Mark A. Reynolds

The Unknown Modulor: the "2.058" Rectangle

Geometer Mark Reynolds examines a ratio that is related to the golden section but is relatively unknown to many architects and designers. Equal to *phi* times its square root, $\phi\sqrt{\phi}$, the ratio *mu*, μ=1:2.058, offers its user an opportunity to work with the golden section but with a different "look."

…Just as Knowledge illuminates the mind, refines the intellect, and pursues universal truths, so out of the love of beautiful things, it quickly conceives and then gives birth to a beautiful daughter, Wisdom…

—Athanasius Kircher, *Ars magna sciendi* [1669, 3r]

I thought that it might be illuminating to look at a ratio that is related to the golden section but is relatively unknown to many architects and designers. It offers its user an opportunity to work with the golden section in a way, and with a "look" that is different from the appearance of the golden section or the square root of the golden section rectangle. The ratio is a distant relative of the more famous phi, ϕ, ratio, but one that still provides its user with easily developed and recognizable golden section relationships within the geometric structure, similar to the way that the internal edge of the square (the line or edge sometimes being referred to as a caesura, denoting a break or change, as in music) places the golden section cut of the golden section rectangle.[1]

The system I will present in this issue may have even pleased Le Corbusier.[2] Corbu's Modulor consists of combinations of squares and golden section rectangles, and the almost limitless combinations of arrangements that the two ratios can generate as compound rectangles. Had he explored the golden section number—1.618—a bit further, he may have found the beautiful number that is created from the golden section and its square root:

$$\phi\sqrt{\phi}=2.058$$

I have long held, and often comment in my writings, that one of the most wondrous aspects of the golden section is that not only do we find its presence in nature and certain mathematical instances, but that by its very essence, the golden section itself grows intriguing geometric relationships. Here then is this issue's subject, the "2.058 rectangle." For the sake of brevity and clarity, I will denote the 2.058 rectangle with the Greek letter *mu*, μ. Also, I am not going to use the misleading notation of three dots to show that a decimal expansion is an approximation (since the reader might incorrectly infer a repeating decimal expansion), choosing instead to state explicitly at the outset that all decimal expansions are correct to the thousandths place.

The ratio of the simple yet fascinating formula $\phi\sqrt{\phi}=2.058$ is a mixed blessing.[3] It is so very nearly a double square that it is almost impossible to distinguish it from its rational neighbor,[4] yet the two are light years apart regarding the geometric armatures produced within each perimeter. On the other hand, for those designers who like the look of the double square but would like to try their hand at a totally different grid to work with and develop, in my opinion, the μ rectangle cannot be surpassed.

To understand the μ rectangle, it will help to first look at the rectangle based on the square root of the golden section, $\sqrt{\phi}$, whose ratio is 1.272:1.[5] Because this ratio is irrational, this rectangle is best made by construction, not measure.

In Fig. 1, rectangle ASTZ is a golden section rectangle. When the long side of the rectangle, AS, is dropped to its opposite long side to make point M and the length TZ, it generates the $\sqrt{\phi}$ ratio at M. AS = AM.

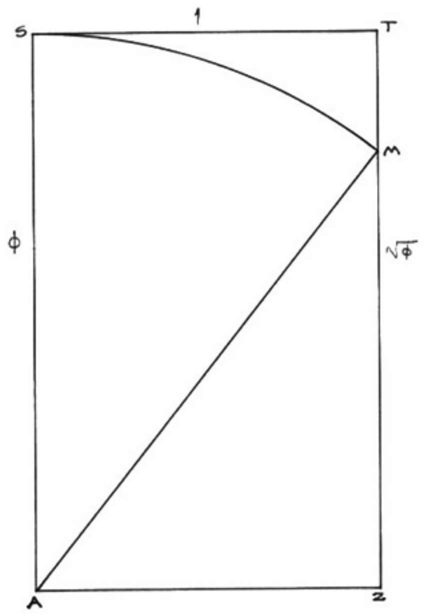

Fig. 1

Rectangle AKMZ in Fig. 2 is a $\sqrt{\phi}$ rectangle. Fig. 3 shows the application of the double square, KXYM, onto the side KM for the purpose of determining the golden section of KM, which is at E. When line EF is drawn parallel to AK and MZ, the μ rectangle AKEF is generated. It is important to note that when the golden section of the long side is found, a μ rectangle will also be generated in the other direction. (These two possibilities are demonstrated in Figs. 10 and 11 below.) The two additional golden sections of the rectangles are implied by the potential for the application of mirror symmetry. It is sometimes overlooked that there are two positions for the golden section of a line, although only one is always used to demonstrate "the cut". Rectangle CBRM is also a μ rectangle. So then, the μ rectangle can be created by finding the golden sections of both long and short sides.

Fig. 4 demonstrates a second method in which the golden section of side MZ can be found. It has always been of interest to me that the $\sqrt{\phi}$ rectangle's long side (here, AK) becomes its own reciprocal! This will be demonstrated in Fig. 5 below. In rotating the long side of the rectangle, AK, onto its opposite long side, MZ, in the same way that the $\sqrt{\phi}$ rectangle was originally made from the golden section rectangle above, we now have the golden section of MZ at N! AK = AN. No other rectangle has these properties.

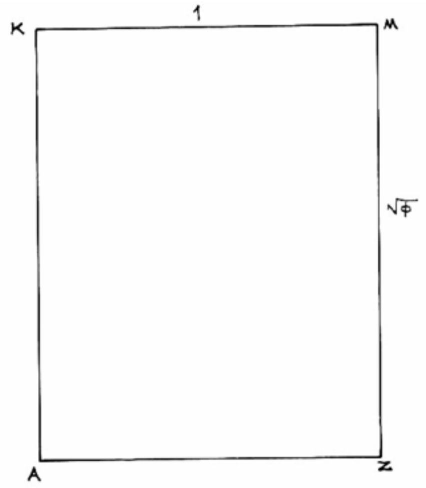

Fig. 2

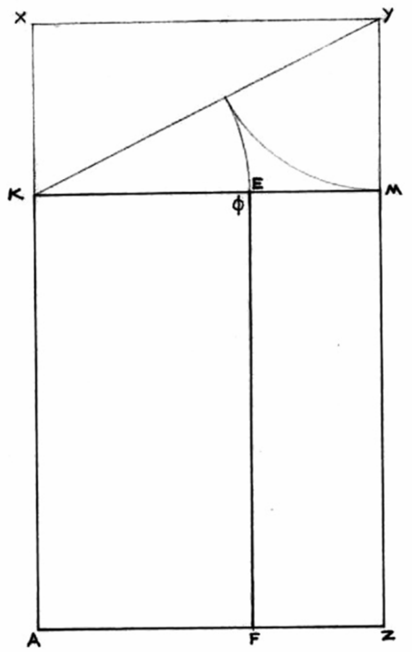

Fig. 3

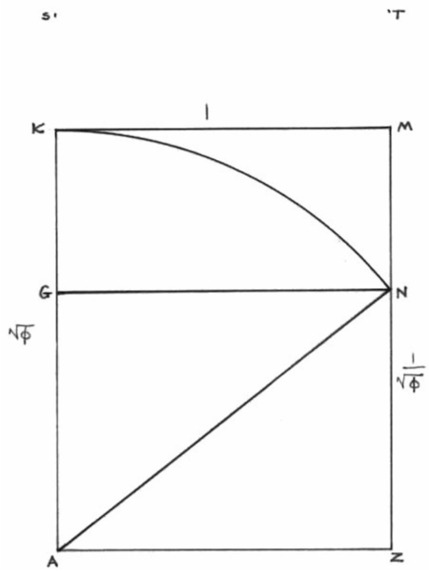

Fig. 4

The method used in Fig. 4 not only locates the golden section of MZ, but it also generates the reciprocal $\sqrt{\phi}$ rectangle, AGNZ, $1/\sqrt{\phi}$, or 0. 786, as seen in Fig. 5. In Fig. 5, ZK is the diagonal, which equals the golden section. AN is the reciprocal, which equals $\sqrt{\phi}$. O is an occult center, and is actually at the golden sections of both rectangle AKMZ and AGNZ! It is the only system where the occult centers are at the golden sections of the perimeter measures. It should also be understood that the line GN, the side of the reciprocal angle AGNZ, is the "caesura", or break, that defines the golden section of rectangle AKMZ as well.

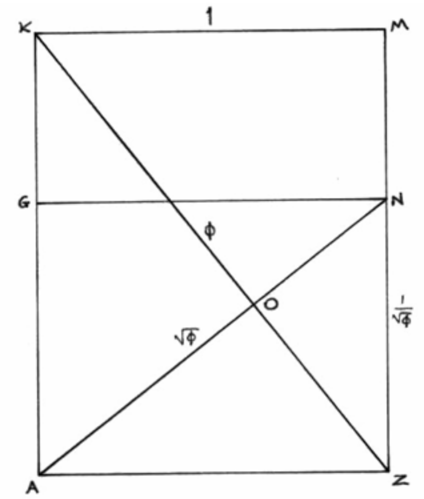

Fig. 5

We can see from the above drawings that we have a very wonderful tool for golden section work.[6] The generation of the golden section within the square root of the golden section rectangle yields the μ rectangle, and vice versa; the two are inseparable.

I have spent some time on this rectangle because it is by the combination of two of these rectangles that we arrive at the μ rectangle. Figs. 6 through 12 will present the developments of the 2.058 or μ rectangle.

Fig. 6 shows the μ rectangle, APRZ. There are several ways to observe its structure:

- Rectangle AKMZ and rectangle KPRM are both $\sqrt{\phi}$ rectangles;
- Rectangle KPRM is the reciprocal $\sqrt{\phi}$ rectangle to $\sqrt{\phi}$ rectangle AKMZ;
- Rectangles AGNZ and KPRM are equal $\sqrt{\phi}$ rectangles, and rectangle GKMN is a μ rectangle that separates them. Of course, rectangle GKMN does not have to be in this position. It could be placed on top, at the bottom, or anywhere internally for that matter, but at the risk of certain harmonies inherent in the system;
- When the rectangle GKMN is placed in this position, its two sides, GN and KM, are at the two golden sections of AP and ZR, the heights of the μ rectangle.

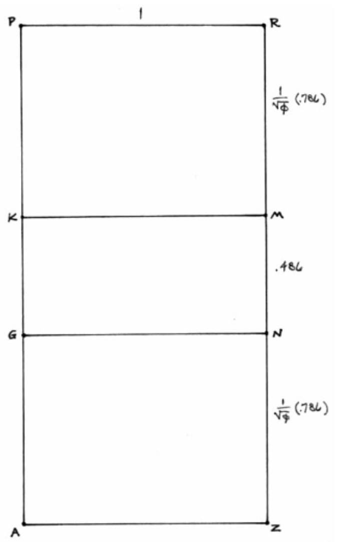

Fig. 6

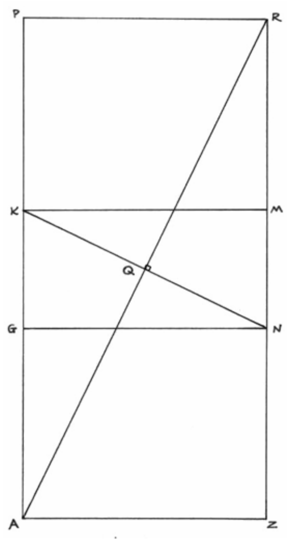

Fig. 7

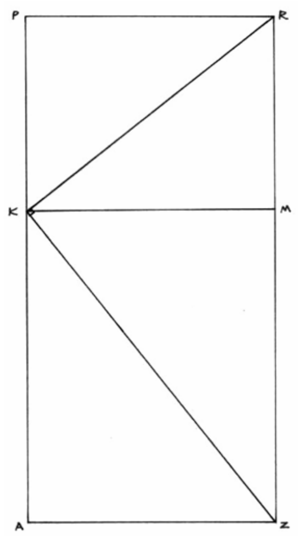

Fig. 8

In Fig. 7 we see that μ rectangle GKMN is a reciprocal to μ rectangle APRZ. Because rectangle GKMN is not at the top or bottom of the master rectangle, there are no occult centers. Here, the diagonal and reciprocal lines intersect at Q, in the dead center! (As is always the case, Q must always be a 90° angle. This is <u>the</u> rule for diagonals and reciprocals.)

Fig. 8 shows the other extreme (the first being the intersection Q in the very center) of the diagonal reciprocal relationship. Here, the intersection is at K, ZK is the diagonal, and KR is the reciprocal. This is yet another way to view the μ rectangle: the $\sqrt{\phi}$ rectangle and the reciprocal $1/\sqrt{\phi}$ rectangle separated yet joined, with the short side KM of rectangle AKMZ becoming the long side of rectangle KPRM.

Fig. 9 demonstrates the result of the diagonal AR being introduced into the system. The "power of the diagonal"[7] is used here to generate μ rectangle CBRM, where AR intersects KM at C. (There is another in the lower left corner that has not been drawn.) Because KM is at a golden section of the height, the power of the diagonal generates the line HB, which is at a golden section of the width. The introduction of the diagonal has now created a grid of $\sqrt{\phi}$ and μ rectangles in the field as well as defining additional golden sections. By the nature of geometry, these progressions will continue to develop as the same procedures are applied to individual units within the grid.

Fig. 10 illustrates a $\sqrt{\phi}$ rectangle, AKMZ and Fig. 11 illustrates KPRM. It demonstrates yet another unique quality of this system.

The $\sqrt{\phi}$ rectangle is the only rectangle to contain exactly three different sizes of itself, LCMN, GKCL, and AGNZ. All three are in a golden section progression in their sizes and their edges. CL, LN and GL are at the golden sections of the master rectangle, and their specific orientation and placement combine to make a complete $\sqrt{\phi}$ rectangle![8] By finding the golden section of the rectangle AKMZ, at GN, we have now generated the μ rectangle, GKMN as well. As mentioned, the μ rectangle is composed of two reciprocal $\sqrt{\phi}$ rectangles.

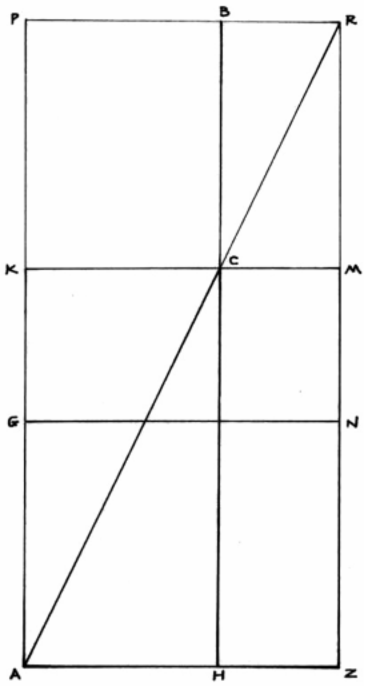

Fig. 9

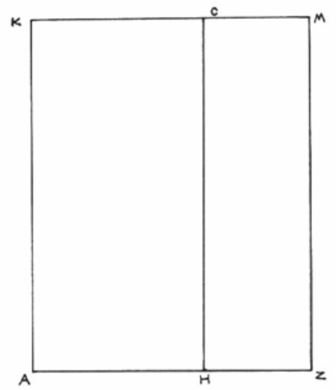

Fig. 10

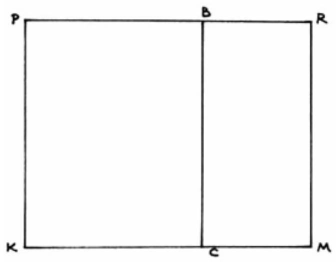

Fig. 11

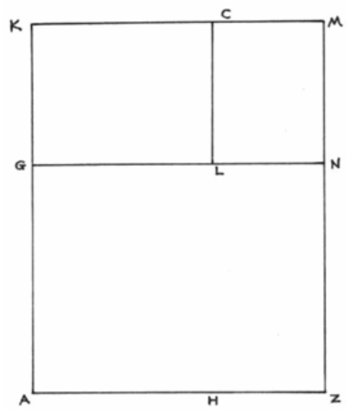

Fig. 12

I have spent quite some time working with the 2.058 rectangle in my art because it is so enjoyable to generate the grid systems found in the ratio, and I have some examples to show.

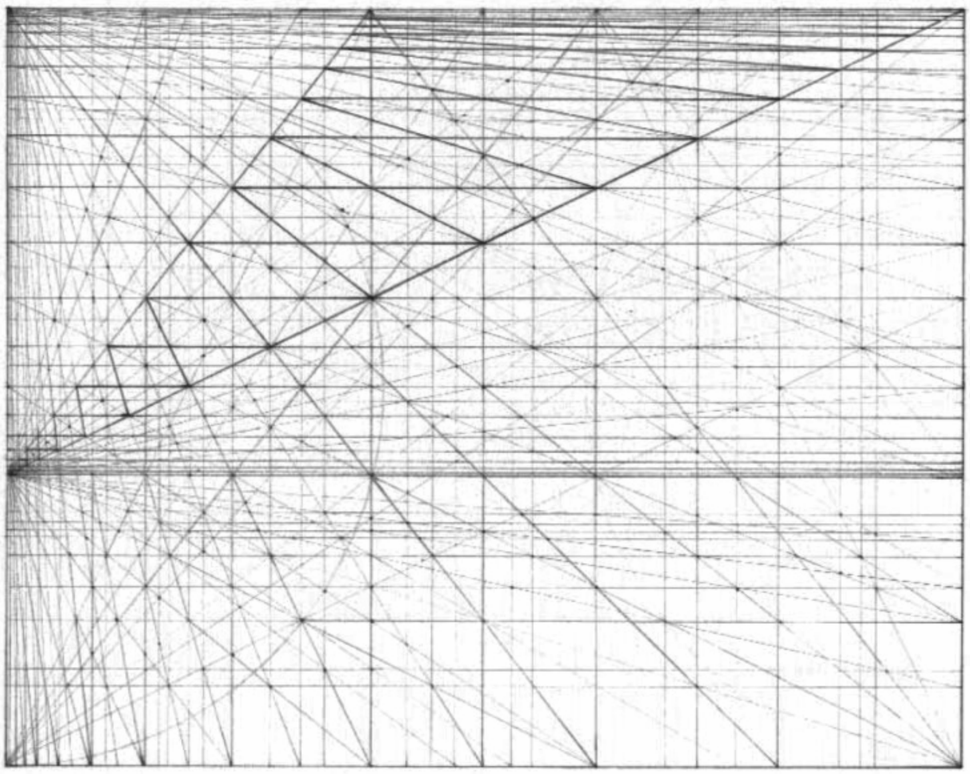

Fig. 13. Some readers might recognize Fig. 13 from the column in the *NNJ* vol. 5, no. 1, a graphite drawing of a dense grid.

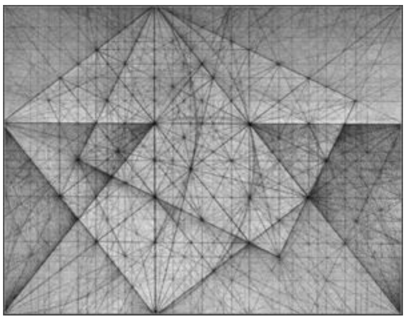

Fig. 14. is a geometric image constructed in the square root of the golden section rectangle, and where we can see the μ rectangle generated by combinations of these square root phi rectangles within the grids.

A Second Method for the μ

There is a second method for constructing the 2.058 rectangle that has a basis found in a construction presented by Matila Ghyka in his well known book, *The Geometry of Art and Life* (New York: Dover, 1977). On page 67, one finds the construction for the geometric mean by the semi-circle method, the Rule of Thales. It so happens in this particular drawing that the two extremes making the diameter of the semicircle are in the golden section ratio, making the geometric mean the square root of the golden section. I have modified the drawing only very slightly in what follows.

Fig. 15 shows the construction of the golden section from the double square AKMZ.

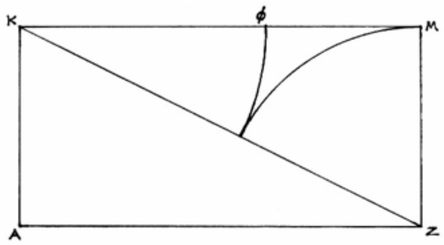

Fig. 15

In Fig. 16,

- MP = 1;
- PK = ϕ=1.618;
- PR = $\sqrt{\phi}$=1.272

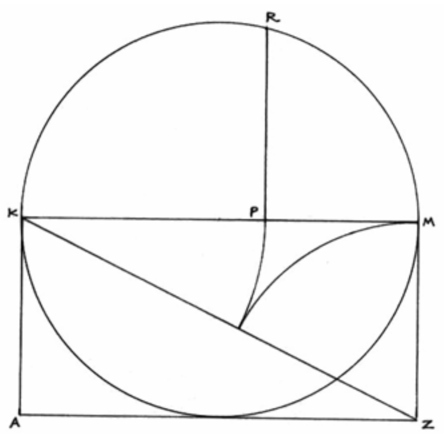

Fig. 16

In Fig. 17, lines KR and RM have been added; now,

- KR = $\sqrt{\phi}$ =1.272;
- RM = 1;
- Point R, by the Rule of Thales, is a 90° angle;
- The diameter of the great circle, KM, as a result = ϕ = 1.618;
- This construction will now allow for rectangles to be drawn.

In Fig. 18, by drawing the line GR through the center O, we now have GR = KM.[9]

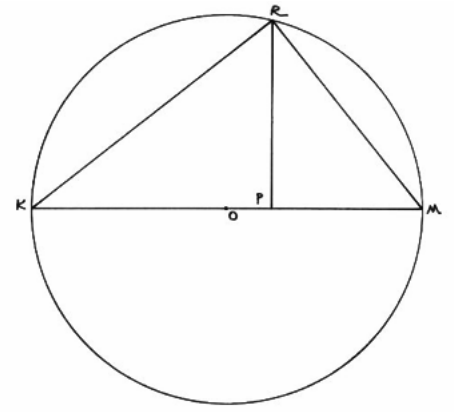

Fig. 17

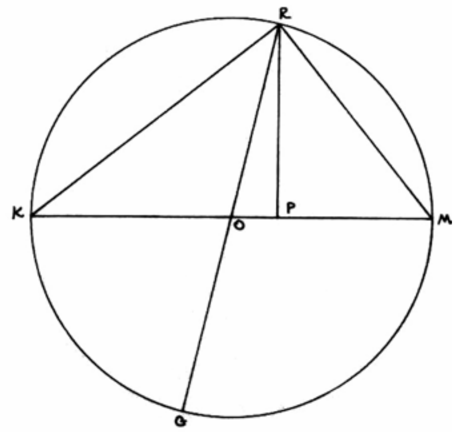

Fig. 18

As Fig. 19 shows, we can now draw the $\sqrt{\phi}$ rectangle GKRM and, therefore, rectangle GKCN is a μ rectangle.

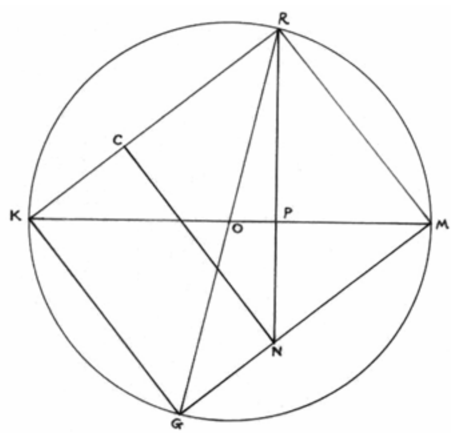

Fig. 19

Having established the $\sqrt{\phi}$ rectangle, we are now able to construct a variety of μ rectangles.

A couple of interesting examples can be found in Figs. 20 and 21. In both cases, P on the line KM, which, as a reminder, is on the golden section, acts as pivot for the side of either of the 2.058 rectangles.

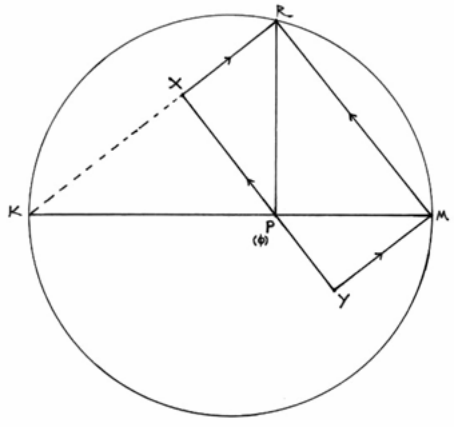

Fig. 20

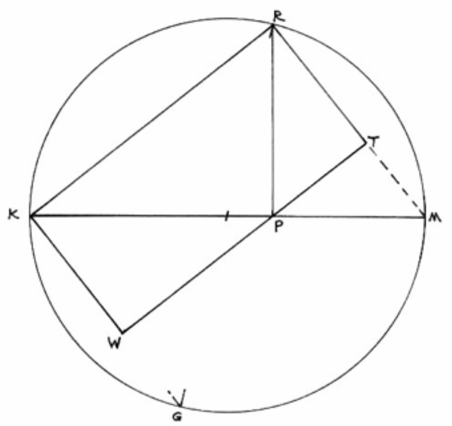

Fig. 21

Certainly, any of these grids could form the basis for plans, elevations, layouts, compositional grids, tilings (as I showed with the "R-Tiles" in *NNJ* vol.4 no.1), and so on. For the past several years, I have been busy at work on a series of drawings in mixed media using this ratio. Fig. 22 shows a pastel and graphite drawing that combines the μ and $\sqrt{\phi}$ rectangles, and that also includes a spiral of $\sqrt{\phi}$ rectangles breaking the perimeter of the drawing. I include the piece to show that the rectangles in this system can be simple, as shown in the column, or can be developed into very complex systems, as can be seen from the artwork. The system is just a wonderful tool to work with.

I will conclude our Autumn column with an interesting and curious aside regarding the number, 2.058. Our number μ minus one is very nearly equal to the "frequency constant" of the even-tempered scale in music. When an instrument is tuned in the even-tempered scale—the twelve equal parts of an octave—the frequency constant equals 1.059, which is equal to twelfth root of the square root of two. When multiplied or divided by this constant, the specific number of vibrations per second of any particular note will be carried to the next note's vibrational speed that is on either side of the original note in the scale by multiplying (going up the scale) or by dividing (going down the scale) by 1.059. The number of vibrations per second needed to produce the note B, for example, must be carried up to the note C, accelerated by a factor of 1.059 times to produce the vibrations/second needed to hear C. C contains B's vibrations per second, times 1.059. Comparing then μ to the frequency constant, we see that 2.058-1=1.058, a difference of .001 from 1.059.

As always, it is my hope that the information I present can be used by architects and designers, and that it may also spark some interest from the mathematical audience. If any readers know of uses for the 2.058 rectangle, please let me know about them. Thanks for looking at the column. Keep the long nights bright by the drawing tables. Have a warm Winter, and we will meet again!

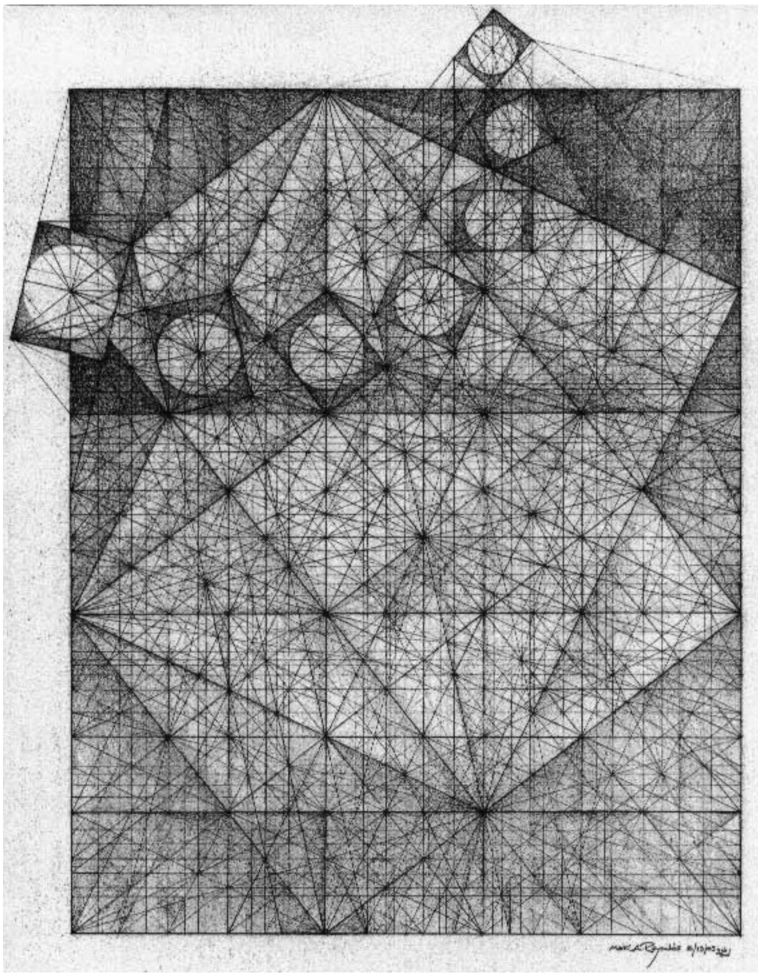

Fig. 22

Notes

1. The system we will be examining can also be found in the series of R-Tiles I created and wrote about in *NNJ* vol. 4 no. 1.
2. I was surprised on my revisit to Corbu's book, *The Modulor*, that this ratio was not to be found anywhere. It is possible that on pp. 94-95 of the 1971 MIT (second) edition that the ratio may be found by measuring all the rectangles there, but I seriously doubt it, and an explanation for this thinking will be found in the column.
3. In an organic sense, this union of the two golden section ratios found in the natural world would logically produce an offspring that should have some stunning qualities, and it does.

Another beautiful aspect of the 2.058 ratio is that when the number is squared, it equals root-4 + root-5: $(\sqrt{\phi} + \sqrt{\phi})^2 = \sqrt{4} + \sqrt{5}$.

4. This issue is discussed in my paper, "A New Geometric Analysis of the Pazzi Chapel in Santa Croce, Florence", presented at the Nexus Conference held in Ferarra in 2000 (pp. 105-121 in *Nexus III: Architecture and Mathematics*, Kim Williams, eds. (Pisa: Pacini Editori, 2000).
5. This is the ratio sometimes referred to when studying the Great Pyramid of Khufu.
6. In the United States, the $\sqrt{\phi}$ rectangle is very nearly identical to 8 ½″ x 11″ paper! Our standard stationery is 1.294. Cutting off a mere 1/8″ from the long side would give us a paper ratio of 1.279 to 1; very close indeed!
7. Among other things, the diagonal can move a measurement from one side to another in a rectangle.
8. Designers may well find this an extraordinary and exquisite example of classical Eurythmy! Perfect *harmonia*.
9. This is known as glide reflection in symmetry operations.

About the Author

Mark A. Reynolds (aka Marcus the Marinite) is contributing editor for the Geometer's Angle column of the *Nexus Network Journal*. He is a visual artist who works primarily in drawing, printmaking and mixed media. He received his Bachelor's and Master's Degrees in Art and Art Education at Towson University in Maryland. He was also awarded the Andelot Fellowship to do post-graduate work in drawing and printmaking at the University of Delaware. He is also an educator who teaches sacred geometry, linear perspective, drawing, and printmaking to both graduate and undergraduate students at the Academy of Art College in San Francisco, California. He was voted Outstanding Educator of the Year by the students in 1992. Additionally, he is a geometer, and his specialties in this field include doing geometric analyses of architecture, paintings, and design. For the past decade, Mr. Reynolds has been at work on an extensive body of drawings, paintings and prints that incorporate and explore the ancient science of sacred, or contemplative, geometry. He is widely exhibited, showing his work in group competitions and one person shows, especially in California. His work is in corporate, public, and private collections, and he is represented by the Mill & Short Gallery in San Francisco. He is a member of the California Society of Printmakers, the Los Angeles Printmaking Society, and the Marin Arts Council. He published "A Comparative Geometric Analysis of the Heights and Bases of the Great Pyramid of Khufu and the Pyramid of the Sun at Teotihuacan" in the *NNJ* vol.1 (1999).