at D. Draw an arc of radius DB that intersects the right circle at point F, as shown.
- Connect points A, C, E, F, D, B and O.

The result is a heptagon that approximates a precise regular heptagon (fig. 25).[13]

- Repeat the process within the left circle, as shown.

Angle AOB equals 128°10′22″ (128.17277…). The interior angles of a true regular heptagon equal 128°34′17″ (128.57142…).

The constructed heptagon approximates a true heptagon within 0.3% (fig. 26).

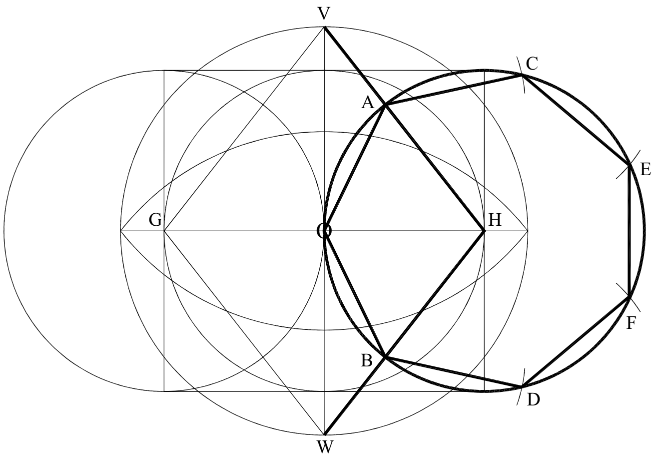


Fig. 25

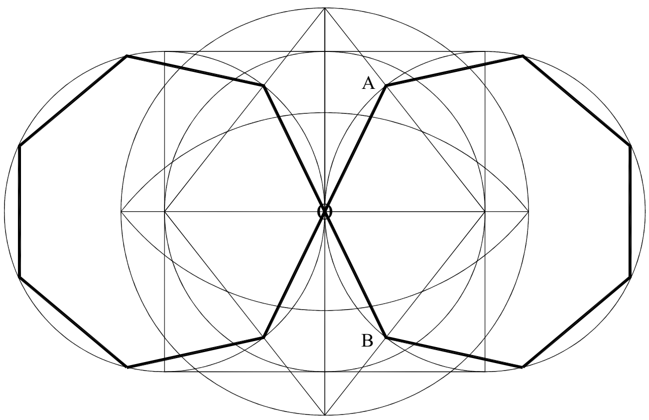
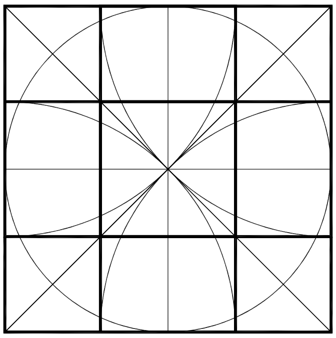


Fig. 26

VIII Brunés Sacred Cut

In a previous column, we examined the "sacred cut," so named by Tons Brunés for its ability to generate a circle and square of nearly equal perimeters and to divide the side of a square into seven nearly equal parts. The square grid contains a center square, four smaller corner squares, and four 1 : √2 rectangles (fig. 27).[14]



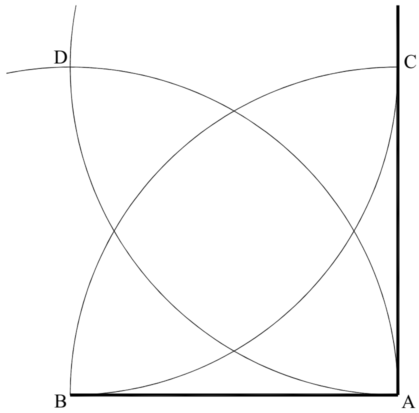


Fig. 27

Fig. 28

Brunés' technique for squaring the circle is based on the observation that a quarter-arc drawn on half the diagonal of a square and the diagonal of half the square are equal in length within 0.6% [Brunés 1967: I, 73-74, 93-94; Watts 1987, 268-269; Watts 1992, 309-310].

- Draw a horizontal baseline AB equal in length to one unit.
- From point A, draw an indefinite line perpendicular to line AB that is slightly longer in length.
- Place the compass point at A. Draw a quarter-arc of radius AB that intersects the line AB at point B and the open-ended line at point C.
- Place the compass point at B. Draw a quarter-arc (or one slightly longer) of the same radius, as shown.
- Place the compass point at C. Draw a quarter-arc (or one slightly longer) of the same radius, as shown.
- Locate point D, where the two quarter-arcs taken from points B and C intersect.
- Place the compass point at D. Draw a quarter-arc of the same radius that intersects the line AC at point C and the line AB at point B (fig. 28).
- Connect points A, B, D and C.

The result is a square (ABDC) of side 1.

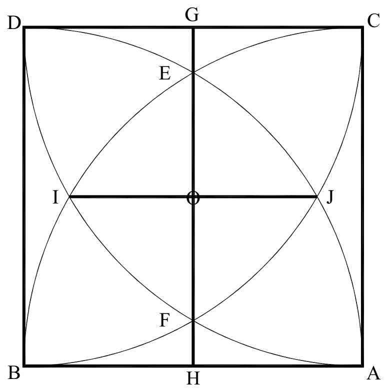
- Locate points E and F where the quarter-arcs intersect above and below, as shown.
- Draw the line EF.

- Extend the line EF in both directions to points G and H on the square.

Line GH divides the square (ABDC) in half.

- Locate points I and J where the quarter-arcs intersect on the left and right, as shown.
- Draw the line IJ.
- Locate the point O where the lines GH and IJ intersect.

Point O marks the midpoint of the line GH and the center of the square (fig. 29).



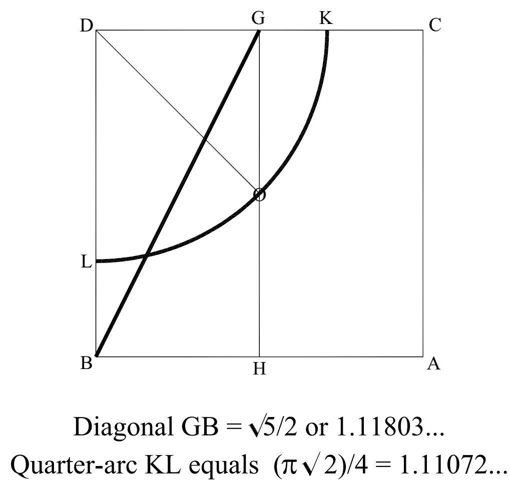


Fig. 29

Fig. 30

- Remove the four quarter-arcs.
- Mark the location of point O and remove the line IJ.
- Draw the semi-diagonal GB.
- Locate points D and O.
- Draw the line DO.
- Place the compass point at D. Draw a quarter-arc of radius DO that intersects the line DC at point K and the line DB at point L.

If the side (AB) of the square is 1, the diagonal (GB) equals $\sqrt{5}/2$, or 1.11803... .

The radius DO equals $1/\sqrt{2}$ or $\sqrt{2}/2$.[15]

If π equals 3.14159, the quarter-arc (KL) drawn on radius DO equals $(\pi\sqrt{2})/4$ or 1.11072….

The diagonal (GB) and the quarter-arc (KL) are equal in length within 0.6% (fig. 30).

Diagonal GB = $\sqrt{5}/2$ or 1.11803...

Quarter-arc KL equals $(\pi\sqrt{2})/4$= 1.11072...

Figure 31 presents a square (ABDC) of side AB, a square of side GB, and a circle of radius OD.

If the side (AB) of the square ABDC is 1, the perimeter of the square of side GB equals 4.47213… and the circumference of the circle of radius OD equals 4.44288….

The circumference of the circle and the perimeter of the square (of side GB) are equal in length within 0.6% (fig. 31).

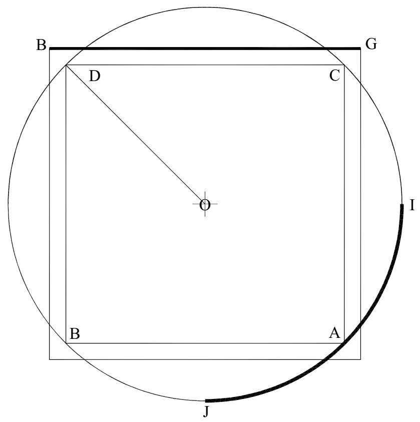


Fig. 31

IX Leonardo's Vitruvian Man

In 1490, Leonardo da Vinci produced the illustration we know today as "Vitruvian Man." The study depicts the set of ideal human proportions proposed by Vitruvius in *De architectura* (*Ten Books on Architecture*), in which the adult male figure is proportioned to a circle and a square. Neither Vitruvius nor Leonardo propose a circle and a square of equal measure, but in Leonardo's interpretation the two are superimposed, to dramatic effect.[16]

The Vitruvian canon of human proportion is well known:

> The center and midpoint of the human body is, naturally, the navel. For if a person is imagined lying back with outstretched arms and feet within a circle whose center is at the navel, the fingers and toes will trace the circumference of this circle as they move about. But to whatever extent a circular scheme may be present in the body, a square design may also be discerned there. For if we measure from the soles of the feet to the crown of the head, and this measurement is compared with that of the outstretched hands, one discovers that this breadth equals the height, just as in areas which have been squared off by use of the set square [Vitruvius 1999: III, i, 47].

Following Vitruvius, Leonardo locates the navel of the human figure at the center of a circle whose circumference bounds the figure's outstretched arms and legs. The figure's total height and arm span are equal in length and measure the edge lengths of a square. The center of the square locates the genitals. Quarter divisions locate the nipples, the base of the knees, the junction of forearm and upper arm, and the width of the shoulders. An eighth division locates the bottom of the chin. When the arms are raised in line with the top of the head, the middle fingers indicate where the circle and square intersect (fig 32).[17]

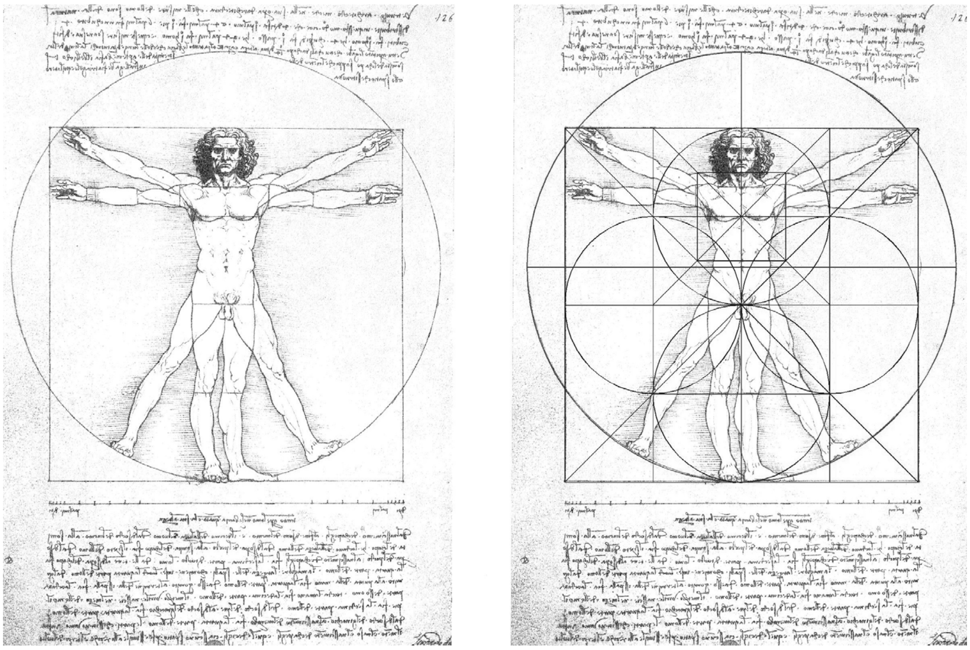


Fig.32. Image: Canon of proportions from Vitruvius's *De architectura* (*Ten Books on Architecture*). Leonardo da Vinci, c. 1490. Venice: Accademia. Geometric overlay: Rachel Fletcher.

The intended result is a harmony of individual parts and the whole:

> And so, if Nature has composed the human body so that in its proportions the separate individual elements answer to the total form, then the ancients seem to have had reason to decide that bringing their creations to full completion likewise required a correspondence between the measure of individual elements and the appearance of the work as a whole [Vitruvius 1999: III, i, 47].

The Vitruvian scheme may be applied to habitats and dwellings of every kind—from houses, temples and cities to the cosmos itself—and is one of several attempts throughout history to express human proportions in precise geometric terms.[18] Whether the results are exact, such efforts reflect a basic human need to perceive a coherent, harmonious world. If geometry is the art of reconciling diverse spatial elements, the quest to square the circle is an art of the highest order.

Notes

1. [See Fletcher 2005b, 44]. Thales of Miletus is said to have learned this technique in Egypt [Padovan 1999, 60-61].
2. J. J. O'Connor and E. F. Roberson provide a concise history, beginning with the Egyptian Rhind papyrus that was scribed by Ahmes and was based on an original dating from 1850 B.C. or earlier. A square nearly equal in area to that of a circle is accomplished when the square is constructed on 8/9 of the circle's diameter [O'Connor and Roberson 1999; van der Waerden 1983, 170-172].

   To E. W. Hobson and others, "squaring the circle" is the name for circles and squares of equal areas and is also known as a circle's quadrature. Hobson's term for circles and squares of equal perimeters is "rectifying the circle." [Hobson 1913, 4-5]. Occasionally, the *ad quadratum* construction, in which a circle is inscribed within a square, or a square within a circle, is named "squaring the circle," even though this does not produce figures of equal measure. [See Fletcher 2005b, 45-49.]
3. In some ways, the quest to square the circle parallels the history of π, which Hobson traces through three historical periods. The first "geometric period" from prehistory through the sixteenth century consisted of producing approximate values for π from geometric constructions. One method attributed to Socrates' contemporary Antiphon begins with a square inscribed in a circle. The sides of the square are bisected, then those of the octagon that results, then the 16agon and so on, until the polygon is indistinguishable from a circle. Socrates' contemporary Bryson improved on this method by considering circumscribed and inscribed polygons together. Another method derives from a theorem by Archimedes (287-212 B.C.), which states that the area of a circle equals the area of a right-triangle whose short side equals the radius of the circle and whose long side equals the circumference [Hobson 1913, 10-11, 15-19; Smith 1958, 302-307].

   Hobson's second "analytical" period, from the mid-seventeenth century, applied analytical processes, specifically trigonometric functions, to solve the problem of π. Not until the third period, from the mid-eighteenth until the late nineteenth century, was π shown to be truly irrational or transcendental [Hobson 1913, 12-13, 43-57].
4. See [Fletcher 2004, 95-96] for more on the circle and [Fletcher 2005b, 35-37] for more on the square.
5. [1969, xxxi]. See [Fletcher 2004, 96] for more on the vesica piscis.
6. [1982, 74-76.] See [Fletcher 2006, 67] for more on the Golden Ratio.
7. See [Fletcher 2005a, 151-153], to derive a regular pentagon from this construction.
8. Note this construction is based on fact that $\sqrt{\phi}$ and $4/\pi$ are nearly exact.

   $$\sqrt{\phi} = 1.272019\ldots$$
   $$4/\pi = 1.273239\ldots$$

   Put another way,

   $$\pi = 3.14159\ldots.$$
   $$4/\sqrt{\phi} = 3.144605\ldots$$
9. To construct the isosceles triangle, draw a vesica piscis from two circles, as shown, such that the center of one circle coincides with the circumference of the other. Next, draw a larger vesica piscis whose short axis equals the width of the two circles. Inscribe a rhombus within the larger vesica piscis. The height of the isosceles triangle equals the long axis of the smaller vesica piscis. The base of the isosceles triangle passes through two points where the rhombus and the small circles intersect.
10. Modern estimates for astronomical distances vary, but their averages are nearly identical to Michell's figures, whose calculations utilize two approximate values of π: the Archimedean value of 22/7 (= 3.14285…) and Fibonacci's approximation of 864/275 (= 3.14181…) [Beckmann, 1971, 66,84]. Michell's measure for the mean radius of the earth is based on a value of π equal to 864/275. For a full account see [Michell 1988, 100-106].

11. Ancient Greek and Hebrew numbering systems do not recognize zero or "0." Therefore, the numbers 108, 1080 and 10,800, although different in quantity, share the same qualitative value [Bond 1977, 6].
12. [Michell 1988, 181.] The individual letters in ΤΟ ΑΓΙΟΝ ΠΝΕΥΜΑ (το αγιον πνευμα), "the Holy Spirit" are: [(300)Τ + (70)Ο] + [(1)Α + (3)Γ + (10)Ι + (70)Ο + (50)Ν] + [(80)Π + (50)Ν + (5)Ε + (400)Υ + (40)Μ + (1)Α]. These add to (370 + 134 + 576) and distill to 1080. The individual letters in ΤΟ ΓΑΙΟΝ ΠΝΕΥΜΑ (το γαιον πνευμα), "the Earth Spirit" are [(300)Τ + (70)Ο] + [(3)Γ + (1)Α + (10)Ι + (70)Ο + (50)Ν] + [(80)Π + (50)Ν + (5)Ε + (400)Υ + (40)Μ + (1)Α]. These add to (370 + 134 + 576) and distill to 1080 [Bond 1977, 6].
13. Side EF is slightly shorter than the others.
14. To derive fig. 26, see [Fletcher 2005b, 56-61].
15. The calculation of the diagonal GB is based on the Pythagorean theorem, such that $BH^2 + HG^2 = GB^2$ $[(1/2)^2 + 1^2 = (5/4)^2]$. Thus, GB = $\sqrt{5}/2$. The calculation of DO is based on the fact that the diagonal of a square of side 1 is equal to $\sqrt{2}$. See [Fletcher 2005b, 44-45] for more on the Pythagorean theorem.
16. Lionel March reconciles the circle and square in the Vitruvian figure through a regular octagon whose base equals the base of the square, and whose half-chord equals the diameter of the circle. The margin of error is approximately 0.57%. If the base of the square and the base of the octagon share the exact location, the center of the octagon and the circumference of the circle nearly coincide. Robert Lawlor proposes a less precise interpretation based on the Golden Section [Lawlor 1982, 59; March 1998, 106-108].
17. Leonardo notes these and other alignments in *The Theory of the Art of Painting*. In addition, he says, "If you open your legs so much as to decrease your height 1/14 and spread and raise your arms till [*sic*] your middle fingers touch the level of the top of your head you must know that the center of the outspread limbs will be in the navel and the space between the legs will be an equilateral triangle" [Richter 1970, I, 182]. March observes the equilateral triangle in an analysis of his own [March 1998, 107].

    For this analysis, the source image of Vitruvian Man was manipulated to correct for distortion in aspect ratio, possibly the result of irregular paper shrinkage. The manipulated image presents a true circle.
18. See [Fletcher 2006, 83-84] for more on human proportions.

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Rachel Fletcher is a theatre designer and geometer living in Massachusetts, with degrees from Hofstra University, SUNY Albany and Humboldt State University. She is the creator/curator of two museum exhibits on geometry, "Infinite Measure" and "Design By Nature". She is the co-curator of the exhibit "Harmony by Design: The Golden Mean" and author of its exhibition catalog. In conjunction with these exhibits, which have traveled to Chicago, Washington, and New York, she teaches geometry and proportion to design practitioners. She is an adjunct professor at the New York School of Interior Design. Her essays have appeared in numerous books and journals, including "Design Spirit", "Parabola", and "The Power of Place". She is the founding director of Housatonic River Walk in Great Barrington, Massachusetts, and is currently directing the creation of an African American Heritage Trail in the Upper Housatonic Valley of Connecticut and Massachusetts.