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Teaching Geometry to Artists

Jack Rees discusses his experience teaching geometry to artists. The aim is to introduce scientific ideas to arts students through the visualizations that are such an important part of discourse in science. Described are the intellectual context, define selected concepts using geometry and introduce elementary mathematical formulae—all relying on graphic visualizations to make fundamental ideas clear. The goal is to provide a means by which visually sophisticated persons may think *with* geometry *about* culture

But it should always be insisted mathematical subject is not to be considered exhausted until it has become intuitively evident...

Felix Klein [1893:243]

1 Introduction

It is my privilege to teach geometry to artists. Until recently, I have offered one elective per semester to undergraduates through the liberal arts department of a small art institute. The courses count as science distribution requirements for a bachelor of fine arts degree. Students are drawn from schools of painting, sculpture, ceramics, textiles, illustration, and new media. My best students are bright, which is to say open to being influenced; tenacious, which is to say requiring clear explanations; and tough-minded, which is to say they will not be patronized. Five students like this in a class of twenty is a joy. I had such a class the last semester I taught.

The courses I teach are all designed to introduce scientific ideas to arts students through the visualizations that are such an important part of discourse in science. I describe the intellectual context, define selected concepts using geometry (classically, a liberal art) and introduce elementary mathematical formulae--all relying on graphic visualizations to make fundamental ideas clear. My goal is to provide a means by which visually sophisticated persons may think *with* geometry *about* culture. On good days I am a storyteller in the history of ideas.

The following paper is part report, part methodological speculation on a class offered during the fall semester of 2003. The title of the course, *Advanced geometry from an elementary standpoint: Topology*, is a play on words, a variation on the title of a famous work by Felix Klein.[1] The title announces both my indebtedness to Klein and the content of the class.

Topology is the geometry of continuity, the last in a series of geometries whose definitions of equivalence become progressively more difficult to describe to students with little formal mathematical education. Topology, conventionally rendered as "rubber sheet geometry", is the geometry of stretching, squeezing, or extruding but not of cutting, folding or tearing *as long as neighboring points remain neighboring points* [Huggett and Jordan 2001]. This course is designed specifically for graphically sophisticated students in the arts[2] and is intended, in the main, to introduce geometry as a discipline of great visual and intellectual beauty. (It helps that we can visit the rare book room of The Linda Hall Library of Science and handle a dozen antique books renowned for their scientific and artistic significance; see http://www.lhl.lib.mo.us.) In class, graphic visualizations and geometrical demonstrations (mostly) take the place of a postulational, or, if you will, axiomatic, presentation. In the end it is hoped that students will unite intellectual inquiry and artistic endeavor according to their own interests.

This essay offers samples of class content highlighting the visual approach in sections 2 and 3. Section 4 details my assumptions about teaching geometry, by which I mean "things being tested in the classroom," and a course outline. Section 5 records some observations based on my experience teaching over the last seven years. Section 6 returns in detail to the content of the topology class.

2 Felix Klein's geometry schema

The nineteenth century in mathematics was named by historian Carl Boyer the "heroic age in geometry" [Boyer 1968: 572-595]. Among the giants of that age stand Felix Klein (1849-1925), who is often praised for his magisterial grasp of the whole of mathematics. This is faint praise among mathematicians. Here is how Constance Reid, author of *Hilbert*, paraphrasing Richard Courant (who organized Klein's final papers) puts it:

> And yet Klein's life had not been without its inner tragedy. The power of synthesis had been granted to him to an extraordinary degree. The other great mathematical power of analysis had been to a certain extent withheld. His ability to bring together the most distant, abstract parts of mathematics had been remarkable, but the sense for the formulation of an individual problem and the absorption in it had been lacking. ... Certainty he had perceived 'that his most splendid scientific creations were fundamentally gigantic sketches, the completion of which he had to leave to other hands' [Reid 1970: 178-179].[3]

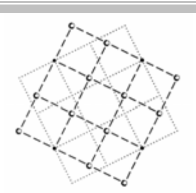
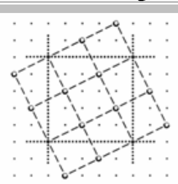
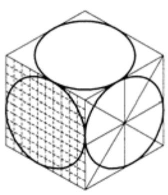
What makes for a great mathematician may not be exclusive of what makes for a great teacher and Klein was, by all accounts, a great teacher. He wielded considerable influence over one of the great mathematical schools of the late nineteenth and early twentieth centuries, the University at Göttingen. He established a research center there that was, for a time, a focus of the mathematical universe, attracting David Hilbert from Königsberg. During his tenure the student body included Hermann Weyl, Richard Courant, and Max Born. The first woman *D.Phil.*, Grace Chisholm Young, graduated in mathematics from a German university, graduated under his auspices.

It is perhaps telling that Klein regarded as his most notable achievement the unification of geometry in what is widely known as his *Erlangen Programm* of 1872 [Klein 1893].[4] Based on the concept of a mapping, Klein showed how the geometries of his age (metrical, projective, line) could be joined into a single geometry using the theory of groups.

A group is a set of elements filtered through an operation. To be a group the elements and their mapping must be closed, associative, contain an identity element and have an inverse. For instance, the integers are a group with respect to addition. The integers are closed since an integer plus an integer is always another integer; associative because (a+b)+c = a+(b+c); they have an identity element—zero; and the inverse of an integer is its negative. Therefore; the integers are said to be mapped onto themselves. Pregnant with promise, the theory of groups unified geometry, unified discrete and continuous mathematics and forecast new approaches in algebra and number theory.

Following Klein's lead, filtered through Lord and Wilson [1968] I take the logical progression of geometric groups to be: congruent, similar, affine, projective, inversive, differential, and topological.[5] This is a logical arrangement because the operations at the core of each group are progressive. In other words congruence is a special case of similarity, is a special case of affinity. etc. Each geometry is simpler than the one that comes before. *Simpler* means that "within the hierarchy of possible geometries, affine structure is more primitive than Euclidean structure because it is based on a smaller set of underlying assumptions, and is therefore invariant over a larger set of possible transformations" [Todd 2001: 195]. What is specifically true for affine in the context of Euclidean geometry is generally true of topology in which there are the fewest restrictions on what constitutes a legal mapping.

Fig. 1 proposes emblems for each geometric group. The emblems, besides being simple place holders, store information about the nature of each geometry. Far less arbitrary than icons, the emblems must be associated with some conceptual content to be effective. A fragment of the content for each group is presented in the accompanying Glossary of the geometries. (Please make every effort to synthesize the graphic descriptions and the textual descriptions in what follows. The intelligence of the material and the efficacy of the method depends on it.)

 *Euclidean congruence*	*Congruence* is the geometry of Euclid whose fundamental theorem is named after Pythagoras. Taking the Pythagorean theorem as an axiom, modern geometers prove the invariance of distance and angle for all rigid motions [Kreamer 1982: 414]. The group of rigid motions include identity, translation, rotation and reflection.[6]
 Euclidean similarity	*Similarity* is the geometry of Euclid concerned with scale, or more accurately: similarity as a transformation that relaxes distance and preserves angle [Forder 1962]. Distance, no longer absolute, is expressed as a ratio of lengths designated "k." Angles remain invariant. Congruence is a special case of similarity where k=1.
 *Affine*	*Affine* is the geometry of Galileo [Yagolom 1979] and, in architectural studios, is known as axonometric projection. This is where figures are stretched (and/or compressed) along parallel lines (area may or may not be preserved). Angle, as a property of equivalent figures, is relaxed. The proportions of angles between figure and transformed figure are expressed by a ratio designated "k." Similarity is a special case of affine geometry where k=1.

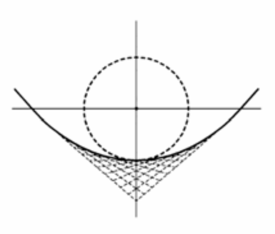
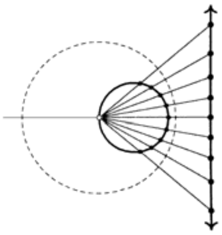
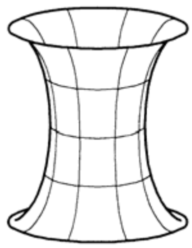
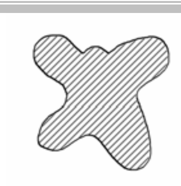
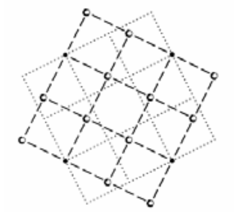
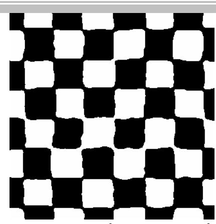
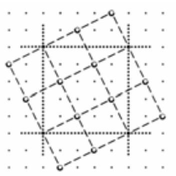
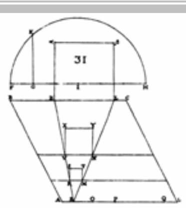
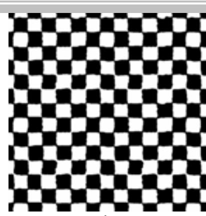
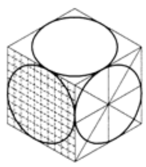
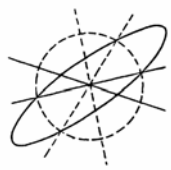
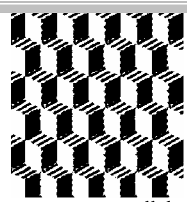
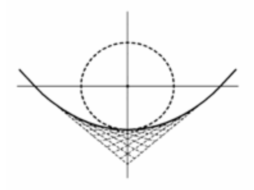
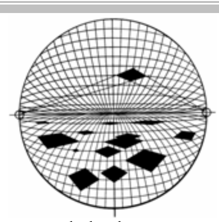
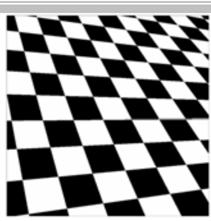
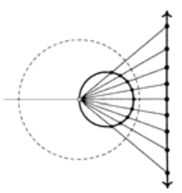
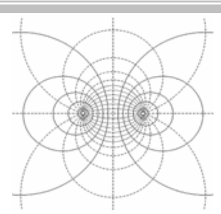
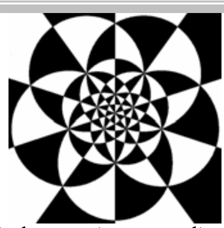
 *Projective*	Informally, *projective* geometry is perspective; a geometry which unifies a figure and its shadows [Ivins 1964]. Strictly speaking, however; all perspectivities are projective but not all projectivities are perspectival. In other words projective geometry includes anamorphosis and most map projections, not just lines converging in an optic field. Affine geometry is a special case of projective geometry, where the vanishing point[7] is located at infinity.
 Inversive	*Inversive* geometry is the first non-Euclidean geometry in the sense that it violates Euclid's assumption that parallel lines never meet.[8] Inversive geometry shows the way the interior of a circle is symmetrical to its exterior [Ogilvy 1969]. In such a transformation "line" no longer means "a straight line" because lines through the center of the circle of inversion map to circles. A mapping of every point inside to every point outside a given circle relaxes collinearity (straightness) and preserves angular relations.
 *Differential*	*Differential* geometry studies surfaces according to their divergence from a tangent plane located at a given point of the surface. These surfaces are said to be curved either positively or negatively depending on if the surface is some variety of cup (positive) or some variety of saddle (negative). Surfaces of zero curvature are (flat) planes. Curvature is quantifiable as an intrinsic relation between geodesics (lines of shortest distance) on small patches of a surface [Lanczos 1965]. Such ideas lead to logically consistent anti-Euclidean[9] geometries.
 *Topological*	*Topology* is the geometry of continuity—perfect elasticity—which preserves only connectedness in a transformation and its inverse [Huggett and Jordan 2001]. Topology studies the most general properties of a figure where there are the greatest number of parameters and the fewest number of invariants.[10]

Fig. 1. Klein's geometry schema in emblems and a glossary of the geometries

3 The array: organized logically

The array is a "graphic essay." No less a definition of geometries than the discourse above, the array presents different cuts at related information, linked through emblems. (Please take a moment to regard the array, fig. 2.)

Emblem	Characterizing Transformation	Transformation Of The Square [In Field]
 Euclidean congruence	 Rigid motions preserve length	 Squares remain squares in a different orientation
 Euclidean similarity	 Shape preserved through scaling	 Squares remain squares of a different size
 Affine	 Parallel lines are preserved	 Squares remain parallelograms
 Projective	 Figure and shadows preserved	 Squares remain quadrilaterals
 Inversive	 Orthagonality is preserved	 Circles remain perpendicular

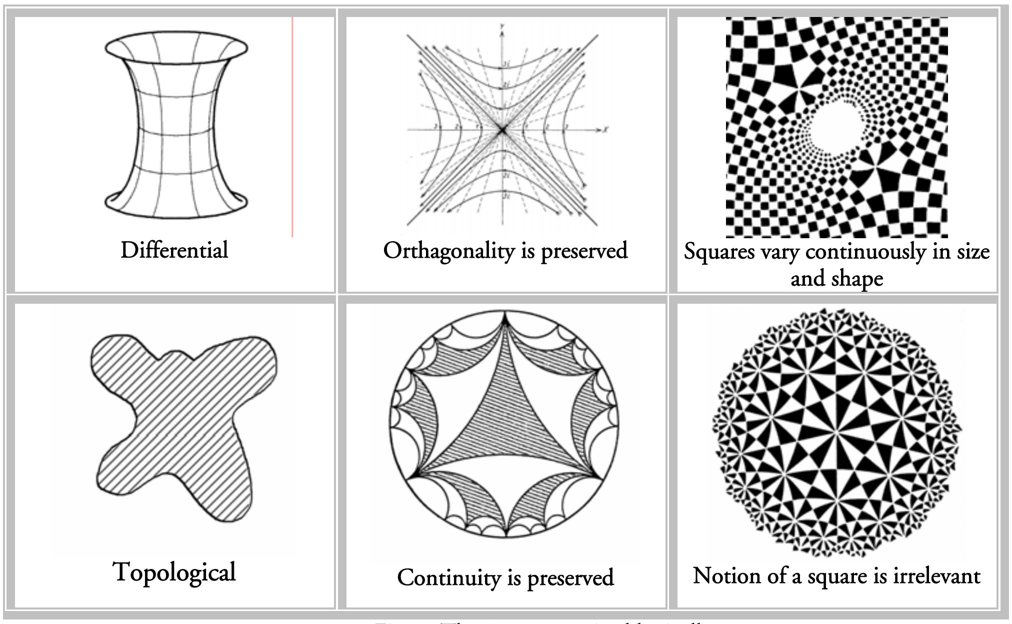
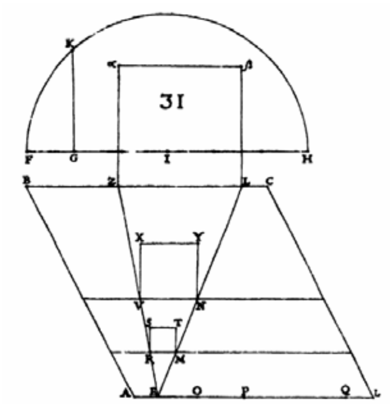


Fig. 2. The Array organized logically

The array demonstrates how the emblems are used to structure the presentation of information and how class content is delivered in a memorable order. The column labelled "characterizing transformation" is intended—working hand in glove with the emblems—to elucidate the nature of transformation in each geometry. For instance, similarity is described using the idea of scale, a concept with which students are (already) well acquainted. I use the emblem to show that similarity contains congruence as a special case (note the rotation and scaling of the dashed square relative to the dotted square). I use the "scaling" graphic from an historic source (Scheiner's *Pantographice* (Rome, 1631) which the students get to inspect first hand at The Linda Hall Library) to demonstrate what property remains invariant (shape or more accurately angle) and what property is relaxed (length).



From: Christoph Scheiner, *Pantographice,* Rome, 1631.

The column "transformation of the square" is an attempt to engage the students' considerable patterning skills. A square chessboard is presented as a gauge figure and "deformed" in a way that is consistent with the rules of transformation for each geometry. These patterns are intended to be evocative, rather than rigorously mathematical, a shameless appeal to students' design sensibilities.

The important idea regarding the array is that Klein's logical progression of geometries can be elaborated in any of a number of ways, depending on what needs to be presented in order to clarify and extend geometrical and scientific concepts. Other columns of information that are included in extended versions of the array are: fundamental theorems, analytical expressions, associated geometers, representative transformations and/or optical analogs. This is to name only a few of the possible themes that may be included in a class.

4 Course outline for "Advanced Geometry from an Elementary Standpoint: Topology"

Up to this point the presentation of class material has been motivated by descriptive exposition of the individual geometries. It is in the outline that one can evaluate just how the course opens out. (Please take a moment to peruse the outline directly below.) Note that Klein's schema still structures the presentation and that the "related ideas" (RI) subhead allows the introduction of topics that may be only tangentially linked to a particular geometry but, that are critical to the exposition of geometrical ideas. In this way the logical progression of geometries may become the backbone for a variety of science related subjects. The key is in how one relates ideas that are not geometry (strictly speaking) to the progression of geometric transformations. In this class "related ideas" are useful as advance preparation for information to come. For instance, introducing Gödel's incompleteness theorem in the context of a discussion about real numbers, is preliminary to defining *groups* and illustrating *closure*. This may seem to be in the wrong order except that "this proposition is true but unprovable" and the analogy of Gödel's infinite regress to the infinite regress of real numbers provides the students a paradox with which they are comfortable. Clearly, they do not have the tools to understand the theorem on a technical level (which they appreciate) and it makes them hungry for some of the technical detail.

Sometimes related ideas can be about the rules behind the rules. Under congruent geometry, I develop symmetry as a related idea. One way to define symmetry is through a demonstration of proper (identity, translation and rotation) and improper (reflection[11]) rigid motions. This approach is always effective because students can visualize the processes that lead to a superimposition of figures and thereby strengthen their geometric intuition. This Euclidean notion of symmetry (I), however, is not very robust. Therefore, it is important to present symmetry (II) more abstractly, as one of the three conditions that has to be met in order for there to be an equivalence relation between sets. Equivalence relations require sets to be *reflexive* (a set D must be equal to itself, D=D), *symmetric* (if transformation t maps E' to E" then t' maps E" to E') and *transitive* (if A is congruent to B and B is congruent to C then C must be congruent to A). Linking the superimposition of figures (symmetry I) to the idea that the mapping must be reversible (symmetry II) is one instance of the way concepts are "grounded" in geometry. It is an example of what makes geometry so beautiful—the evolution of ideas towards their simplest, oftentimes most abstract expression.

Finally, related ideas allow themes to be developed over the course of a semester. The theme developed in the topology class was infinity. The infinity of points on a line; the twin infinities of the very small and the very large; the role of infinity in the development of projective geometry; infinity as a point in the complex number plane; the infinity of a figure that is bounded but not closed, etc. Related ideas add density to the course. More importantly the model of 1) a taxonomy of geometries providing the structure and 2) a stew of related ideas providing the variety, is adaptable to a cluster of science courses. For instance a class with a kinematic emphasis would present physical concepts as related ideas. As a bonus, students who elect two different classes may begin to appreciate geometry as the "language of physics" and (possibly) begin to compare that to the role geometry plays in art.

Course outline

I. Preliminaries (note: RI = Related Idea)

0. Definition of topology as the geometry of continuity
1. The real numbers (RI: Gödel's incompleteness theorem)
2. Definition of Geometry (RI: Definition of transformation)

II. Klein's Schema (see figs. 1 and 2)

3. Congruence (RI: Symmetry)
4. Similarity (RI: The concept of a group)
5. Affine Geometry (RI: Conic sections)
6. Projective Geometry (RI: Interlude devoted to William Ivins, *Art & Geometry*)
7. Inversive Geometry (RI: The complex number plane)
8. Differential Geometry (RI: Intrinsic geometry)
9. Topology as Homomorphism (RI: Euler Formula, Orientability and Metric Spaces; Cut Points, Components, Compliments, and Closure)

5 Assumptions

a. Audience. I believe there is a substantial audience of artists curious about science, non-scientists interested in science and proto-scientists whose interest is yet to bloom, to continue developing this program. The subject is intrinsically interesting if only we can capture those who are not disposed towards analytical methods. I assume further, we need to develop science and geometry courses that teach differently those who are to be trained and those who are to be fascinated. Presented correctly, the material itself will do the enticing.

b. Mnemonic. I assume that a mnemonic association of images and concepts in a structured hierarchy fosters assimilation of:

1. strange, often counter-intuitive ideas;
2. the ideological, historical and disciplinary context of the information;
3. their unexpected, myriad relations.

The approach is spatially organized, graphically demonstrated, as technically accurate as the audience allows, conceptually sophisticated and flexible.

c. Methodological. I assume the presentation of content may be adjusted to fit the audience without "dumbing down" the material. The method triangulates among *graphical*, *technical* and *synthetic* information.

1. *Graphical* information leads the presentation and treats geometry as a species of visual art. There is a great deal of evidence concerning a basic human competence that might be described as using images to think *with* (rather than merely *about*). That this capability is disrespected in the academy is scandalous. As Barbara Maria Stafford has written:

> In the widespread postmodern denigration of the aesthetic, what is forgotten is that from Leibniz to Schiller, the term connoted the integration of mental activity with feeling.

> *Aisthesis*, as perception or sensation, has in post-Cartesian and especially post-Kantian thought become separated from cognition. Rediscovering its pragmatic capacity to bridge experience and rationality, emotion and logic, seems all the more important in the era of virtual reality and seemingly nonmediated media. The awareness that images can sustain the continuity of thinking, not merely serve as fictionalizing counterfeits or pseudo-intellectual goods, brings both an ethical and aesthetic dimension to the computer age [Stafford 1996:52].[12]

2. The *technical* detail may be as elaborate or as elemental as outside factors allow and I think it is important to present as much geometrical detail as possible. During the course of the semester I was able to rigorously define continuity, equivalence, closure, group, and homeomorphism based on less thorough definitions of transformation, symmetry, infinity, cross ratio, curvature, function and I suspect what held the students' interest was the unfolding story. It is a narrative[13] in which many of the details were only glimpsed yet; are we as teachers measured by the questions we inspire as well as the facts we impart?
3. *Synthetic* in the technical geometrical definition means deduction: building a system proposition by proposition from general principles. Klein's system is synthetic in this way, (even though others filled in the details). In this sense Klein's work exists squarely in the grand tradition of Euclid. I mean synthetic in a slightly different sense, as "combining ideas so as to form a whole that is greater than the sum of its parts."[14] Klein's schema is also synthetic in this sense. Big ideas in science and mathematics are synthetic in that one gets to do more with less.[15]

To be sure, the approach to teaching geometry herein described is synthetic for students in so far as they can appreciate the connective tissue unifying geometry, but it may be more than that. On good days I see flashes of insight that joins geometry and physics and, every now and again, a glimmer that promises an implementation of geometric techniques and scientific ideas in their own work. By this measure there remains more to be done in adapting advanced mathematical ideas for artists, and I derive inspiration from the students, who often turn out to be excellent teachers.

d. Voice. I think the voice in which geometry is presented is important. I favor the voice of revelation trading on geometry as an hermetic tradition, with one important caveat. The beauty in economy that proofs display, the elegant foundations and of the chain of logic, is mystery enough. There is no need to trade in some variety of Rosicrucian mysticism because the details of modern scientific geometry are as demanding, as hidden from untutored consideration and as full of wonder as any esoteric teaching.

6 Observations

I have three observations concerning my teaching experience so far.

1. Some minor yet significant percentage of art students want to know about matters geometrical. Of the thirteen classes I have taught only one was *not* over-subscribed. Partially this is because I teach trendy subjects like chaos theory and partly, I think, it is because there is a pent-up demand for science-related courses. Mathematicians sneer at the idea that mathematics could be a spectator sport but I think that too much is at stake to allow the professionals to have the last word.
2. The flexibility of the schema is most promising. I adapt it to a variety of geometry related courses. In *C is for Chaos* I draw on the schema to develop ideas of scaling, fields, symmetry and continuity in order to explain dynamical systems, sensitive dependence, irreversibility,

confinement, and periodicity. The Klein schema is tailor made for the course *Space from Aristotle to Einstein*. In that class I concentrate on the geometries from projective on in order to explain, at the end, how objects follow straight lines in curved space. A topics in "Western Thought" class, *Art, Science and Rhetoric*, dwells on the geometries from Euclid to Alberti, concluding with a detailed exposition of perspective: locating painting in a scientific tradition and statics in an artistic tradition. I have also taught *Paraline Drawing*, a studio class filtered through Euclidean and Galilean geometry disguised as the tools and techniques of drafting.

3. Finally, I think it important to acknowledge that not everything fits the schema. In another course, *Color from Aristotle to Newton*, the Klein schema is irrelevant. I harbor hopes that differential geometry might provide some techniques germane to the perception of color and that a class on color in psychology and physics could be founded on a geometrical exposition, but enough speculation. As evidence of the actual class content for *Advanced Geometry* the I present the final from that class, with and without answers. I think there is no better way to convey a sense of the course content than to see for what information the students were held accountable. Just for fun, take the test yourself!

To download a .pdf file of the final exam from *Advanced Geometry* with and without answers, go to http://www.nexusjournal.com/Rees.html

Notes

1. The tile *Elementary Mathematics from an Advanced Standpoint: Geometry*, was originally published as volume 2 of *Elementarmathematik: vom hoeheren Standpunkte aus* (Lepizig, 1909; 3rd ed. Berlin, 1924); translated into English in 1931 (from the 3rd edition), it has been reprinted by Dover Publications (2004).
2. The design for this course is a product of the process I went through (in fits and starts) to grasp simple mechanical concepts. Graced with a facility for geometry, I was often frustrated in my attempts to understand analytical physical expositions. Unwilling to give up the appreciation of statics for obvious reasons (I am an architect) I found the giants of physics often presented their insights in geometrical forms relatively easy to understand (**F**=m**a** is due to Euler not Newton). My experience suggests that if there is a reason (and a rational) to teach physics geometrically it may also be possible to teach geometry graphically. I am not proposing a reform of technical education for scientists or mathematicians, just a different emphasis, one that may play to the strengths of non-specialists more effectively.
3. The clear-eyed, no holds barred, appraisal of the work of mathematicians by other mathematicians has always delighted me. It stands very much in contradistinction to the relativistic discussion of art common in American schools and is often harsh even by architectural school standards of critique.
4. See the MacTutor History of Mathematics archive of the University of St. Andrews, Scotland, for an excellent biography of Klein and his intellectual accomplishments and mathematical context: (http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Klein.html).
5. This stratification of Klein's schema is by no means unproblematic. Many mathematicians omit congruence and similarity as discrete geometries, subsuming the whole of Euclidean geometry in affine. I do not favor this approach because it is important to ease students into the details of mappings using transformations with which they already have experience. Since I take pains to show how similarity is a special case of affinity, no harm is done.
   Other mathematicians exclude differential geometry from the schema altogether. I have never been sure exactly why, but presumably because it is a quantitative operation in a field of qualitative transformations. I profoundly disagree with this exclusion on pedagogical grounds. Klein specifically included metrical geometry in his schematization and because so much physics is founded upon differential geometry, we (teachers) desperately need ways to introduce a visualization students can use to help make sense of change over time, at an instant. The qualitative insights of Gauss (using the *Abstract* as a primary source) provide just such an opportunity.

6. This is well trod territory in science museum displays and I have never seen the whole schema played out with anything near the graphical sophistication accorded the rigid motions. I expected to find this kind of a detailed presentation, since most historians call attention to the central role of Klein's schema in teaching mathematics, yet I found such a schema only in [Lord and Wilson 1986] whose simple diagrams became touchstones. As my grasp of the geometries developed I took to revising the diagrams, often formalizing a sketch from my notes or re-drawing graphics from particularly helpful sources. The discipline required to construct an hyperbola or draw, with construction lines, a pair of inverse points, is an important part of the method here espoused. I often give drawing problems as homework and revise lectures according to students' progress measured by their drawings.
7. Vanishing point in this context refers to the point where converging lines intersect. Converging lines in perspective constructions are parallel. When the vanishing point is moved infinitely far away, lines that appear to converge in a finite field are said to intersect at infinity but they no longer converge and are therefore said to be parallel. As a consequence affine geometry is established as a special case of projective geometry. The necessity for this unexpected reformation of Euclidean geometry—any two lines, in a plane, always intersect (at an imaginary point if necessary)—has to do with the reformulation of geometric foundations by David Hilbert and with the introduction of homogeneous coordinates. However I often refer to it as a strategy to preserve the duality of lines and points.
8. I think it important to distinguish between strong and weak forms of non-Euclidean geometry. Inversive is weakly non-Euclidean because it shares every fundamental geometric characteristic but that of the fifth postulate. Strong non-Euclidean geometries violate the principle of rigid motion. (cf. [Hartshorne 1997]).
9. Gauss called non-Euclidean geometry anti-Euclidean [Gauss 1965], a usage I favor because curved spaces violate the spirit of Kant's *a priori* regarding Descartes's coordination of Euclid, by which I mean the automatic assumption of embeddedness.
10. I think it impossible to overestimate the importance of the tendency to ever greater generalization often evident in geometry. To express this idea as a gross generalization: in the humanities intellectual progress is often evident as the differentiation of ever narrower domains. Art history is divided into Ancient, Renaissance and Modern. Renaissance Art history is divided into Proto-, High-, Baroque and Mannerist. High-Renaissance is distinguished according to its Venetian and Florentine varients—and so it goes, ever narrower, ever more specialized. It seems that in mathematics there are (at least) more instances of major intellectual breakthroughs that unite discrete practices than in any other discipline. Klein's *Erlangen Programm* is such a breakthrough. Another example is the way projective geometry provides a unified treatment of circles, ellipses, parabolae and hyperoblae as conic sections. Another example, drawn from geometrical physics, is the way Newtonian relativity (itself a generalization of Galilean relativity) is a special case of a more general rule—Special relativity. "Synthetic," as I use the word later in the essay, is akin to this process.
11. Reflection is an improper rigid motion because it requires the figure to move outside of its plane. I make much of this distinction early on so that when discussing attitude transformations (in differential geometry) as translations and rotations only, it is clear that reflections are excluded because they change the handedness of the coordinate system.
12. By the way, I think the "Institute" (art education) has made the inverse error, ignoring "mental activity" and fixating the "feeling" component of aesthetics. It is my conviction that architectural education presents a "third way," combining intellection and emotion, aesthetic and scientific education in an effective synthesis.
13. I am careful in class to draw a distinction between "a" story and "the" story. There is no question that I am only telling one of many possible stories.
14. Herbert Simon refers to what I am calling synthetic as a "pragmatic" response to complexity:

    > ...by a complex system, I mean one made up of a large number of parts that interact in a nonsimple way. In such systems, the whole is more than the sum of the parts, not in an ultimate, metaphysical sense, but in the important pragmatic sense that, given the properties of the parts and the laws of their interaction, it is not a trivial matter to infer the properties of the whole. In the face of complexity, an in-principle reductionist may be at the same time a pragmatic holist [Simon 1962:468].

15. For example, before Klein developed the concept of a group, Euclidean and non-Euclidean geometry were treated as fundamentally different geometries. After he developed the group concept they can be treated as parts of a greater whole, the geometry of invariants. A large part of his stated motivation for the Erlangen Programm was, in fact, this unification of geometry.
    But it has seemed the more justifiable to publish connective observations of this kind because geometry, which is after all one in substance, has been broken up in the course of its recent rapid development into a series of almost distinct theories, which are advancing in comparative independence of each other [Klein 1893:216].

Bibliography

BOYER, Carl B. 1968. *A History of Mathematics*. New York: Wiley.

BRANNAN, D. A., Matthew F. ESPLEN, and Jeremy GRAY. *Geometry*. Cambridge: Cambridge University Press, 1999.

DOBLIN, Jay. 1958. *Perspective: A New System For Designers*. New York: Whitney Library of Design. (1st ed. 1956).

FORDER, Henry G. 1962. *Geometry: An Introduction*. 2nd ed. New York: Harper.

GAUSS, Carl Friedrich. 1965. Gauss's abstract of the *Disquisitiones Generales Circa Superficies Curvas* (1825). Pp. 45-49 in *General Investigations of Curved Surfaces of 1827 and 1825*. Hewlett, NY: Raven Press.

HARTSHORNE, Robin. 1997. *Companion to Euclid: a Course of Geometry Based on Euclid's Elements and its Modern Descendants*. Providence; Berkeley: American Mathematical Society.

HUGGETT, Stephen A. and David Jordan. 2001. *A Topological Aperitif*. London and New York: Springer.

IVINS, William Mills. 1964. *Art & Geometry: A Study in Space Intuitions* . New York: Dover Publications. (1st ed. 1946).

KLEIN, Felix. 1893. "A comparative review of recent researches in geometry." *Bulletin of the New York Mathematical Society* 2: 215-249.

———. 2004. *Elementary Mathematics from an Advanced Standpoint: Geometry*. New York: Dover (1st ed. 1909).

KLEIN, Felix and W. ROSEMANN. 1968. *Vorlesungen Uber Nicht-Euklidische Geometrie*. Berlin: Julius Springer. (1st ed. 1928.)

KRAMER, E. E. 1982. *The Nature and Growth of Modern Mathematics*. Princeton: Princeton University Press.

LANCZOS, Cornelius. 1965. *Albert Einstein and the Cosmic World Order*. New York: Interscience Publishers.

LEE, T.D. 1988. *Symmetries, Asymmetries and The World of Particles*. Seattle: University of Washington Press.

LORD, Eric A. and C. B. WILSON. 1986. *The Mathematical Description of Shape and Form*. New York: John Wiley and Sons. (1st ed. 1984).

MAXWELL, James Clerk. 1954. *A Treatise on Electricity and Magnetism*. New York: Dover. (1st ed. 1873.)

OGILVY, C. Stanley 1969. *Excursions in Geometry*. New York: Oxford University Press.

REID, Constance. 1970. *Hilbert*. New York: Springer-Verlag.

SCHEINER, Christoph. 1631. *Pantographice, seu Ars delineandi res quaslibet per parallelogrammum lineare seu cavum*. Rome: ex typographia Ludouici Grignani.

SIMON, Herbert A. 1962. "The architecture of complexity." *Proceedings of the American Philosophical Society* **106**: 467-482.

STAFFORD, Barbara Maria. 1996. *Good Looking: Essays on the Virtue of Images*. Cambridge, MA: MIT Press.

TODD, James T., Augustinus H. J. OOMES, Jan J. KOENDERINK and Astrid M. L. KAPPERS. 2001. "On the affine structure of perceptual space." *Psychological Science* 12: 191-196.

YAGLOM, I.M. 1979. *A Simple Non-Euclidean Geometry and its Physical Basis*. New York: Springer.

YATES, Frances. 1966. *The Art of Memory*. Chicago: University of Chicago Press.

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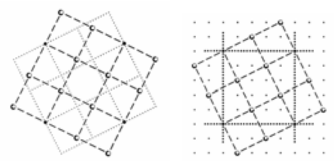

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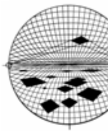

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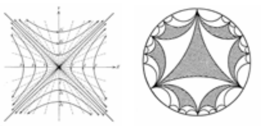

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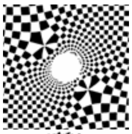

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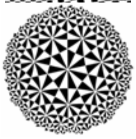

Adapted from D. A. Brannan, Matthew F. Esplen, and Jeremy Gray. *Geometry*. Cambridge: Cambridge University Press, 1999.

About the author

J.M. (Jack) Rees is an architect who specializes in designing architectural modifications to existing structures. He started his career in the textile design studio of Jack Lenor Larsen. Trained as an architect, professionally accomplished as an interior designer, experienced in book arts, he is a spatial database designer conversant with ArcINFO and SpanFM. Exhibitions include *The Architecture of Paper* (2004), *The Virtual Nomad* (1999), and *Manhattan Miniture Golf* (1978). He is guest editor of the *2006 REVIEW Architecture and Urban Planning Annual*. He has projects under construction in Colorado and Missouri. He is a generalist in what feels like a world of specialists.

He writes about himself: "As I often find myself a purveyor of unpopular ideas, allow me a short apology. My avocation is history of geometry which means that I occasionally read geometric proofs for entertainment. Architecture is entertaining precisely opposite the way mathematical proofs are entertaining. Geometry (like painting) requires a highly focused contemplation towards an occasionally ecstatic reward. Architecture, on the other hand, received in a "state of distraction," is thicker than the eureka moment geometry and painting share. Architecture is thick the way play is deep, a somatic thrill as opposed to an intellectual reward."