John Sharp

Spirals and the Golden Section

The author examines different types of spirals and their relationships to the Golden Section in order to provide the necessary background so that logic rather than intuition can be followed, correct value judgments be made, and new ideas can be developed.

Introduction

The Golden Section is a fascinating topic that continually generates new ideas. It also has a status that leads many people to assume its presence when it has no relation to a problem. It often forces a blindness to other alternatives when intuition is followed rather than logic. Mathematical inexperience may also be a cause of some of these problems. In the following, my aim is to fill in some gaps, so that correct value judgements may be made and to show how new ideas can be developed on the rich subject area of spirals and the Golden Section.

Since this special issue of the *NNJ* is concerned with the Golden Section, I am not describing its properties unless appropriate. I shall use the symbol ϕ to denote the Golden Section ($\phi \approx 1.61803$).

There are many aspects to Golden Section spirals, and much more could be written. The parts of this paper are meant to be read sequentially, and it is especially important to understand the different types of spirals in order that the following parts are seen in context.

Part 1. Types of Spirals

In order to understand different types of Golden Section spirals, it is necessary to be aware of the properties of different types of spirals. This section looks at spirals from that viewpoint.

Clarifying types of spirals. A spiral is a plane curve that arises as a result of the movement of a point away from (or towards) a centre combined with a rotation about the centre. The centre is called the *pole*.

The rules that relate the movement from the pole relative to the rotation affect the shape of the spiral. Many errors that arise in relation to descriptions of the spiral arise because of lack of appreciation of this fact.

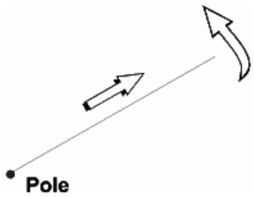

Figure 1

Throughout this article r denotes the distance of a point on the spiral from the pole and θ the angle of rotation of a line through the centre and that point. Different kinds of spirals are the result of the different ways that r can depend on θ.

Archimedean spirals. Archimedean spirals are the simplest type of spiral. An everyday example is a coil of rope. The distinguishing feature of this type of spiral is that if you draw a line from the pole, the spiral cuts the line in equal divisions.

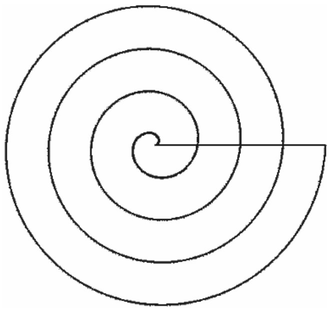

Figure 2

It can be thought of as being based on addition or subtraction since the spiral is formed from the rule that for a given rotation angle (like one revolution) the distance from the pole has a fixed amount added.

The relationship between the movement and rotation is that the movement is directly proportional to the angle of rotation. Thus the dependence of r on θ is given by

$$r = a\theta \ ,$$

where a is the constant of proportionality. The multiplication of θ by a is due to repeatedly adding the aforementioned fixed amount. For example, if θ is measured simply in revolutions, then the fixed amount is simply a: after one revolution the distance from the pole, r will be a; after two revolutions, r will be $2a$; etc. It may be more convenient to measure θ in degrees (where one revolution corresponds to 360°) or radians (where one revolution corresponds to 2π radians), but the same reasoning still applies: multiplication is the result of repeated addition.

Different values of the constant a give Archimedean spirals that are tightly wound (small a) or loosely wound (large a).

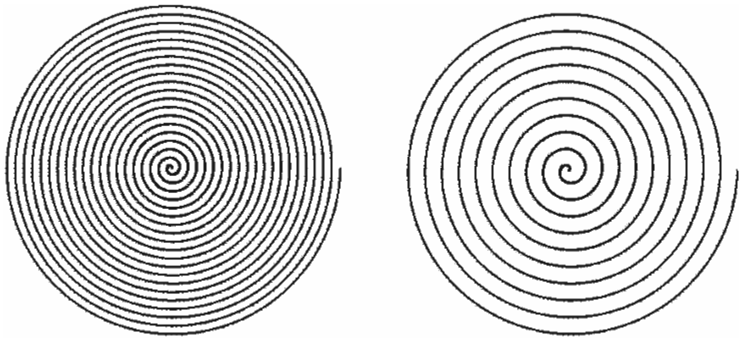

Figure 3

If you magnify one of these spirals the correct amount, it will fit directly on the other one *without any other movement like rotation*. This is an important property that needs to be remembered as a contrast to the logarithmic (equiangular) spiral described below. Another important property is that the pole is a point on the spiral, and it is easy to draw that point.

This spiral has nothing whatsoever to do with the Golden Section. The constant of proportionality a does not confer any special properties on the spiral. *There is only one Archimedean spiral.*

The logarithmic or equiangular spiral. This spiral can be thought of as being based on multiplication or division since the spiral is formed from the rule that for a given rotation angle (like one revolution) the distance from the pole is multiplied by a fixed amount. A special case of this multiplication factor is when it is the Golden Section ϕ (or powers thereof).

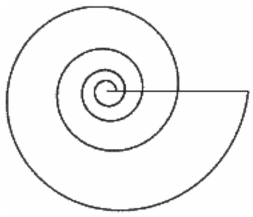

Figure 4

The relationship between the movement and rotation is more complicated than the Archimedean spiral. As the line rotates so the distance of the point increases exponentially (due to repeated multiplication instead of repeated addition). In this case, the dependence of r on θ is given by

$$r = ab^{\theta},$$

with θ measured in revolutions, or alternatively,

$$r = ae^{\theta \cot \alpha}$$

with θ measured in radians, where the constant a is a simple magnification factor, the number e is a universal constant (like π is), specifically the base of the natural logarithm ($e \approx 2.718$), and α is a constant angle. This could be written more simply as $r = ae^{c\theta}$ by setting $c = \cot \alpha$, but that would not explicitly show α.

The reason to make α explicit is the following important property of the logarithmic spiral: a line from the pole of a logarithmic spiral makes a constant angle with the tangent to the spiral, namely α. Hence it is often called the equiangular spiral.

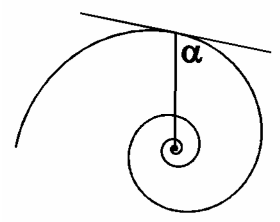

Figure 5

The mathematical relationships between α and the aforementioned multiplication factor will be further examined and exploited in Sections 3 and 4. For now, suffice it to note that different values of the constant a simply magnify the spiral, but different values of α determine the shape of the spiral. That is, there are an infinite number of logarithmic spirals, one for each choice of α. Conversely, given a member of the 'family' of logarithmic spirals, sufficient information about the particular spiral will allow you to determine the corresponding α. This will be done with a ubiquitous spiral based on the Golden Section, in Part 3.

By considering the spiral in terms of the tangent angle α, two curves can be seen as part of the family of logarithmic spirals. When α is 0° or 180° the spiral becomes a straight line, and when it is 90° it becomes a circle. This latter result is important when considering the Golden Section spiral produced approximately as a sequence of circular arcs. In characterising various Golden Section spirals, the tangent angle will be used to compare spirals produced in different ways.

If you simply magnify a logarithmic spiral, it will not fit on top of itself, ***but if you then rotate it about the pole an appropriate amount, it will.*** It is this property that is important for any logarithmic spiral, but it is especially important for the Golden Section logarithmic spiral. This is one point of contrast with the Archimedean spiral, the other being that it is not possible to draw the curve all the way to the pole since you need infinite division to reach it.

Approximations to spirals. Because of the way Golden Section spirals are often drawn, it is important to be aware that some logarithmic spirals can be drawn ***approximately.*** This usually means drawing them using arcs of circles.

For curves made up of portions of other curves to 'flow' without kinks or cusps, the tangents of two curves at the intersection point must be the same line. For two circles to join as if they are one curve, the intersection point must be on a line through their centres, so two arcs can be made to appear as if they are one continuous line as follows.

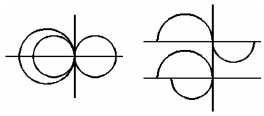

Figure 6

Such a 'piecewise' curve cannot be described by a single mathematical equation, and, as we shall see when we draw a Golden Section logarithmic spiral using this technique, it can lead to properties which are different from the those of single-equation curves. Note that the circles do not have to be the same size. As long as the line joining the centres goes though the intersection point, the curve will appear smooth.

Spiral similarity. There is another aspect of spirals that is important, particularly because of the property of logarithmic spirals in which a combination of rotation and magnification allows the curve to be superimposed on itself. If this transformation is carried out on a motif, with a suitably matched rotation and magnification, a pattern can be produced which is comparable to frieze patterns or wallpaper patterns. The following figure gives an example.

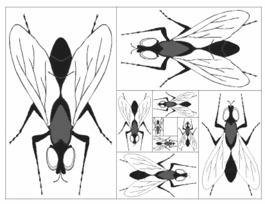

Figure 7

Note that although this uses rectangles, it has nothing to do with the Golden Section, although the techniques described in parts 2 and 3 for producing a Golden Section spiral use a similar drawing. There are two types of rectangles in figure 7, with sides in the ratio 4:3 and 12:7 (the latter being the ones containing the flies).

Part 2. Spirals From The Golden Rectangle

Before going into the deeper mathematics of the Golden Section spirals, this section looks at spirals drawn as approximations using circles, together with variations on these approximations to create new spirals.

Golden rectangle spirals. We are now in a position to draw the first Golden Section spiral and develop new spirals using the same technique:

The spiral approximated with arcs of circles. Using the technique for spiral similarity shown in figure 7, if you take a Golden Section rectangle, that is one with its sides in the ratio $1{:}\phi$, and append a square to the longer side, you end up with another golden rectangle. If you continue adding another square, then you produce another one and can do this indefinitely.

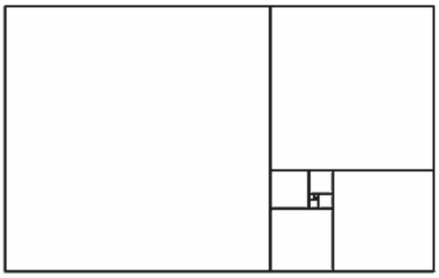

Figure 8

You can, of course, do the reverse, and subtract. Each time, if you keep the same orientation of the figure, you need to rotate the drawing. This brings home the rotation and magnification (or dilation if you are subtracting). This is the so-called "whirling squares" so named by the art historian Jay Hambidge [1926].

Figure 8 was drawn with a true Golden Section rectangle. You can draw a similar figure if you use a rectangle with sides in the ratio of two successive terms of the Fibonacci sequence. The rectangles and squares then have integral sides that are Fibonacci numbers.

The most well known Golden Section spiral is drawn from figure 8, using arcs of circles. A quarter circle is drawn in each square so that the line joining the centres goes though the touching point to give a smooth curve.

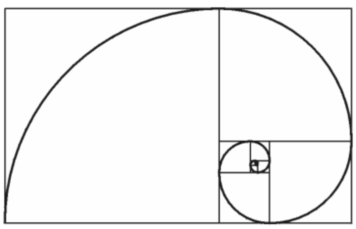

Figure 9

The pole for the spiral is found by drawing the diagonals of the Golden Section rectangles.

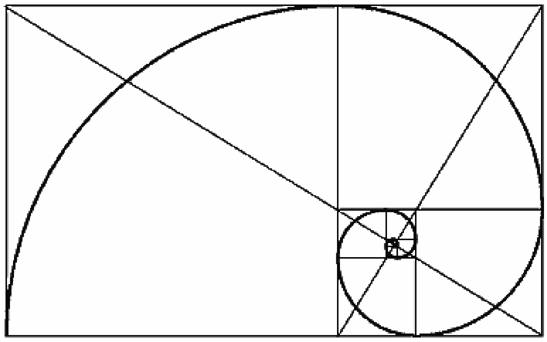

Figure 10

This allows us to unravel the mathematics of this spiral, but before doing that I would like to show that it is possible to use this technique for creating other spirals and designs using this set of whirling squares.

Doubling the spirals. Reflecting the spiral about the diagonal of the largest square yields a very satisfying design. The first stage with the construction squares is:

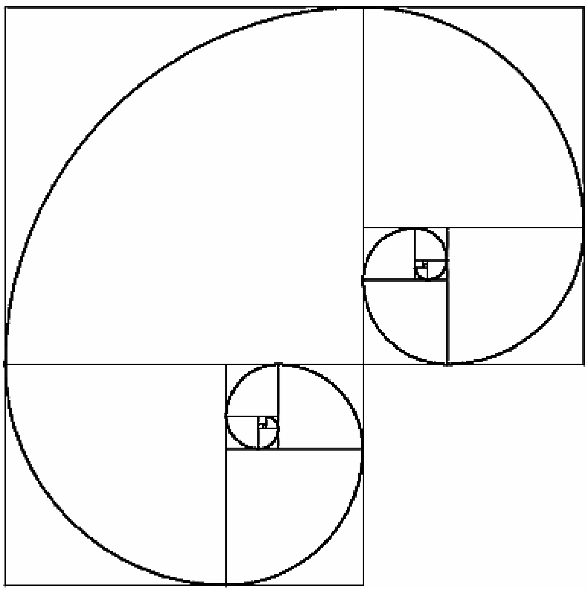

Figure 11

and then rotated by 45° on its own gives a smooth curve because of the way the arcs flow:

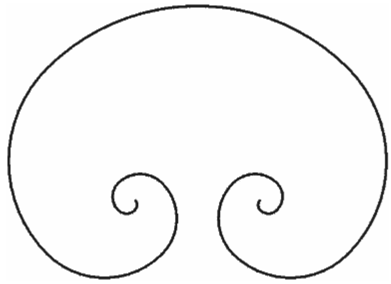

Figure 12

The anti-spiral from arcs of circles. The quarter arcs can be drawn in each square in different ways. Not all form a continuous curve. The following one forms a series of cusps.

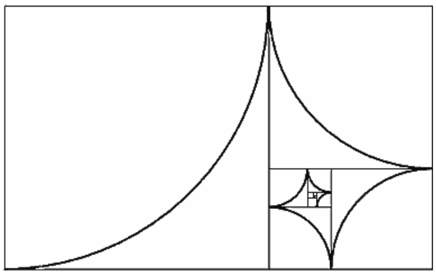

Figure 13

Using the technique shown in figures 12 and 13, the arcs no longer appear to be formed from quarter circles.

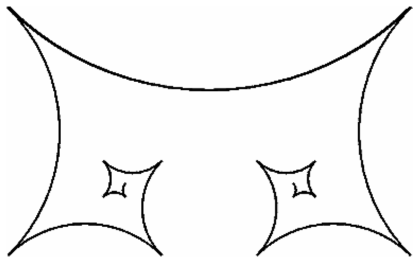

Figure 14

The "wobbly" spiral. I "discovered" this one by accident when I used a CAD program to draw the spiral of figure 9. When you draw an arc with such a program, using the centre and two points on the arc, the program sometimes decides to go in the opposite direction from the one you want. Instead of drawing a quarter circle, I found it produced the corresponding three quarter one. This is the result on the second from largest square of the whirling squares.

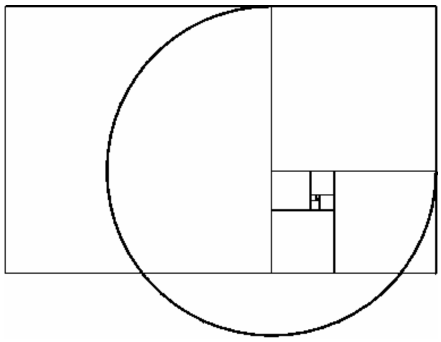

Figure 15

The full drawing, with the three-quarter circles drawn for each square, then becomes:

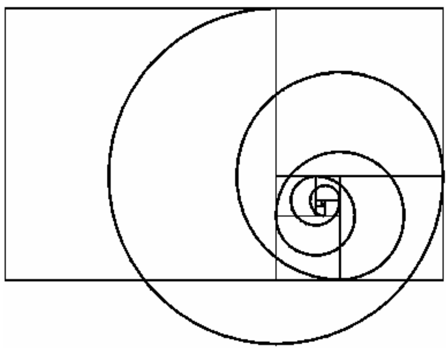

Figure 16

I called this the "wobbly" spiral, since it appears to wobble back and fore. Without the construction squares it looks like this.

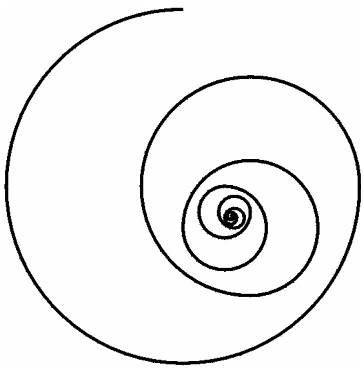

Figure 17

The "wobbly" anti-spiral. Using the technique described above (as in figure 13), it is also possible to create an anti-spiral version of the wobbly spiral, although it does not quite have the wobble you would expect.

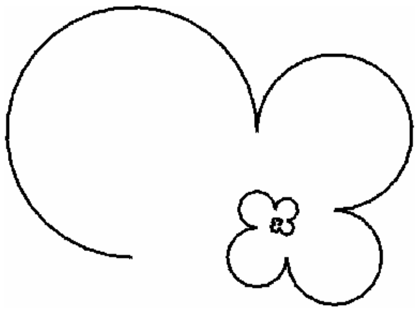

Figure 18

Golden Section spirals and triangles. There are two Golden Section isosceles triangles formed by having two long or two short sides equal, and the only right-angled triangle with its sides in geometric progression is also a Golden Section one. It is also possible to produce Golden Section spirals using equilateral triangles.

The Golden Section triangles and their relationships. For ease of writing in the following, I will call the two parts of the Golden Section division Long and Short and use the first letter of these names to describe the triangles I am working with.

The two types of isosceles triangles having the Golden Section ratio of sides then become LLS and SSL. (An LLS triangle has angles of 72, 72, and 36 degrees; and an SSL triangle of 36, 36, and 108 degrees.) Dividing a side L in each of them in the Golden Section creates more isosceles triangles.

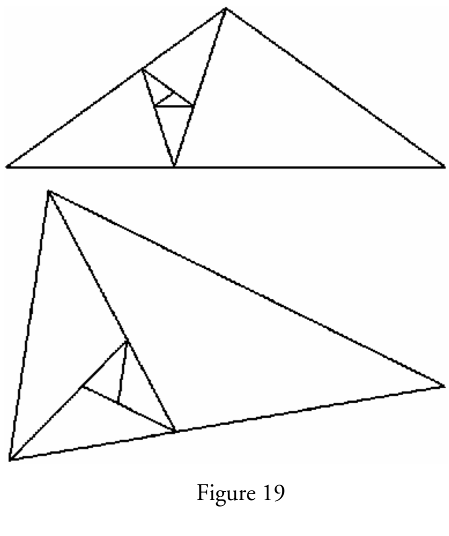

Figure 19

With the SSL triangle at the top of figure 19, each division produces a LLS and another SSL, which when subdivided in the same way creates the spiral of triangles. Similarly the LLS triangle below it when subdivided creates an SSL and another LLS and so on.

It is easier to draw the spiral using arcs of circles with the LLS triangle (figure 20). They are not quarter circles as with the Golden Section rectangle but arcs subtending 108°. The centre and ends of the arcs are clearly defined, with the centre as the division point on one L side and the ends on the opposite L side.

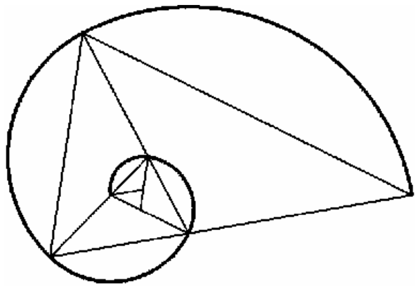

Figure 20

The equivalent arc-spiral on the SSL triangle looks like this:

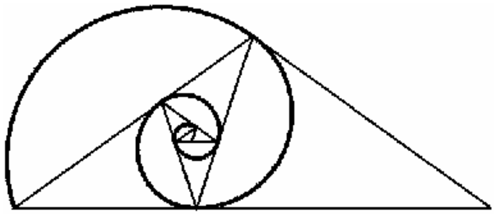

Figure 21

At first glance, the arcs seem to be drawn with centre on the point of division on the L side and ends at the endpoints of the S side. But the rule shown in figure 6 does not apply if this is the case because the line joining the centres does not pass though the joining point of the two arcs. If you try to draw a spiral this way, then the spiral looks odd because it is not smooth.

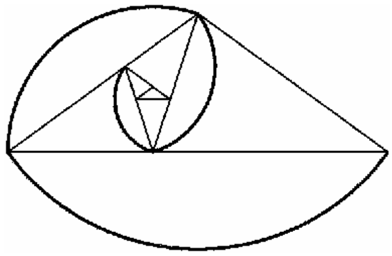

Figure 22

In order to draw the correct spiral, the arcs must be drawn with centre at the 'centroid' (centre of mass) of the corresponding triangle, which is the point of intersection of the angle bisectors of the three angles (actually, just two angle bisectors will suffice); the ends of the arc are at the endpoints of the L side.

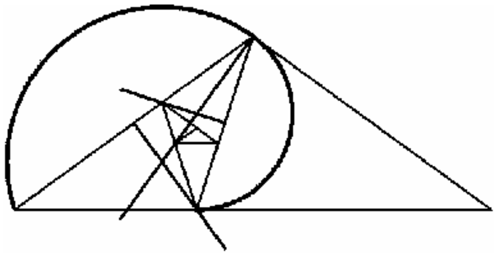

Figure 23

The arcs in this case subtend an angle of 144°. Because it is a more complex diagram, it is well worth drawing and studying how the centres and ends of the arcs are related and how the bisectors come together.

Escher's Golden Section triangle. This triangle does not seem to appear in the Golden Section literature, and I have named it after Escher since it is the only case I can find of his using the Golden Section consciously. It is drawn in his notebooks as a tessellation (see [Schattschneider 1990: 83]). It is a right-angled triangle with sides in the ratio $1:\sqrt{\phi}:\phi$. If you write equations for a right-angled triangle with sides in geometric progression, you see that this is the only such triangle. (Specifically, this requires the sides of the triangle to satisfy the Pythagorean Theorem, $a^2+b^2=c^2$, as well as the geometric progression $a/b=b/c$. Since the scale of the triangle is inconsequential, we can take $a=1$ for convenience, and then solving simultaneously yields $b=\sqrt{\phi}$ and $c=\phi$.)

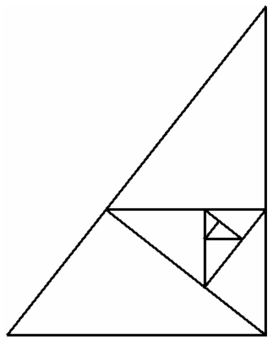

Figure 24

Although you can see a set of spiralling triangles, there is not the symmetry of the standard Golden Section triangles, and thus creating a spiral using circular arcs is not possible.

Double spirals from triangles. The double spiral of equilateral triangles in figure 25 has been drawn using Fibonacci numbers, but the Golden Section can be used directly just as easily. If the drawing were created using the Golden Section, then the repeating parallelogram has sides that are in the Golden ratio. A long side of the parallelogram is divided in the ratio of the square of the Golden ratio.

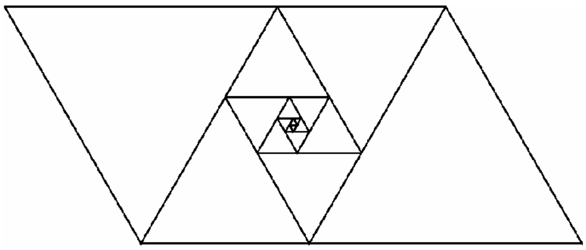

Figure 25

A pair of spirals can be drawn using arcs of circles. The arcs used to create the spiral are drawn as follows. Consider the line on which an arc is drawn as the side of an equilateral triangle. Then the arc is part of the circumcircle of that triangle.

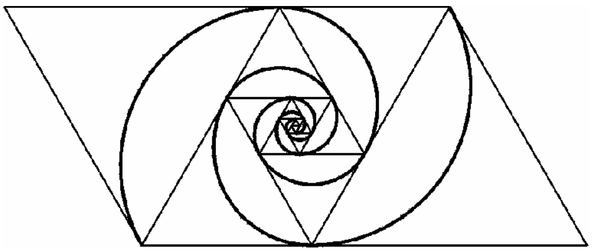

Figure 26

Because triangles tile the plane very easily, it can be adapted to use other triangles, for example with Golden Section isosceles triangles it looks like this:

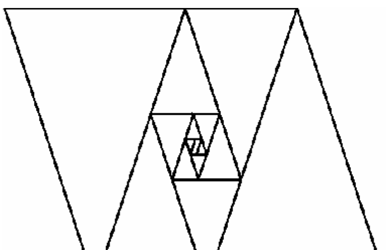

Figure 27

Spirals and the pentagon. The pentagon can be treated in a similar way. For example, the centre of the largest arc in figure 28 is the point marked C.

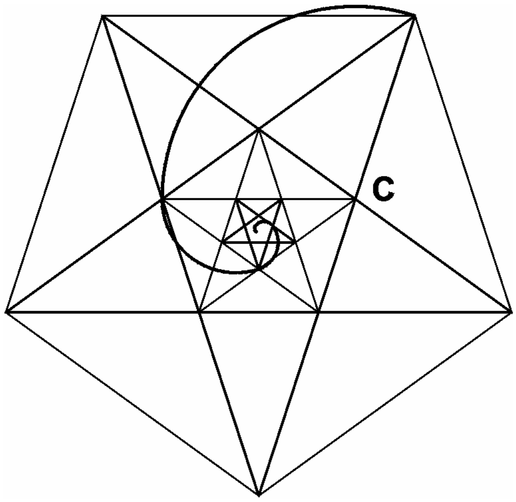

Figure 28

There are a number of spirals that can be drawn in this way. The set of ten spirals is very much like natural forms composed of Golden Section spirals as you would see in the seeds of a sunflower, for example.

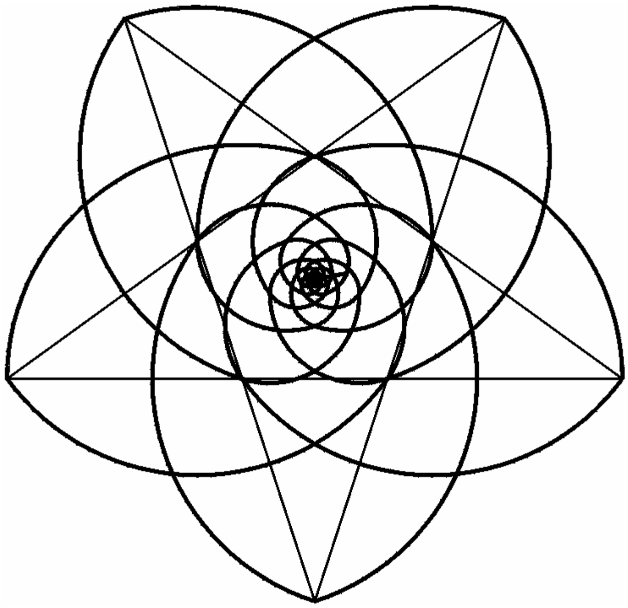

Figure 29

Part 3. Mathematics Of True Golden Section Spirals

The approximate methods, described in part 2, show the structural way of creating the spirals. This section looks at the mathematics of Golden Section spirals as it relates to the approximate methods using arcs of circles in part 2 and shows how to find the equations of the exact spirals.

Mathematics of the true Golden Section spirals. The spirals drawn using arcs of circles are approximations, but each one has a corresponding true spiral — in most cases a logarithmic spiral — that can be represented by an equation. Since they are all generated by using properties of the Golden Section, the question arises as to how many different spirals are there and indeed, if they are the same.

The spiral and the Golden Section rectangle. The spiral drawn using quarter circles in the set of whirling squares is a like a logarithmic spiral since each rotation of 90° means the radius of the circle is multiplied by the Golden Section. (Thus, the multiplication factor for a full rotation of 360° is ϕ^4.) It is not a true logarithmic spiral, however, because each quarter circle of the spiral has a different centre, whereas a logarithmic spiral rotates about a single pole (see the log spiral in part 1). To analyse the mathematics, we have to go back to the whirling squares diagram and look at its spiral similarity.

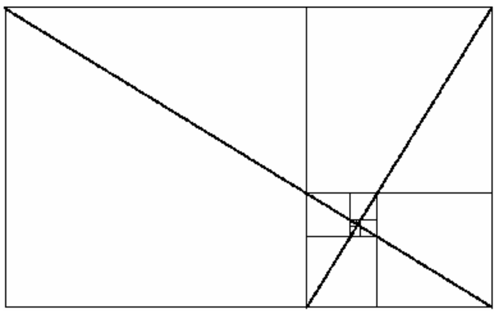

Figure 30

As shown in figure 30, the centre of rotation is found from the whirling squares diagram by intersecting diagonals of the Golden Section rectangles. These two diagonals are at right angles. That the centre is the intersection of successive diagonals of the Golden Section rectangles can be seen by the spiral symmetry of the squares and Golden Section rectangles. Although these diagonals have been drawn for the largest two rectangles, they are also the diagonals of the other Golden Section rectangles.

Knowing this centre, to find the true mathematical spiral for which we have an approximation made up of arcs of circles, there are now two properties that can be used to find its equation:

1. the spiral goes through the point where the arcs of circles meet, that is where the squares are cut off;
2. the spiral is tangent to the side of the rectangle.

These are in fact two problems yielding different results, although, as we shall see, the same spiral.

The essential mathematics of the logarithmic spiral is embodied in its polar equation, previously examined in section 1:

$$r = ae^{\theta \cot \alpha} .$$

Recall that θ is measured in radians, and the constant α defines the angle between the radius r (the line from the pole to a point on the curve) and the tangent, as shown in figure 31.

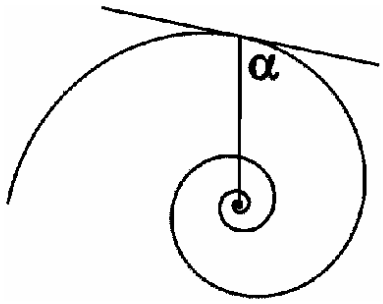

Figure 31

If we know the value of the radius at two different values of θ, we can determine the value of the tangent angle α and thus define the equation.

Adding some more lines to Fig 30 to create the radial vectors to points *D*, *E*, *F* and *G* of the Golden Section spiral gives us figure 32:

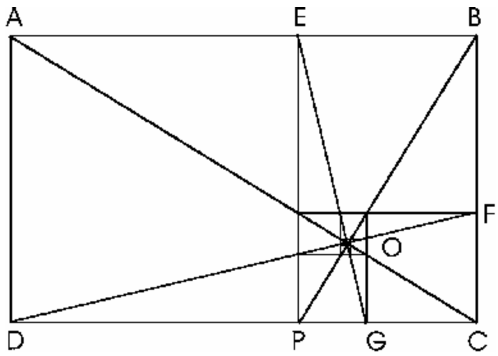

Figure 32

Since *OE* and *DF* are perpendicular, triangles *FOE* and *EOD* are similar; they are right triangles with hypotenuses in the ratio ϕ. Thus

$$\frac{EO}{FO} = \frac{ED}{EF} = \phi$$

so that the radii after 90º (i.e., π /2 radian) rotations are in the ratio of the Golden Section. This means that if we draw a logarithmic spiral through these points, then $r = ae^{\theta \cot\alpha}$ and $\phi r = ae^{(\theta+\pi/2)\cot\alpha}$, which after division and taking natural logs, gives $\cot\alpha = 2(\ln\phi)/\pi$, and hence $\alpha \approx 72.9676$ degrees.

For future calculations, we note the that the formula for α – given the multiplication factor, *M*, for a full 360º (i.e., 2π radian) rotation – may be obtained analogously, yielding $\alpha = \cot^{-1}((\ln M)/2\pi)$; in fact, the equation for the spiral, $r = ae^{\theta \cot\alpha}$, can then be written $r = aM^{\theta/2\pi}$.

While this last equation is simpler, it does not explicitly show α.

Now that we have the equation and we know the pole, we can plot the true spiral through the points where the squares divide the sides of the rectangle, which gives the spiral shown in figure 33.

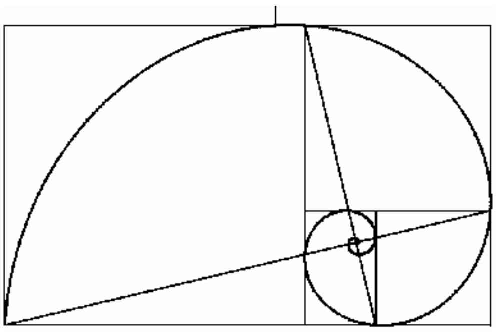

Figure 33

This looks remarkably similar to the one from the arcs of circles, and if the two are included in the same diagram then it is difficult to see the lines apart unless the diagram is magnified and the lines are thin.

The short line sticking up at the top of figure 33 is not a mistake. It shows an important property of the true logarithmic spiral through the points: it goes outside the rectangle, although only slightly. The short line is located at the position where the spiral cuts the rectangle when it returns inside. It is sometimes, but rarely, mentioned in descriptions of this diagram, but I have never seen either a calculation, or description, of how much. The amount it goes out is very small (only 0.165% of the shortest side of the rectangle) and so does not show up without high magnification. The area around the marked point at the top of figure 33 shown in figure 34. The arc of the spiral which is outside the rectangle is barely apparent, and not enough for a pencil drawing to show up.

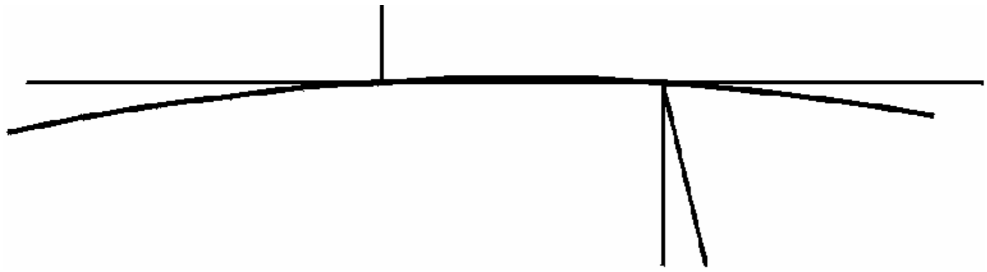

Figure 34

The spiral to touch the rectangle side. We have seen that although the rectangle goes through the points, it does not fit neatly into the rectangle. If the spiral were to touch the sides of the rectangle, the line from the pole would need to make a tangent angle of 72.9676° with the side of the rectangle. So if we take the spiral that goes though the points and rotated it, would it touch all the sides in the same way? It would, because any four radii at right angles from the pole are successively in the ratio of the Golden Section. This may be seen from the following diagram.

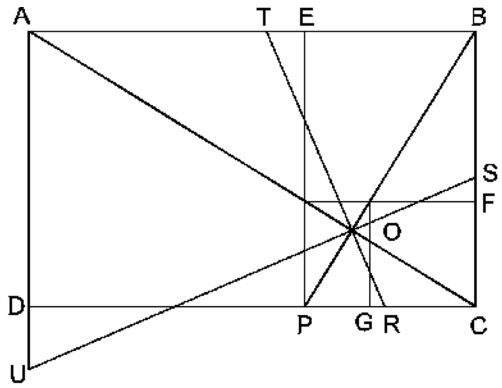

Figure 35

(Note that this diagram is for illustrative purposes and is not an accurate representation of how the spiral is placed in figure 33.)

The right angles triangles *POC*, *COB*, *BOA* and *TOU* are all similar, with hypotenuses in the ratio of the Golden Section, so the other corresponding sides are also in this ratio. If *R*, *S*, *T* and *U* are the points of intersection of any four other radii formed by rotating the lines *DF* and *GE*, then triangles *ROC*, *SOB*, *TOA* and *UOZ* (where *Z* is the intersection point of *BP* extended with *AD* extended) are all similar, so that the sides *OR*, *OS*, *OT* and *OU* are successively in the Golden ratio.

This means that the spiral that touches the four sides of the rectangle is the same one as the one in figure 32, except that it is rotated slightly, so that it touches a little way along the side and not at the point where the vertex square sits. The touching point (the point equivalent to point *U* in figure 35) is 8.228% of the short side of the rectangle (that is the ratio of *DU* to *AB*). The spiral in position then looks like this:

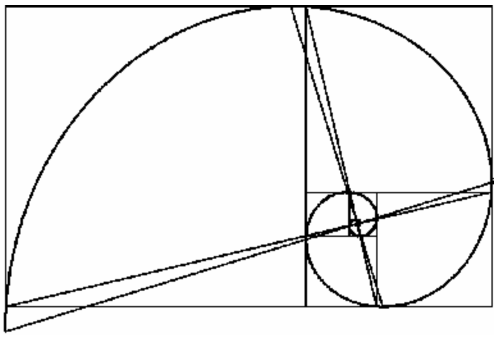

Figure 36

The calculation of the angle of rotation is too detailed to include here, but it is slightly over 3.75°.

The triangular and pentagonal spirals. Similar techniques can be used to find the equations of the spirals from the triangles (figures 20 and 21) and the pentagonal one (figure 28). Finding the centre is not as obvious and requires knowledge of the centre of spiral similarity of a triangle. This is a special point of a triangle known as the Brocard point; for full details see my explanation [Sharp 1999], available on the Association of Teachers of Mathematics website,
(http://www.atm.org.uk/resources/articles/geometry/brocard/point.html).

For the Golden Section spiral for the triangle LLS (figure 20), the radial vector has a ratio of ϕ after a rotation of 108° (and hence $\phi^{10/3}$ after a full rotation, since 360° = (10/3)×108°). This gives the tangent angle $\alpha \approx 75.6788$ degrees.

For the Golden Section spiral for the triangle SSL (figure 21), the radial vector has a ratio of ϕ after a rotation of 144°. This gives the tangent angle $\alpha \approx 79.1609$ degrees.

For the pentagon case (figures 28 and 29), the spiral has a radial vector has a ratio of ϕ^2 after a rotation of 108°. This gives the tangent angle $\alpha \approx 62.9520$ degrees.

Comparing the tangent angle of 72.9676° for the Golden Section spiral described by the rectangle, this shows that these four spirals, while all defined by the Golden Section, are very different.

The equation of the wobbly spiral The approximate methods for drawing spirals in part 2 gives rise to some complicated spirals. The only one whose equation is relatively easy to find is the wobbly spiral for the Golden rectangle. Full details for derivation of the equation are given in [Sharp 1997].

The equation is $r = (1 + 2k\sin(4\theta/3))ae^{k\theta}$, where $k = 2(\ln\phi)/3\pi$.

If you have software for plotting curves, I welcome you to try using this equation.

Part 4. The Myth Of The Nautilus Shell

Having been able to compare the different Golden Section spirals in Part 3, this section looks at the most enduring myth of Golden Section spirals: because a spiral is a logarithmic spiral it is a Golden Section one.

"A *little* knowledge is a dangerous thing." Using the mathematics of the Golden Section spirals in part 3, and the description of spirals in part 1, I now want to show that people with a slight knowledge of mathematics can make leaps of deduction, which can then be perpetuated as myths.

The simplest case of a mistake was one I heard on the radio recently by an artist who had been brought into one of a series of programmes on numbers. In the one on the Golden Section he went on to describe the spiral in the Golden Section rectangle. This is not an easy thing to do on the radio. It was made harder that his train of thought was:

1. The Greeks were the geometers who knew about and discovered the Golden Section.
2. Archimedes is a well-known Greek geometer who has a spiral named after him.
3. Therefore the spiral is a spiral of Archimedes.

This is quite rare. The most common misconception follows the logic:

1. Spirals of shells, particularly the Nautilus are logarithmic spirals.
2. The Golden Section can be used to draw a logarithmic spiral.
3. Therefore shell spirals are related to the Golden Section and in particular, the Nautilus shell spiral is a Golden Section spiral.

I will also describe how a spiral design extrapolated from the rose window of Chartres cathedral has been misattributed in the same way.

One of the amazing things about such misconceptions is that they are so widespread, even by mathematicians who should know better. It is a prime example of why geometry needs to be taught more widely and not only geometry, but the visual appreciation of shape and proportion. This is especially true for the education of artists, graphic designers and architects.

I can explain one way the errors about the Golden Section spiral are perpetuated. This anecdote affected me personally. I wrote an article on Golden Section spirals for a mathematics magazine a few years ago that included a section on why the Nautilus shell was not a Golden Section spiral. The page proofs came back with a heading that was quite bland. I was horrified, however, when I received my sample copy. The designer had drawn a Nautilus shell on the cover and put the same drawing at the heading of the article. The editorial in the next issue had an interesting apology and retraction. However, readers do not always see the next issue.

Why the Nautilus shell is NOT a Golden Section spiral. The mathematics of the Golden Section spirals in part 3 allows the spirals to be quantified. We have seen that this has enabled us to say that each one is different. They are all logarithmic/equiangular spirals, but the tangent angle α is different.

The Nautilus shell is a logarithmic spiral. Such a shape arises because a growing animal has the same proportions as it grows and the spiral fits the requirement to protect this shape as it gets larger.

When trying to measure a Nautilus shell to determine the shape of the spiral, you can either work from a photograph or a sectioned specimen, which is essentially equivalent (though photos may add distortions, of course). In either case, there are numerous experimental difficulties. For example:

- the section may not be exactly in the right direction, that is it might not be in the right plane;
- the thickness of the shell means that there is a considerable error in deciding where to take the measurement;
- finding the centre (pole) of the spiral is not always easy.

The specimen in figure 37 was measured as follows.

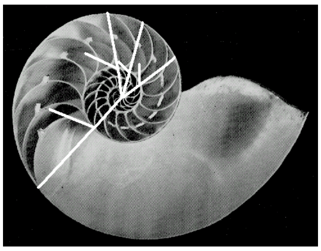

Figure 37

In order to approximate the multiplication factor for the Nautilus logarithmic spiral, measurements were taken for four different 360° rotations of the spiral and the ratio of the radial vectors calculated for each rotation. The light gray lines show the measurement lines. Values obtained were 2.95, 3.02, 2.83, and 2.97, giving an average of 2.94. Although the errors are quite high it shows a value in the region of 3. Other measurements I have seen are also around 3. Since the shell is a living form, making statements other than this is as far as you can go. The creatures are no more uniformly shaped than you or I are.

A logarithmic spiral with a multiplication factor of 3 has a tangent angle of 80.08°. Compare this with the following table.

Golden Section spiral	tangent angle	multiplication factor
rectangle	72.9676°	$\phi^4 \approx 6.8541$
triangle LLS	75.6788°	$\phi^{10/3} \approx 4.9731$
triangle SSL	79.1609°	$\phi^{5/2} \approx 3.3302$
pentagon	62.9520°	$\phi^{20/3} \approx 24.7315$

Note that a small change in tangent angle corresponds to a large change in the multiplication factor. So although the Nautilus ratio is close to the SSL ratio, this is merely a coincidence. It is a very long way from the one for the Golden Section rectangle. In fact, comparing the Golden Section spiral (the left of figure 38) with the logarithmic spiral having a multiplication factor of 3 (the right of figure 38) and the Nautilus in figure 37, it is clear that the Golden Section rectangular spiral and the Nautilus spiral simply do not match. There just are not enough turns with the Golden Section spiral.

Figure 38

Of course, one could specify any multiplication factor and use it to define a specific logarithmic spiral. In this way, one could define other Golden Section spirals, without appealing to any approximate spiral construction, simply by specifying the multiplication factor to be a chosen power of ϕ (e.g., ϕ, ϕ^2, $\phi^{1/2}$, … – the possibilities are endless). With a suitable choice of the power, one could even produce a spiral very close to the Nautilus spiral. This would be quite contrived, however, and it must be stressed that the spiral traditionally associated with the Nautilus is the one corresponding to the Golden Section rectangle (i.e., multiplication factor ϕ^4), which is clearly far from a match.

There is a much longer and more detailed discussion on this subject in [Fonseca 1993]. He also finds the ratio for one turn of the Nautilus as very close to 3. He describes how an artist makes errors in order to justify her assumption that the Nautilus is a Golden Section spiral. It is a classic description of how the Golden Section is misused.

The north window of Chartres. In his excellent work *Rose Windows* [1979], Painton Cowen superimposes a geometrical diagram over the north window of Chartres cathedral and in doing so makes the leap that since the Golden Section can be used to draw a logarithmic spiral, so this is a Golden Section spiral.

The diagram Cowen uses is shown in figure 39. Such a set of spirals bears a superficial resemblance to the sets of pentagon spirals shown in figure 29. He has probably made the visual leap having seen a diagram of the spirals in a sunflower since it looks so close.

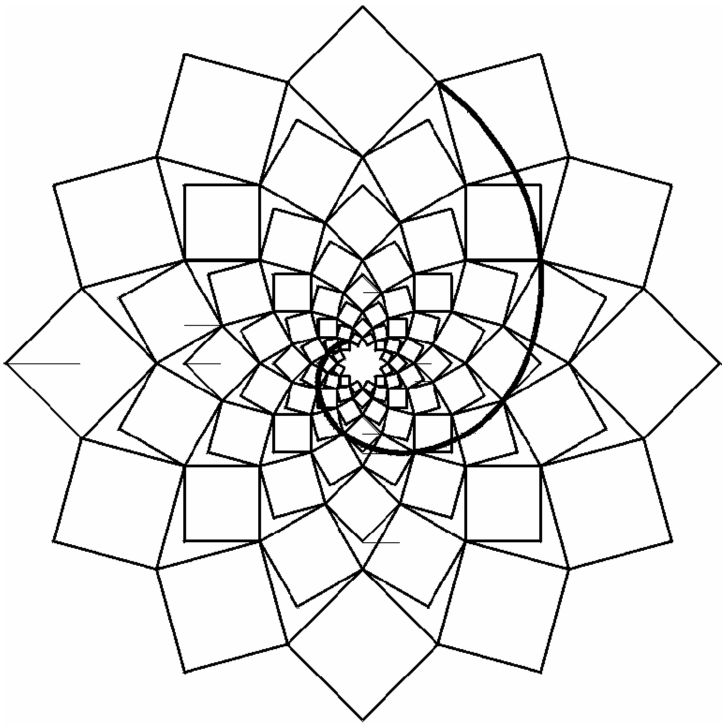

Figure 39

The overall spiral pattern has a twelve-fold symmetry that matches the symmetry of the window, which reflects the twelve apostles. He has drawn many auxiliary spirals like the one shown above, as well. The auxiliary spiral has a multiplication factor of $\left(\sqrt{2}\right)^8 = 16$, which leads to $\alpha \approx 66.1895$ degrees, whereas the spirals of the overall pattern have multiplication factor $\left(\sqrt{2}\right)^{24} = 4096$, which leads to $\alpha \approx 37.0672$ degrees. There is no reason to call either of these spirals a Golden Section spiral.

References

COWEN, PAINTON. 1979. *Rose Windows*. London: Thames and Hudson.

FONSECA, RORY. 1993. Shape and Order in Organic Nature: The Nautilus Pompilius. *Leonardo* **26**: 201-204.

HAMBIDGE, JAY. 1926. *The Elements of Dynamic Symmetry*. Rpt. 1953, New York: Dover.

SCHATTSCHNEIDER, DORIS. 1990. *M C Escher: Visions of Symmetry,* New York: W H Freeman.

SHARP, JOHN. 1997. Golden Section spirals. *Mathematics in School* 26, 5: 8-12.
SHARP, JOHN. 1999. The Brocard point. A response to a Challenge, *Micromath* 15, 3. Republished on the Association of Teachers of Mathematics website, http://www.atm.org.uk/resources/articles/geometry/brocard/point.html

For further reading

LAWRENCE, J. DENNIS. 1972. *A Catalog of Special Plane Curves*. New York: Dover.
LOCKWOOD, E. H. 1967. *A Book of Curves*. Cambridge: Cambridge University Press.
MAOR, ELI. 1994. *e: The story of a number*. Princeton: Princeton University Press.
YATES, ROBERT C. 1947. C*urves and Their Properties*. Rpt. 1974, Washington: National Council of Teachers of Mathematics.

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John Sharp has researched and taught Geometry and Art for over 20 years in Adult Education in and around London. He is the illustrator of David Wells's *Penguin Dictionary of Curious and Interesting Geometry* and has written his own book on modelling geometrical surfaces called *Sliceforms*, some of which are in the "Strange Surfaces" exhibit in the Science Museum in London (http://www.sciencemuseum.org.uk/on-line/surfaces/index.asp).