Mark A. Reynolds

R-Tiles

The square root of the Golden Section, $\sqrt{\phi}$, is at times not given its due in discussions regarding the Golden Section. This is an unfortunate situation because its geometric properties provide us with a unique and rarely applied design tool. In this column, geometer Mark Reynolds focuses on the $\sqrt{\phi}$ rectangle and its abilities to create an exciting tiling, R-Tiles.

The square root of the Golden Section, $\sqrt{\phi}$, is at times not given its due in discussions regarding the Golden Section. This is an unfortunate situation because the geometric properties of $\sqrt{\phi}$ provide us with a unique and rarely applied design tool. In this current Geometer's Angle column, we find it in Scott Olson's article, "The Indefinite Dyad and the Golden Section: Uncovering Plato's Second Principle", but it comes and goes quickly and perhaps could be one of the most easily overlooked aspects of the article. For these reasons, I have decided to use $\sqrt{\phi}$ as the focus of my column this issue.

In my too-numerous-to-count studies of the Golden Section and its related geometric structures (particularly its square root) as design tools, I came across an irregular pentagram (*ASPTR* in Figure 1) in the gridwork of a $\sqrt{\phi}$ rectangle ($AZ : AK = \sqrt{\phi}$). Even though I had been working with the multitude of grids within this perimeter for years, I had never noticed it before. I had begun work on a $\sqrt{\phi}$ construction on a large sheet of drawing paper (about 36" by 48", large by geometric drawing standards in any event), and I had taken a coffee break. The sheet was affixed to the wall, and as I came back into the studio, I looked at the sheet from the side. In its foreshortened position, I could see the star clearly. I immediately began a series of studies on the subject. *R-Tiles* is the name I have given to a specific group of shapes, tiles, patterns, tessellations, and relationships that make up a unique geometric system based on $\sqrt{\phi}$.

As a way of understanding the tilings, it is first necessary to understand the $\sqrt{\phi}$ rectangle, as it has a few of its own unique properties. In Figure 2, *AKMZ* is once again a $\sqrt{\phi}$ rectangle; for convenience, let us take $AK = 1$ so that $AZ = \sqrt{\phi}$. The diagonal of rectangle *AKMZ*, that is, line segment *KZ*, is then equal to the Golden Section: $KZ = \phi$. The line segment *AP* is a "reciprocal" to diagonal *KZ*, meaning that it intersects *KZ* at 90° (this 90°angle is a mandatory factor); we label the intersection point *G*. The fact is that in any rectangle, a reciprocal to a diagonal is itself a diagonal to a second rectangle equal in ratio to the original rectangle; in the present case, since *AP* is a reciprocal, it makes rectangle *AKPR* also a $\sqrt{\phi}$ rectangle.[1] Importantly, point *R* cuts *AZ* at the Golden Section, i.e., $AZ : AR = AR : RZ = \phi$ (this is true only because *AKMZ* is a $\sqrt{\phi}$ rectangle).

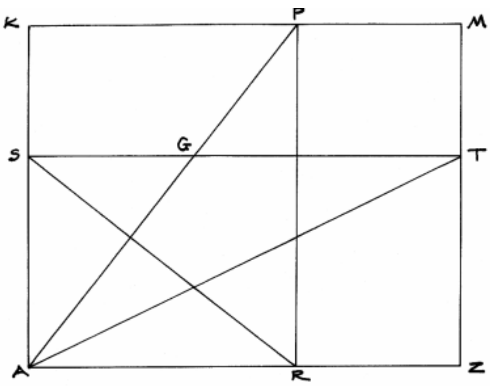

Figure 1.

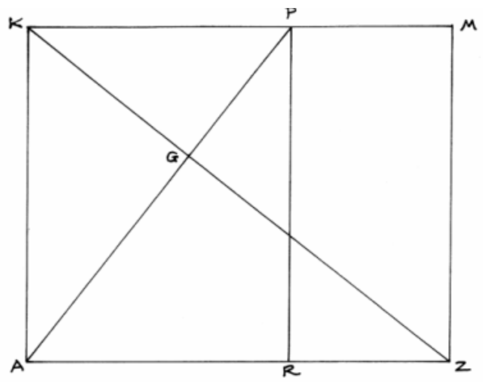

Figure 2.

Point G has, at times, been called an "occult center"[2] because it is "hidden from the eye" (not because of anything magical), and is not as easily approximated, that is, "seen", as the true center is. These occult centers can be generated only when the reciprocal (here, AP) is drawn from a corner of the rectangle. This means that all (non-square) rectangles have four occult centers.[3]

These occult centers have a unique role to play in the geometry of any rectangle. The four line segments that radiate from a given occult center will form a geometric progression in the same proportion as the original rectangle. In the present case, focusing on occult center G, we find that $GZ : GA = GA : GK = GK : GP = \sqrt{\phi}$, the same ratio as $AZ : AK$ and $AK : AR$. Of great importance in the present case (only) is that the four occult centers of the $\sqrt{\phi}$ rectangle are positioned at the Golden Sections of the rectangle's sides, both the short and long! (See Figure 4 below.)

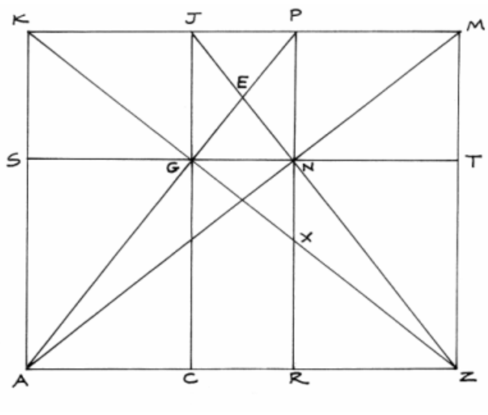

Figure 3.

Also, only in the $\sqrt{\phi}$ rectangle will the reciprocal rectangles' long sides (here, for example, *RP*) pass through the two other occult centers on the opposite side. In Figure 3, the long side, *RP*, of the reciprocal rectangle *AKPR*, passes through occult centers *N* and *X*; and we already know that *R* (respectively, *P*) cuts *AZ* (respectively, *KM*) at the Golden Section. By symmetry, so does *CJ* cut the horizontal sides of *AKMZ* at the Golden Section. Line segment *ST* passes through occult centers *G* and *N*, and it cuts the two vertical sides of *AKMZ* at the Golden Section. Of note as well is that in Figure 3, isosceles triangle *AEZ* is the approximate elevation of the Khufu Pyramid at Giza, and right, scalene triangle *ARP* is a "Triangle of Price" ($AR : RP: AP :: 1 : \sqrt{\phi} : \phi$) [Reynolds 1999].

The picture is completed in Figure 4, which shows all four occult centers, *W*, *G*, *N* and *X*. Line segments *CJ*, *RP*, *UV* and *ST* cut the four sides of the master rectangle *AKMZ* at (numerous) Golden Sections. In fact, each side of the rectangle is now cut in precisely the same way as in Scott Olsen's Figure 2 (match his points *adcb* with, for example, points *KJPM*).

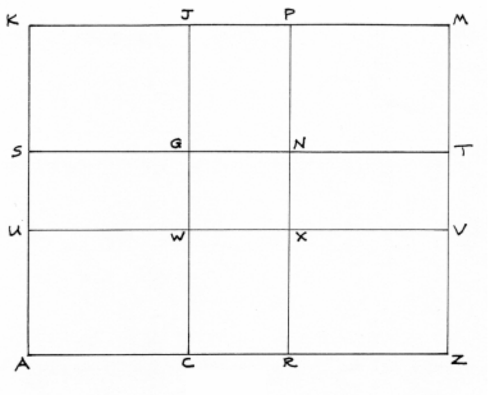

Figure 4.

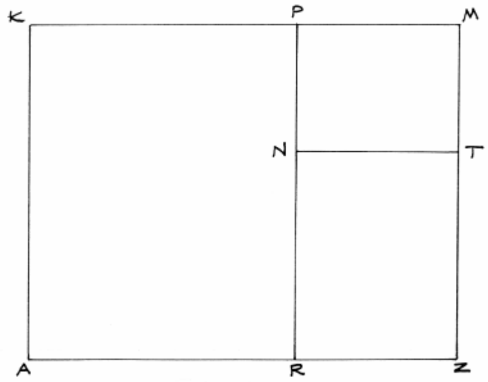

Figure 5.

In Figure 5, *N* is an occult center of rectangle *AKMZ*, and it divides the rectangle into precisely three $\sqrt{\phi}$ rectangles: *AKPR*, *RNTZ* and *NPMT*. All three are different sizes, and two of the three have the same directional orientation (here two are vertical, *AKPR* and *RNTZ*).

No other rectangle can claim all of these properties and qualities regarding the Golden Section, not even the Golden Section rectangle itself.

Figure 6 illustrates the great variety and quality of the pentagonal structures that arise from the $\sqrt{\phi}$ rectangle (as does the one in Figure 1), and it is worthy of note to observe that all of these tilings exist within a $\sqrt{\phi}$ rectangle, i.e., that the grid itself provides the designer with the opportunity to work with this great variety of tiles.

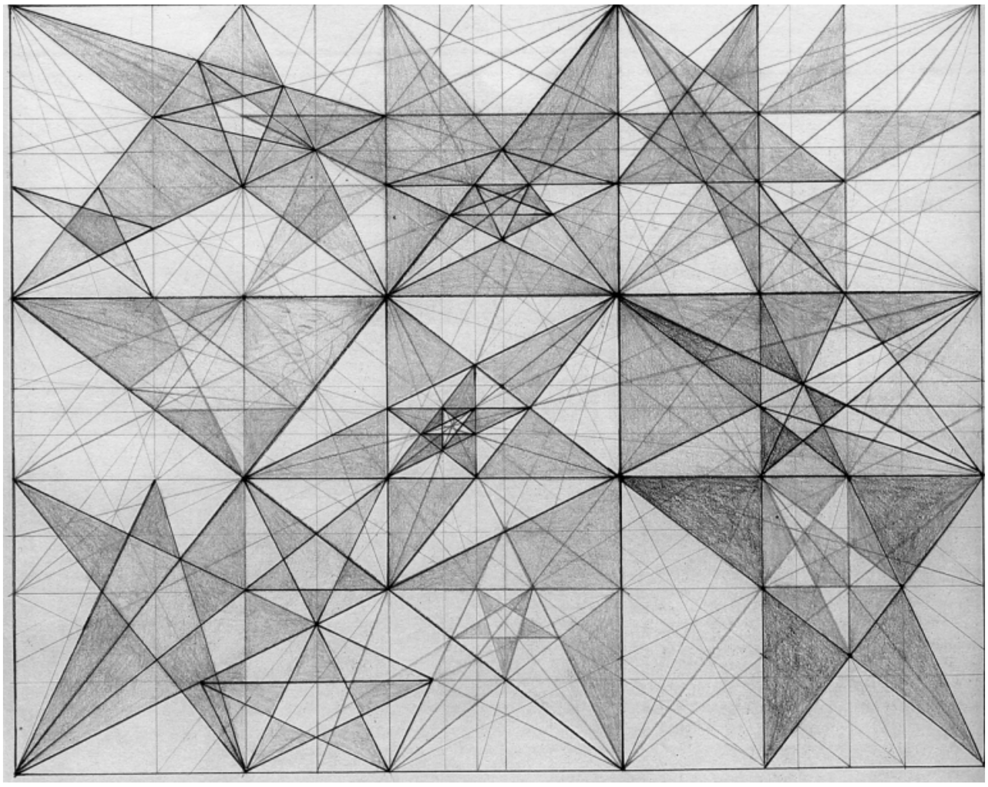

Figure 6.

It is certainly always possible to find irregular polygons and stars in virtually any grid, especially if that grid is a more developed and complex one. Also, pentagons, hexagons and octagons can all be found mingling together within one perimeter. I believe that what makes this gridwork so unique and fascinating, however, is that it uses a variety of numbers and geometric structures that are all directly related to its original generator, ϕ. In other grids, although we may find irregular polygons and stars in randomly chosen grids, their numerical and geometric structures quickly evaporate into a hodgepodge of numbers not associated with their origins, often not even remotely. For example, the irregular hexagons that may be found in a $\sqrt{3}$ rectangle, or the octagons that result from the $\sqrt{2}$ (not even to mention any pentagonal systems that can be seen) quickly move off into unrelated numbers, or numbers that may even be associated with other, unrelated systems. In the case of R-Tiles, the ϕ family group of numbers is to be found. Most notably, we find ϕ, $1/\phi$, ϕ^2, and $\phi\sqrt{\phi} \approx 2.058$. So within the grid, the integrity of the original number, ϕ, is maintained throughout the system, while remaining clearly pentagonal in nature, visually as well. For me, this is another aesthetic aspect of the tiling - it does not present a chaotic or messy appearance, and the pentagons and pentagrams maintain their identity. They do not get lost in a sea of other unrelated structures.[4]

Another aspect of their interest is in their flexibility, whether a periodic pattern or an integrated individuality is sought within the plane. Both work equally well. All that is required is to construct of the $\sqrt{\phi}$ rectangle, and then to begin work. The various tiles can easily be found and developed. Figure 1 is the "master tile" for the series, and Figure 6 is a study of R-Tiles within a rectangle, which demonstrates the great variety of tilings to be found within this rectangle.

There is infinite variety and beauty in tessellations, and many cultures have explored their possibilities and mathematical relationships. It is my hope that this small group of tiles can join the ranks of those wonders that have been with us for millennia.

Theremainder of the figures further illustrate the properties discussed above. (In the studies, it may be helpful to look for the $\sqrt{\phi}$ rectangles - the "master rectangles" - and their grids for this series - in all their variety.)

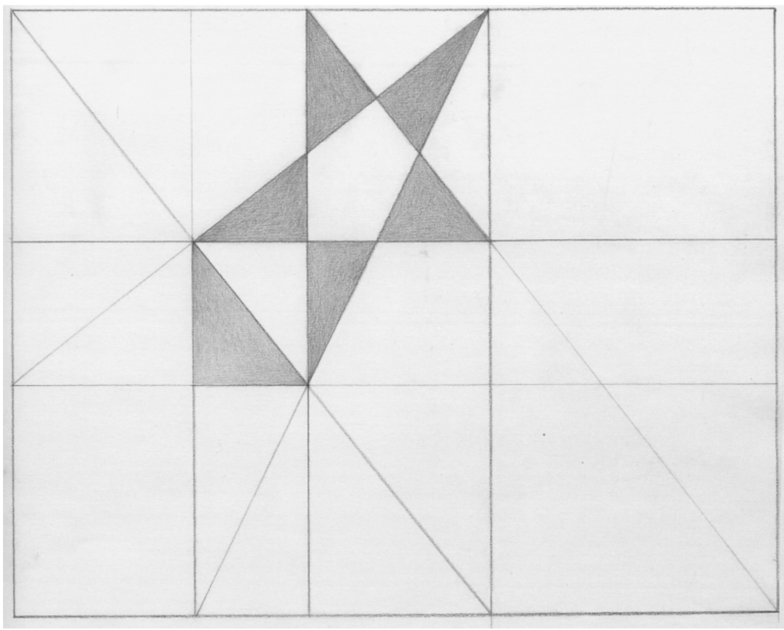

Figure 7. Study of a pentagram and a Triangle of Price within a $\sqrt{\phi}$ rectangle and one of its basic master grids.

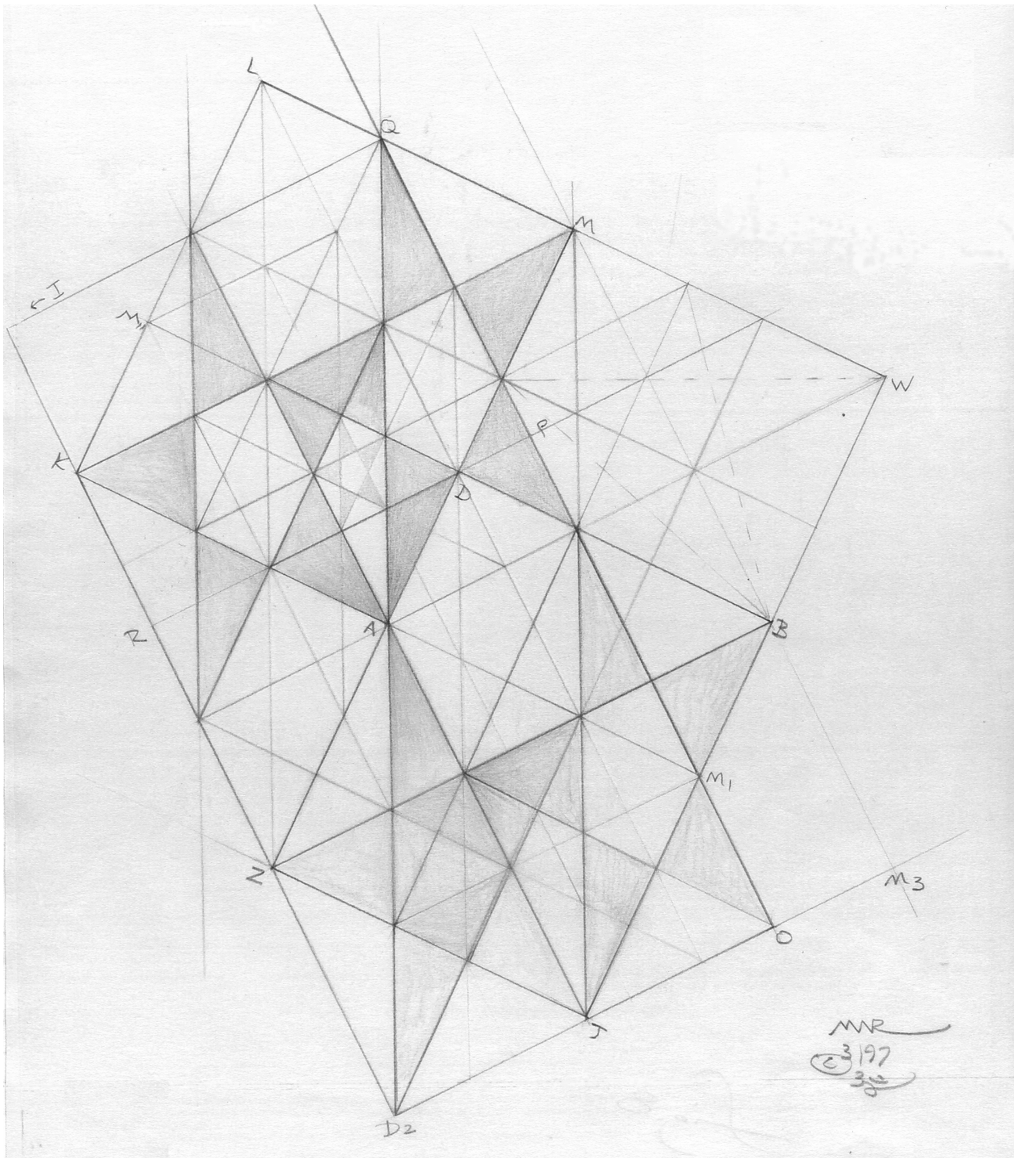

Figure 8. R-Tiles in a tessellation

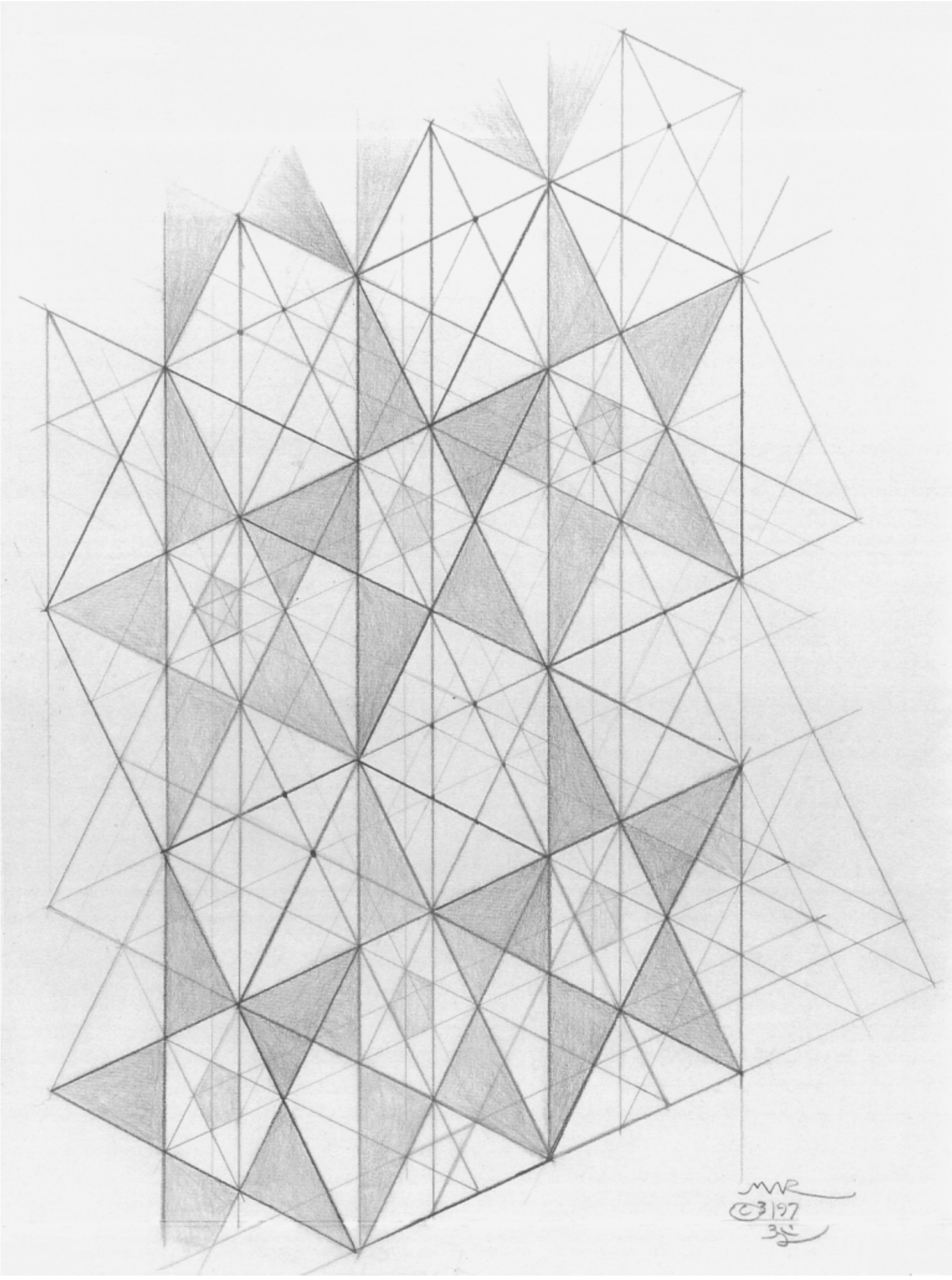

Figure 9. Another example of R-Tiles in a tessellation.

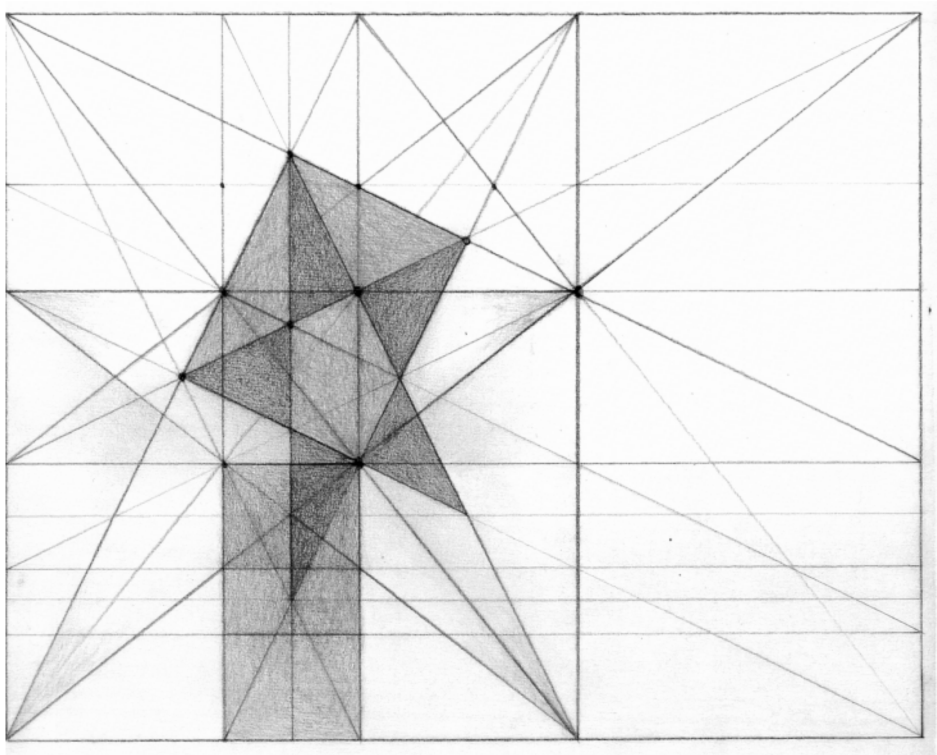

Figure 10. Study of pentagram and tiles, including a slanted $\sqrt{\phi}$ standing on the $\phi\sqrt{\phi}$. Here is an example of a more developed $\sqrt{\phi}$ grid, but still basically a master grid.

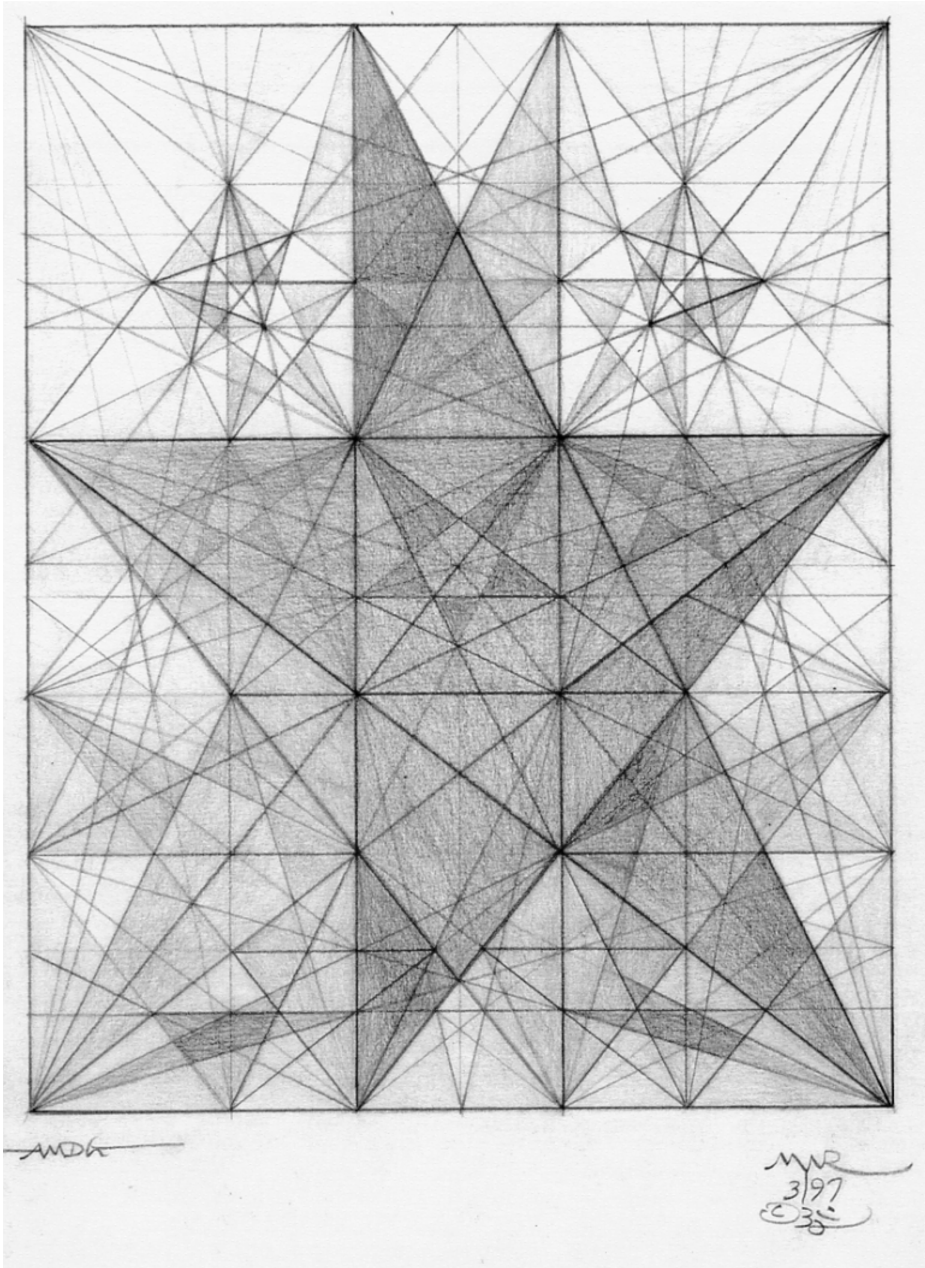

Figure 11. Reflected and reversed pentagonal tiling system, demonstrating a great variety of pentagonal structures within the rectangle. The infinity can be further expounded by the fact that as any rectangle will completely tessellate the plane, so these $\sqrt{\phi}$ rectangles can be drawn out and away in all directions from this master rectangle.

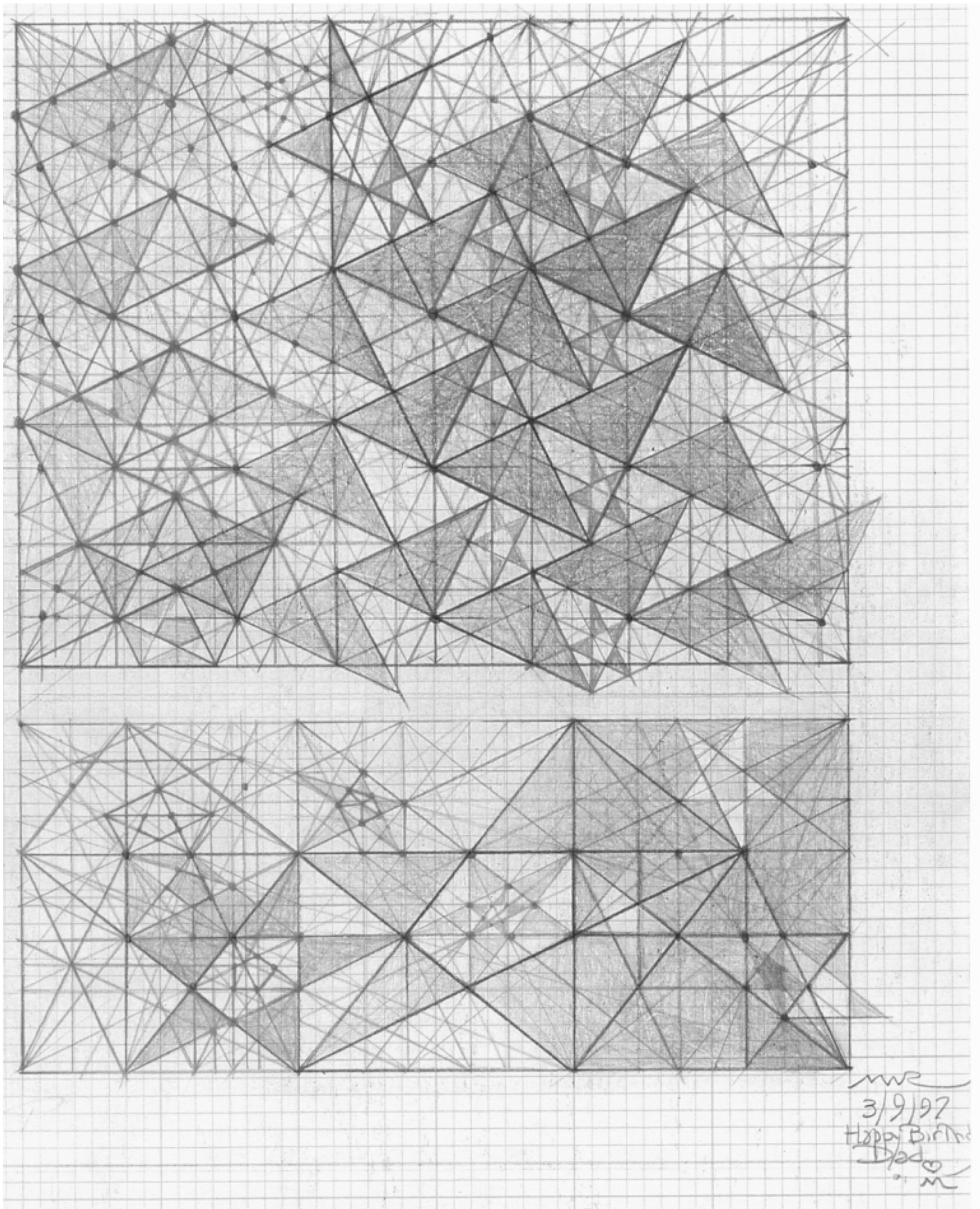

Figure 12. Tiling studies with an emphasis on the Triangle of Price and the $\sqrt{\phi}$ rectangle as a key design tools in this series.

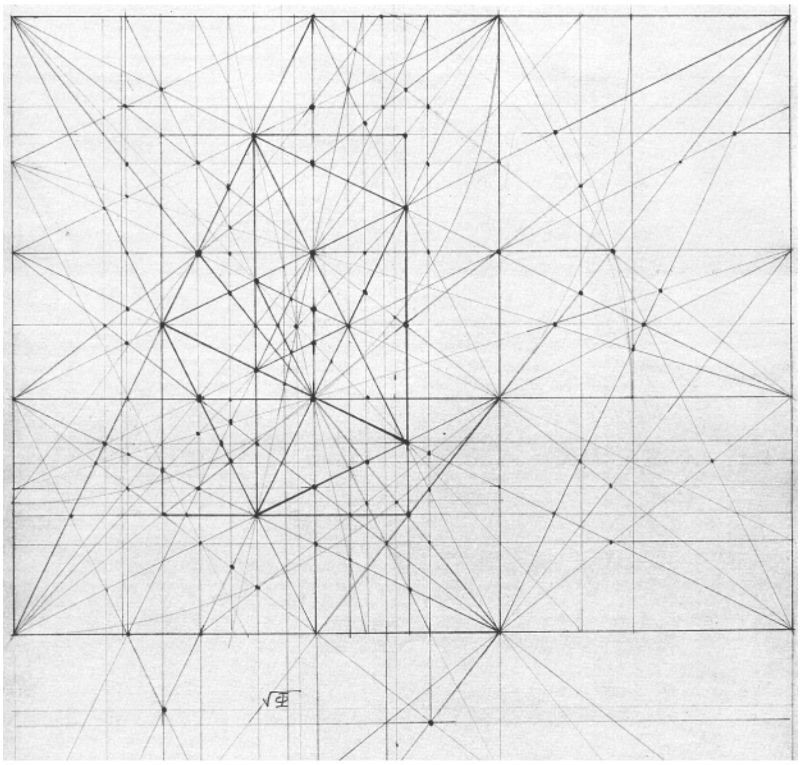

Figure 13. Compositional study for a painting.

Notes

1. It is useful that a diagonal to any given rectangle has an infinite number of reciprocal line segments, since they can be generated anywhere along the diagonal line (here, KZ), including outside the master rectangle, and they all will yield a reciprocal rectangle equal in ratio to the master rectangle. The only requirement, again, is that the reciprocal intersect the diagonal at 90°.
2. See [Hambidge 1967] for a clear development of this concept from both a mathematical and geometric point of view. For the sake of discussion, I will adopt this term in the description.
3. In a square the reciprocal to one diagonal is simply the other diagonal (analogously to the fact that the reciprocal of the number 1 is the number 1). In other words, the two diagonals cross at 90° in the center, so the 'four' occult centers all coexist at the center of the square.
4. One could certainly find an infinite variety of polygons, but this is true even with a square grid!

References

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About the author

Marcus the Marinite (aka Mark A. Reynolds) is a visual artist who works primarily in drawing, printmaking and mixed media. He received his Bachelor's and Master's Degrees in Art and Art Education at Towson University in Maryland. He was also awarded the Andelot Fellowship to do post-graduate work in drawing and printmaking at the University of Delaware. He is also an educator who teaches sacred geometry, linear perspective, drawing, and printmaking to both graduate and undergraduate students at the Academy of Art College in San Francisco, California. He was voted Outstanding Educator of the Year by the students in 1992. Additionally, he is a geometer, and his specialties in this field include doing geometric analyses of architecture, paintings, and design. For the past decade, Mr. Reynolds has been at work on an extensive body of drawings, paintings and prints that incorporate and explore the ancient science of sacred, or contemplative, geometry. He is widely exhibited, showing his work in group competitions and one person shows, especially in California. His work is in corporate, public, and private collections, and he is represented by the Mill & Short Gallery in San Francisco. He is a member of the California Society of Printmakers, the Los Angeles Printmaking Society, and the Marin Arts Council. He published "A Comparative Geometric Analysis of the Heights and Bases of the Great Pyramid of Khufu and the Pyramid of the Sun at Teotihuacan" in the *NNJ* vol.1 (1999).