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Reconstruction of Forms by Linear Algebra

Lines and surfaces are boundary elements of objects and buildings: it is very important to give the students a mathematical approach to them. Elena Marchetti and Luisa Rossi Costa present linear algebra (by vectors and matrices) as an elegant and synthetic method, not only for the description but also for the virtual reconstruction of shapes.The aim of our activity is to facilitate—and, at the same time, to develop—the comprehension of crucial mathematical tools involved in the realization of forms and shapes in arts, architecture and industrial design and in computer graphics. Another important aspect of linear algebra to be pointed out to the students is its application in graphics software packages, which work with transformations that change the position, orientation and size of objects in a drawing

Introduction

In this paper we present the didactic experience realized from the academic year 2001-02 on, with students of Mathematical Courses held in the Faculty of Architettura e Società of the Politecnico di Milano.

The aim of our activity is to facilitate and, at the same time, to develop the comprehension of crucial mathematical tools involved in the realization of forms and shapes in arts, architecture and industrial design and in computer graphics.

Lines and surfaces are boundary elements of objects and buildings: it is very important to give the students a mathematical approach to them. We think that linear algebra (by vectors and matrices) is an elegant and synthetic method, not only for the description but also for the virtual reconstruction of shapes.

Another important aspect of linear algebra to be pointed out to the students is its application in graphics software packages [Foley, et al. 1992]. All the programs used in city-planning work with transformations that change the position, orientation and size of objects in a drawing.

The important architectural studios present their projects by using fascinating virtual images or movies that show their projects inserted in the existing context, a town or landscape,. The students must in any case be made conscious of the mathematical background behind those sophisticated software packages even if they are used sight unseen.

The activity developed in our courses is complementary to theoretical lessons and exercises, and offers students a collection of shapes that recur frequently in the contexts mentioned above.

We address the activity to first-year students who have an average mathematical background (they come from a wide spectrum of high schools), so we focus on peculiar, simple, but quite attractive examples.

In Section 1 we describe the activity in details, briefly presenting essential mathematical tools, even if well-known to everybody working with Mathematics; in Section 2 we present a short collection of students' final projects. We conclude with brief comments.

1 Description of the activity

1.01 Mathematical tools: vectors and matrices. We think linear algebra is an easy tool for working with lines and surfaces. Its methods are the traditional subjects of first-year programs. It is crucial for students to learn to use them with efficacy.

The first step is to present the theory of matrices and vector calculus. The second step is to give the students geometrical applications in 2D and 3D Cartesian spaces. The third step is to apply matrices and vector calculus to significant examples in the artistic and architectural field.

Here we introduce briefly the elementary vectors and matrices calculus in 3D Cartesian space, exactly as we approach it in our courses. Naturally specialists can skip this section but those less familiar with the subject may be interested in reading it. We limit our mathematical illustration to 3D space (where we live!), but the definitions can be extended to any n-dimensional abstract space (for a larger theoretical description see for example [Binmore and Davies 2001], [Strang 1998]).

The principal elements to be known are:

- *square* (3,3) *matrices* A= $\mathbf{A}_{(3,3)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$;

  column vectors (or *column matrices*) (3,1) $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$;

  row vectors (1,3) (or *row matrices*) $a = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$;

- *matrix operations* illustrated for square matrices and vectors;
- *bijection* between points P=(x,y,z) in 3D Cartesian space *Oxyz* and column vectors $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix}_T$ (sometimes written in the transposed row-form); in describing transformations in xy-plane we will use vectors having the third component equal to zero;
- *linear transformations* in 3D realized by appropriate matrix operations.

The elementary operations with matrices and vectors are:

- *multiplication* by a scalar k:

$$k\mathbf{A} = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}, \quad k\mathbf{a} = \begin{bmatrix} ka_1 \\ ka_2 \\ ka_3 \end{bmatrix}, \quad k\mathbf{a} = \begin{bmatrix} ka_1 & ka_2 & ka_3 \end{bmatrix};$$

- *addition*:

$$\mathbf{A}+\mathbf{B}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}+\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}$$

$$\mathbf{a}+\mathbf{b}=\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}+\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}=\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{bmatrix}=\begin{bmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \end{bmatrix}_T$$

$$=\begin{bmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \end{bmatrix}_T ;$$

- *product of a square matrix* (3,3) by a *column vector* (3,1) (the result is a *column vector* (3,1))

$$\mathbf{c}=\mathbf{Ab}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}=\begin{bmatrix} a_{11}b_1+a_{12}b_2+a_{13}b_3 \\ a_{21}b_1+a_{22}b_2+a_{23}b_3 \\ a_{31}b_1+a_{32}b_2+a_{33}b_3 \end{bmatrix};$$

- *product of two square matrices* (3,3), known also as “row-column product”, non-commutative and natural extension of the previous matrix-vector product:

$$D=\mathbf{AB}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}=$$

$$=\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{bmatrix}.$$

1.02 Mathematical tools: transformations. In this section we present the transformations of the plane or of the 3D space, most of which are isometries, involved in our practical classes.

The following notation

$$\mathbf{v}'=\mathbf{Av}+\mathbf{h} \qquad (1)$$

can formalize affine transformations in the Cartesian *Oxyz* space. In relation (1), $\mathbf{v}=\begin{bmatrix} x & y & z \end{bmatrix}_T$ and $\mathbf{v}'=\begin{bmatrix} x' & y' & z' \end{bmatrix}_T$ correspond to the starting point and to its transformed respectively;   is the (3,3) matrix related to the transformation, and $\mathbf{h}=\begin{bmatrix} a & b & c \end{bmatrix}_T$ is the translation vector.

In the following we give some simple cases:

identity:	$\mathbf{A}=\mathbf{I}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \mathbf{h}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};$

translation:	$\mathbf{A} = \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{h} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, a, b, c (real parameters);
reflection with respect to one of the coordinate planes; just one component of the vector v changes in sign, consequently:	$\mathbf{A}_{xy} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ or $\mathbf{A}_{yz} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ or $\mathbf{A}_{zx} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{h} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$;
rotation of an angle ϑ, anticlockwise, around the z-axis:	$\mathbf{A} = \begin{bmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{h} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$;
scaling with centre in the origin O: intended as a change of measure in some or all directions:	$\mathbf{A} = \begin{bmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{bmatrix}$ (r, s, t positive real parameters), $\mathbf{h} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (r, s, t positive real parameters), the choice r = s = t gives the homothety.

We can also combine the transformations mentioned above by multiplying the corresponding matrices, but we have to remember that the order must be respected (the product of matrices normally is non-commutative). For example the scaling matrix with at least two non-equal factors does not commute with the matrices of other transformations.

1.03 Practical steps. Having given the students an adequate mathematical background, we carry on showing them a collection of objects and images that present symmetric or similar parts, taken from the fields of industrial design, architecture and the arts.

The teacher invites the students:

- to analyse one of the chosen forms,
- to fix it in a suitable Cartesian coordinate system,
- to pick out the essential part,
- to identify in it some crucial points, basic lines or surfaces with vectors,
- to discover the correct transformations necessary to rebuild the object,
- to associate the corresponding matrices.

Now the students are ready to apply the right matrices to the crucial vectors, realizing the sequential steps (that is, the plane transformations) necessary to rebuild the shape virtually so that the whole object gradually emerges, step by step.

It is essential to operate with a computer, to become familiar with a dedicated software and to create attractive graphic images to compare with the original object.

i) The first simple exercise can be the homothety, evident in the famous painting *Carré noir* by K. Malevich (1929) (Fig.1).

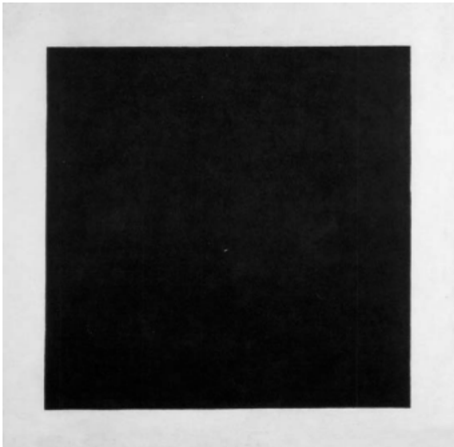

Fig. 1

The painting is centered in the *xy*-plane origin *O*, the plane scaling allowing the transition from the black square inside to the white frame is represented by the matrix

$\mathbf{A}_1 = \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $r > 1$ applied to the internal square sides.

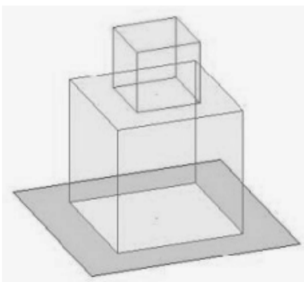

Fig. 2

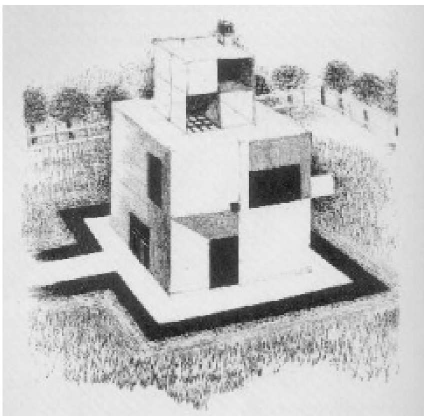

Fig. 3

The same plane transformation is also evident in Fig. 2, the virtual reconstruction of Itten's drawing entitled *The white man's house* (1920), shown in Fig. 3. In fact, the pavement around the house comes from a homothetic transformation of the base.

Fig. 4

In Fig. 4 you can see the *Am Horn house* by G. Muche and A. Meyer (1923), inspired by the Itten's drawing. The parallelepiped on the bottom, P1, is first reduced, then translated along the vertical axis: the mathematical transformation (1) works with

$$\mathbf{A} = \mathbf{A}_2 = \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix}, \quad r < 1 \quad \text{and } \mathbf{h} = \mathbf{h}_2 = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix},$$

where c is the height of P_1.

ii) The second exercise is conceived to illustrate rotation: the frieze in Fig.5, the decoration of a building in an area of Milan from the era of the Liberty style, suggests the use of the rotation matrix having $\vartheta = \pi$.

Fig. 5

The whole frieze is formed by *n* subsequently horizontal translations applied to the left basic motive, centered in the *xy*-plane origin *O*.

In the synthetic formula (1) we put

$$\text{and } \mathbf{A} = \mathbf{A}_3 = \begin{bmatrix} \cos\pi & -\sin\pi & 0 \\ \sin\pi & \cos\pi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{h} = \mathbf{h}_k = \begin{bmatrix} ka \\ 0 \\ 0 \end{bmatrix}, \ k=1,2,\ldots,n$$

(a is related to the width of the basic motive).

Fig. 6

iii) The third exercise, the choir stalls in the Cathedral of Pienza (1642, Tuscany) (Fig.6) collects all the transformations: the pattern is centered in the xy-plane origin O, it is possible to generate the internal "flower" starting from its fourth part (one petal) by subsequent rotations of $(\vartheta = \pi / 2)$. Reducing the "big flower" by a suitable scaling, the new flower undergoes a translation, e.g. in the direction of the x-axis. The wooden marquetry is completed by three rotations of the "small flower" around the z-axis $(\vartheta = \pi / 2)$.

In conclusion the teacher invites the students to discover a different way of reconstruction, involving the symmetry axes.

2 Examples of students' work

As part of the final exams, the students are invited to select for themselves an architectural or artistic form, to study it and to generate a virtual image of it, applying the methods we have shown.

This request is a homework project to be done in one week; since we started this didactic experience, all the students have reacted enthusiastically and creatively, integrating their work with relevant knowledge of art history and of architecture both ancient and modern. We ask them, in fact, to present the virtual reconstruction in the correct artistic context and to round it out with some historical information.

We also suggest that they look for and analyze significant examples in the area where they live.

The following examples represent some of the proposals given by the students, which have now become part of the material we use as teaching supports, because of their validity.

Example 1. One of the most interesting and well-presented applications of our method is the virtual reconstruction of Villa Bianca-Seveso (Milan, Italy), designed by G. Terragni in 1936-37 (Fig. 7).

Fig. 7

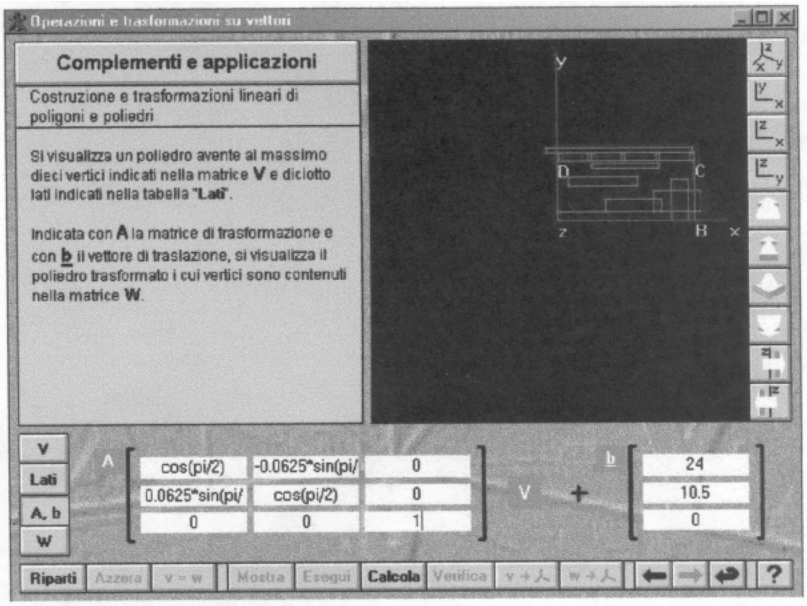

Fig. 8

The young student did it starting from the rectangular frame R of the façade, using scaling matrices and translation vectors; he re-drew windows, basement decoration and top. Finally he completed the virtual front with the main door, drawn by scaling, rotation (angle $\vartheta = \pi / 2$ and translation of R. The result is shown in Fig. 8, as it appears in the computer screen using a software familiar to students.

Along the same lines, other students proposed the German Pavilion in Barcelona (1929) and the Convention Hall in Chicago (1953), both by L. Mies van der Rohe (we don't provide images for these and other buildings mentioned, but these are easily found in texts about architecture).

Example 2. Many students were interested in plans of historical buildings, where symmetries and rotations are very often recognisable. For example, you can immediately discover an axial symmetry in the Kedleston Hall plan (1759), projected by Robert Adam for the elegant manor in Derbyshire, England.

Another interesting plan because of the evident symmetries, scaling and translations is *Le Phalanstère* by the sociologist C. Fourier (1772-1837). At the beginning of nineteenth century Fourier studied the quality of life in the countryside and wrote several treatises on it, complete with projects.

Example 3. Looking for patterns and tessellations, among different works done by the students, we find also a pretty presentation of the pavements of S.Maria in Cosmedin (Rome), where different transformations of the equilateral triangle are recognisable: scaling, translation and rotation. Other students were inspired by the decoration tiles on the façade of S. Maria Collemaggio (L'Aquila, Italy, fourteenth century).

Example 4. Concerning classic English Architecture, students did not overlook the Royal Crescent in Bath. Designed by the son of the architect John Wood, it comprises thirty houses; the basic vertical module (one column and one adjacent small façade) is rotated along the Bath Circus (inspired by the roman Coliseum). The idea of blocks of town houses like this has another previous famous example in Place des Vosges (1605) in Paris, where the houses are translated along the sides of a rectangle and rotated (by an angle $\vartheta = 3\pi / 2$ in the vertices.

Example 5. We now mention, last but not least, one peculiar example in 3D: the cubic houses in Rotterdam, by Piet Blom (1984) (Fig.9).

Fig. 9

These are an intriguing collection of row-houses, obtained by starting from a cube centered in the origin, with faces parallel to the coordinate planes. Appropriate rotations around each axis place one vertex as if it were the support. Having built up the first module, superimposing the rotated cube on a basement, the others are arranged by translation along a line.

Conclusions

The students are surprised to discover that mathematical tools are particularly suitable to describe shapes recurring in architecture and the arts. Consequently they are encouraged to read the forms in a mathematical way and to learn the theory useful for this purpose.

They certainly understand that mathematics, with its rules, is not in antithesis with their thinking but, on the contrary, provides them with a real, effective, support towards becoming more creative.

As teachers, we are very satisfied at the end of the courses because of the feedback from the students. We hope to have been sufficiently clear in our descriptions of the virtual reconstructions, as well as emphasizing the significant interdisciplinary work.

In this way students include mathematics in a group of topics such aesthetics, composition, city planning, sociology, etc., that are indivisible from architecture. They understand that knowledge of mathematics and geometry is an important aid to developing their creativity as well understanding the structural context or the project calculus.

Acknowledgments

Parts of the didactic experience here described was supported in 2002 by the Politecnico of Milan as PID-Progetti Integrativi della Didattica – Legge 370/1999.

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We mention few mathematical books, examples among many others, where you can find technical information on Linear Algebra and Transformations. Every book on the same subject is adequate to learn the necessary mathematical tools.

For our didactic methodology, many books and papers, starting from [Weyl 1982], are crucial in teaching the students the connection between Mathematics, Architecture and Arts; we do not include here a long list, avoiding the risk of forgetting some essential references.
In working with personal computers, we used MATLAB®, Maple® and didactic software realized on purpose for our aims, but you can work with any other suitable software. The software we developed is available free of charge at the address :

http://web.mate.polimi.it/viste/studenti/main.php

Choose the teacher's surname "Marchetti" or "Rossi", and the course "Matematica per l'Architettura".

About the authors

Elena Marchetti received her doctorate in mathematics at the Faculty of Sciences at the Università degli Studi in Milan. She was a researcher of Mathematical Analysis at the Department of Mathematics of the Politecnico of Milan, and since 1988 is an associate professor of Istituzioni di Matematica at the Faculty of Architecture of the Politecnico of Milan. For many years she taught in courses of mathematical analysis to engineering students, and since 1988 she has taught mathematics courses to architecture students. Her research activity is concentrated in the area of numerical analysis, principally regarding numerical integration and its applications. She has produced numerous publications in Italian and international scientific journals. Her participation and collaboration in several conferences dedicated to the application of mathematics to architecture has stimulated her interest in this subject. The experience gained through intense years of teaching courses to architecture students has led her to publish some textbooks, one of which regards lines and surfaces and has a multimedia support package, on the production of which she collaborated. She published papers with the aim to connect arts, architecture and mathematics.

Luisa Rossi Costa earned her doctorate in Mathematics in 1970 at Milan University and she attended lectures and courses at Scuola Normale Superiore in Pisa and at Istituto di Alta Matematica in Rome. Since October 1970 she has taught at the Engineering Faculty of the Politecnico of Milan, where she is associate professor of mathematical analysis. She first developed her research in numerical analysis, on variational problems and on calculating complex eigenvalues. Her interest then changed to functional analysis and to solving problems connected with partial differential equations of a parabolic type. She also studied inverse problems in order to determinate an unknown surface, an unknown coefficient in the heat equation and a metric in geophysics, with the purpose to find stable solutions in a suitable functional space. She published several papers on these subjects. She took part in the creation of lessons for a first-level degree in engineering via the Internet. She also researches subjects regarding teaching methods and the formation of high school students; she collaborates on the e-learning platform "M@thonline". Following a continuing interest in art and architecture, and believing that mathematics contains a strong component of beauty, she tries to connect these apparently different fields. She has published many papers connected to this aim.