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A Note on Geometric Embeddings of Simplicial Complexes in a Euclidean Space

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Abstract. In this note we prove that if a simplicial complex K can be embedded geometrically in $\mathbf{R}^m$, then a certain linear system of equations associated with K possesses a small integral solution.

1. Introduction

In this note we obtain several necessary conditions on a simplicial complex for possessing a geometric embedding in $\mathbf{R}^m$. We start by briefly describing the motivation for the problem.

Let K be an n-dimensional simplicial complex. It is well known that any such K can be embedded (even geometrically) in $\mathbf{R}^{2n+1}$. On the other hand not all n-dimensional complexes are embeddable in $\mathbf{R}^{2n}$. Works of van Kampen [14], Flores [6], Shapiro [13], and Wu [15] provide the necessary conditions for an n-dimensional simplicial complex to possess a piecewise-linear embedding in $\mathbf{R}^m$ for $n \le m \le 2n$. These conditions are also sufficient for the case $m = 2n$, $n \neq 2$. (For $n = 1$, $m = 2$ the sufficiency follows easily from Kuratowski's criterion for planarity of graphs. For $n = 2$, $m = 4$ Freedman et al. constructed a two-dimensional simplicial complex for which the van Kampen–Flores conditions hold but which does not admit an embedding into $\mathbf{R}^4$ [7].) A famous consequence of these conditions is that the n-dimensional skeleton of a $(2n+2)$-dimensional simplex cannot be embedded in $\mathbf{R}^{2n}$. For related results see also [11] and [12].

The question we are interested in is when a simplicial complex which is P.L. embeddable in $\mathbf{R}^m$ admits a geometric embedding in $\mathbf{R}^m$.

Definition 1.1. Let f be an embedding of a simplicial complex K in $\mathbf{R}^m$. f is said to be a geometric embedding iff the image of any simplex $\sigma \in K$ is a geometric simplex in $\mathbf{R}^m$.

It is known that embeddability of a complex does not, in general, imply its geometric embeddability. For example, there is a triangulated Möbius strip constructed by Brehm [3] which possesses no geometric embedding in $\mathbf{R}^3$ (not even an embedding in which all edges are straight line segments). For higher-dimensional examples of this kind see [4]. There is also a three-dimensional manifold constructed by Freedman which is topologically embeddable in $\mathbf{R}^4$, but has no triangulation which is geometrically embeddable in $\mathbf{R}^4$. The most recent result in this area is a closed triangulated orientable two-dimensional manifold of genus 6 with 12 vertices found by Bokowski and Guedes de Oliveira [2] that has no geometric embedding in $\mathbf{R}^3$. Moreover, Bokowski and Guedes de Oliveira showed that this manifold possesses no geometric embedding in $\mathbf{R}^3$ even after deleting one specific triangle, thus solving the problem, *whether for every g there exists a triangulated closed orientable* 2*-manifold of genus g which has no geometric embedding in* $\mathbf{R}^3$ (see [1] and [8]), for $g \geq 6$.

The famous Heawood conjecture settled by Ringel and Youngs [10] asserts that for any integer $n \geq 7$ such that $g = (n-3)(n-4)/12 \in \mathbf{Z}$ there is a triangulation of a closed orientable two-dimensional manifold of genus g whose 1-skeleton is a complete graph on n vertices. One of the open conjectures states that if n is sufficiently large, then such two-dimensional complexes do not admit a geometric embedding in $\mathbf{R}^3$. Note that for these manifolds $n = O(g^{1/2})$. On the other hand, the construction due to McMullen et al. [9] shows that for any g there exists a triangulated two-dimensional closed manifold of genus g on $n = O(g/\log g)$ vertices which is geometrically embeddable in $\mathbf{R}^3$.

In this note we associate with every simplicial complex K and integer m a certain linear system of equations. It follows from the result of van Kampen, Flores, Shapiro, and Wu that if K is P.L. embeddable in $\mathbf{R}^m$, then this system possesses an integral solution. We show that if, in addition, K can be embedded geometrically in $\mathbf{R}^m$, then this linear system possesses a small integral solution. The central idea of the proof is that in the case of geometric embeddability the intersection numbers involved in the van Kampen–Flores conditions cannot be arbitrary, but are rather integers with small absolute values. At present, we have no application of this result (even with some computer experimentation), but we hope that it can be useful in attacking the above and similar problems.

The rest of the note is organized as follows: in Section 2 we review the necessary background on obstructions for P.L. embeddability and in Section 3 we state and prove obstructions for geometric embeddability.

2. Obstructions for P.L. Embeddability

In this section we review the necessary background on obstructions for P.L. embeddability. The presentation here relies mostly on Wu's book [15].

Let K be an n-dimensional simplicial complex on the vertex set $\{a_1, \ldots, a_N\}$. (We assume that the simplexes of K are oriented by listing the vertices in increasing order). Two simplexes of K are said to be *nondiagonic* if they have no vertices in common. The *deleted product of* K, $K * K$, is a subcomplex of $K \times K$ consisting of products of pairs of nondiagonic simplexes.

Let $C_q(K*K) = \bigoplus\{\mathbf{Z}(\sigma_1 \times \sigma_2)\colon \sigma_1 \times \sigma_2 \in K*K, \dim(\sigma_1 \times \sigma_2) = q\}$ be a group of q-dimensional chains of $K * K$ with coefficients in $\mathbf{Z}$. The cells of $K * K$ are assumed to be oriented as the product of oriented simplexes. Let

$$\partial(\sigma_1 \times \sigma_2) = \partial\sigma_1 \times \sigma_2 + (-1)^{\dim\sigma_1}\sigma_1 \times \partial\sigma_2$$

be the ordinary boundary operator $\partial\colon\ C_q(K * K) \to C_{q-1}(K * K)$. Let

$$C^q(K * K) = \mathrm{Hom}_{\mathbf{Z}}(C_q(K * K), \mathbf{Z})$$

be the group of q-dimensional cochains of $K * K$ with coefficients in $\mathbf{Z}$ and let δ be the coboundary operator dual to ∂.

Note that there is an involution $t\colon\ K * K \to K * K$ defined by $t(\sigma_1 \times \sigma_2) = (-1)^{\dim\sigma_1 \dim\sigma_2}(\sigma_2 \times \sigma_1)$. Using this involution, we can define the group of q-dimensional antisymmetric cochains of $K * K$ and the group of q-dimensional symmetric cochains:

$$C_a^q(K * K) = \{\lambda \in C^q(K * K)\colon\ t^{\#}\lambda = -\lambda\},$$
$$C_s^q(K * K) = \{\lambda \in C^q(K * K)\colon\ t^{\#}\lambda = \lambda\}.$$

Since the ordinary coboundary operator δ commutes with $t^{\#}$, we can define the groups of cocycles $Z_a^q(K*K)$ and $Z_s^q(K*K)$, groups of coboundaries $B_a^q(K*K)$ and $B_s^q(K*K)$, and cohomology groups $H_a^q(K * K)$ and $H_s^q(K * K)$ in the usual way. Given $m \in \mathbf{N}$ define

$$H^q_{\rho_m}(K * K) = \begin{cases} H_s^q(K * K) & \text{if } m \text{ is even,} \\ H_a^q(K * K) & \text{if } m \text{ is odd.} \end{cases}$$

In the following we assume that $\mathbf{R}^m$ is endowed with a fixed orientation. Let $f\colon K \to \mathbf{R}^m$ be any P.L. map such that $f(\sigma) \cap f(\tau) = \emptyset$ for any $\sigma \times \tau \in \mathrm{Skel}_{m-1}(K * K)$. Define a *special embedding m-cocycle* $\varphi_f = \varphi_f(K)$ as follows:

$$\varphi_f(\sigma_1 \times \sigma_2) = (-1)^{\dim(\sigma_1)} f(\sigma_1) \cdot f(\sigma_2) \qquad \text{for any } m\text{-cell} \quad \sigma_1 \times \sigma_2 \in K * K,$$

where $f(\sigma_1) \cdot f(\sigma_2)$ is the *index of intersection* (or, *intersection number*) of simplexes σ_1 and σ_2 in $\mathbf{R}^m$. (For the definition and basic properties of the intersection number the reader is referred to Wu's book [15], or, for more modern treatment, to [5]. Roughly speaking, the index of intersection is the algebraic (i.e., including orientation) number of points of intersection of two singular cells of complementary dimension in a Euclidean space. In the special case when σ_1 and σ_2 intersect each other transversely and in at most a finite number of points, we can assign to each intersection point (more precisely, to each point $(x, y) \in \sigma_1 \times \sigma_2$ such that $f(x) = f(y)$) a local index of intersection, $s(x, y) = \pm 1$, which depends on the relative position of $f(V_x)$ and $f(V_y)$ in the oriented $\mathbf{R}^m$, where V_x and V_y are small oriented neighborhoods of x in σ_1 and y in σ_2, respectively. In this case the (total) index of intersection is equal to the sum of local indices.)

The following theorem is a version of the van Kampen–Flores theorem (see [13] and [15]).

Theorem 2.1. *For any simplicial complex K, $\varphi_f(K) \in Z^m_{\rho_m}(K * K)$, and so $\varphi_f(K)$ determines a cohomology class, $[\varphi_f(K)]$, in $H^m_{\rho_m}(K * K)$. Moreover, if K is embeddable in $\mathbf{R}^m$, then $\varphi_f(K) \in B^m_{\rho_m}(K * K)$, that is, $[\varphi_f(K)] = 0$ in $H^m_{\rho_m}(K * K)$.*

It turns out that the class $[\varphi_f(K)]$ is independent of a P.L. map f. Indeed, orient $\mathbf{R}^m$ and $\mathbf{R}^{m+1}$ by the ordered systems of coordinates $(x_1, \ldots, x_m)$ and $(x_1, \ldots, x_m, x_{m+1})$, respectively. Let $g\colon K \to \mathbf{R}^m$ be another P.L. map such that $g(\sigma) \cap g(\tau) = \emptyset$ for any $\sigma \times \tau \in \mathrm{Skel}_{m-1}(K * K)$. Define

$$h\colon |K| \times I \to \mathbf{R}^m \times I \qquad \text{(where } I = [0, 1])$$

by

$$h((x, t)) = (tf(x) + (1 - t)g(x), t) \qquad \text{for} \quad x \in |K|, \quad t \in I.$$

By slightly perturbing the vertices of $f(K)$ and $g(K)$, if necessary, we may suppose that they are in general position. We can then define the cochain $\phi = \phi_h$ by

$$\phi(\sigma \times \tau) = h(\sigma \times I) \cdot h(\tau \times I) \qquad \text{for any } (m-1)\text{-cell} \quad \sigma \times \tau \in K * K,$$

where $h(\sigma \times I) \cdot h(\tau \times I)$ is the index of intersection of cells $\sigma \times I$ and $\tau \times I$ in $\mathbf{R}^{m+1}$. (We orient cells $\{\sigma \times I\colon \sigma \in K\}$ by orienting $I = [0, 1]$ from 0 to 1.) Then the following holds (see p. 180 of [15]):

Proposition 2.1. *$\phi \in C^{m-1}_{\rho_m}(K * K)$ and $\delta\phi = \varphi_g - \varphi_f$, and thus the class $[\varphi_f] \in H^m_{\rho_m}$ is independent of the choice of f.*

The following theorem is another (equivalent) version of the van Kampen–Flores theorem (these versions are equivalent by Proposition 2.1).

Theorem 2.2 [13], [15]. *For a simplicial complex K define*

$$\Phi^m(K) = \begin{cases} (1 + t^{\#}) \sum_1 \{(a_{i_0} \ldots a_{i_{m'}}) \times (a_{j_0} \ldots a_{j_{m'}})\} & \text{if} \quad m = 2m', \\ (1 - t^{\#}) \sum_2 \{(a_{i_0} \ldots a_{i_{m'}}) \times (a_{j_0} \ldots a_{j_{m'+1}})\} & \text{if} \quad m = 2m' + 1, \end{cases}$$

where summations $\sum_1, \sum_2$ are computed over all possible sets of indices (i, j) such that $i_0 < j_0 < i_1 < \cdots < i_{m'} < j_{m'}$, and $j_0 < i_0 < j_1 < \cdots < i_{m'} < j_{m'+1}$, respectively. Then $\Phi^m(K) \in Z^m_{\rho_m}(K * K)$. Moreover, if K is embeddable in $\mathbf{R}^m$, then the cohomology class of $\Phi^m(K)$, $[\Phi^m(K)]$, is equal to 0 in $H^m_{\rho_m}(K * K)$.

Proof. Let $a_1, \ldots, a_N$ be the vertices of K. Let $C(m, N)$ be an m-dimensional cyclic polytope with N vertices $\mathbf{c_1}, \ldots, \mathbf{c_N} \in \mathbf{R}^m$ (that is, $\mathbf{c_1} = x(t_1), \ldots, \mathbf{c_N} = x(t_N)$ are the points on the moment curve $x(t) = (t, t^2, \ldots, t^m)$ and $t_1 < t_2 < \cdots < t_N$), and let $g\colon K \to \mathbf{R}^m$ be linear on each simplex of K which maps the vertices of K to the corresponding vertices of $C(m, N)$. It is well known (and easy to check) that $\varphi_g(K) = \pm\Phi^m(K)$, and so the theorem follows from Theorem 2.1. $\square$

3. Obstructions for Geometric Embeddability

In this section we prove that if a simplicial complex K can be embedded geometrically in $\mathbf{R}^m$, then, in addition to Theorem 2.2, the following holds.

Theorem 3.1. *If K can be embedded geometrically in $\mathbf{R}^m$, then there exists a cochain $\lambda \in C^{m-1}_{\rho_m}(K * K)$ such that $\delta(\lambda) = \Phi^m$ and*

$$|\lambda(\sigma_1 \times \sigma_2)| \leq \left\lceil \frac{m}{2} \right\rceil \qquad \textit{for all } (m-1)\textit{-cells} \quad \sigma_1 \times \sigma_2 \in K * K.$$

Proof. Let $f\colon K \to \mathbf{R}^m$ be a geometric embedding of K in $\mathbf{R}^m$ and let $\mathbf{b_1}, \ldots, \mathbf{b_N} \in \mathbf{R}^m$ be the vertices of $f(K)$. Let $C(m, N)$ be a cyclic polytope with vertices $\mathbf{c_1}, \ldots, \mathbf{c_N} \in \mathbf{R}^m$, and let $g\colon K \to \mathbf{R}^m$ map all vertices of K to the corresponding vertices of $C(m, N)$ and be linear on all simplexes of K. Since slight perturbations of vertices do not change the combinatorics of intersections, we can assume without loss of generality that $\mathbf{b_1}, \ldots, \mathbf{b_N}, \mathbf{c_1}, \ldots, \mathbf{c_N}$ are generic.

Define $h = h(f, g)$ and ϕ_h as in Section 2:

$$h\colon |K| \times I \to \mathbf{R}^m \times I,$$
$$h((x,t)) = (tf(x) + (1-t)g(x), t) \qquad \text{for} \quad x \in |K|, \quad t \in I;$$

$$\phi_h(\sigma' \times \tau') = h(\sigma' \times I) \cdot h(\tau' \times I) \qquad \text{for} \quad \sigma' \times \tau' \in K * K, \qquad \dim(\sigma' \times \tau') = m-1.$$

By Proposition 2.1, $\phi_h \in C^{m-1}_{\rho_m}(K * K)$. Moreover, since f is an embedding, the special embedding cocycle of f, φ_f, is equal to 0, and so Proposition 2.1 implies that

$$\delta(\phi_h) = \varphi_g - \varphi_f = \pm\Phi^m - 0 = \pm\Phi^m.$$

Thus, setting

$$\lambda = \begin{cases} \phi_h & \text{if} \quad \delta(\phi_h) = \Phi^m, \\ -\phi_h & \text{otherwise}, \end{cases}$$

to complete the proof, it is sufficient to show that

$$|h(\sigma' \times I) \cdot h(\tau' \times I)| \leq \left\lceil \frac{m}{2} \right\rceil \qquad \text{for any } (m-1)\text{-cell} \quad \sigma' \times \tau' \in K * K.$$

The proof of this fact relies on two lemmas. The first lemma estimates the number of intersection points of $h(\sigma' \times I)$ with $h(\tau' \times I)$, and the second lemma computes the sign of the (local) index of intersection at each such point.

Lemma 3.1. *For any $(m-1)$-cell $\sigma' \times \tau' \in K * K$ the number of pairs*

$$((x,s),(y,t)) \in (\sigma' \times I) \times (\tau' \times I),$$

such that $h(x,s) = h(y,t)$ is not greater than m.

Proof. Let $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m}$ be the standard basis of $\mathbf{R}^m$. Given two simplexes

$$\sigma' = (a_{i_0}, a_{a_1}, \ldots, a_{i_k}) \quad \text{and} \quad \tau' = (a_{i_{k+1}}, \ldots, a_{i_m})$$

choose an affine transformation $A = A_{\sigma',\tau'}\colon \mathbf{R}^m \times \mathbf{R} \to \mathbf{R}^m \times \mathbf{R}$ which maps

$$(\mathbf{c_{i_0}}, 0), (\mathbf{c_{i_1}}, 0), \ldots, (\mathbf{c_{i_m}}, 0), (\mathbf{b_{i_0}}, 1) \text{ to } (\mathbf{0}, 0), (\mathbf{e_1}, 0), (\mathbf{e_2}, 0), \ldots, (\mathbf{e_m}, 0), (\mathbf{0}, 1),$$

respectively. Then A is a bijective map which preserves $\mathbf{R}^m \times \{0\}$ and $\mathbf{R}^m \times \{1\}$. Thus, it is sufficient to show that

$$|\{((x,s),(y,t)) \in (\sigma' \times I) \times (\tau' \times I)\colon\ A \circ h(x,s) = A \circ h(y,t)\}| \le m.$$

Let $\{(\mathbf{0}, 1), (\mathbf{d_1}, 1), (\mathbf{d_2}, 1), \ldots, (\mathbf{d_k}, 1)\}$ and $\{(\mathbf{d_{k+1}}, 1), \ldots, (\mathbf{d_m}, 1)\}$ be the vertices of $A \circ h(\sigma' \times \{1\})$ and $A \circ h(\tau' \times \{1\})$, respectively (here $\mathbf{d_i} = (d_{i1}, \ldots, d_{im}) \in \mathbf{R}^m$). Since the vertices of $A \circ h(\sigma' \times \{0\})$ and $A \circ h(\tau' \times \{0\})$ are $\{(\mathbf{0}, 0), (\mathbf{e_1}, 0), \ldots, (\mathbf{e_k}, 0)\}$ and $\{(\mathbf{e_{k+1}}, 0), \ldots, (\mathbf{e_m}, 0)\}$, respectively, we obtain that if $A \circ h(x,s) = A \circ h(y,t)$, then $s = t$ $(0 < s < 1)$ and the barycentric coordinates of x in σ' and of y in τ' (we denote them by $\alpha_0, \alpha_1, \ldots, \alpha_k$ and $\beta_{k+1}, \ldots, \beta_m$, respectively; here $k = \dim \sigma'$) satisfy the following equation:

$$s \sum_{i=1}^{k} \alpha_i \mathbf{d_i} + (1-s) \sum_{i=1}^{k} \alpha_i \mathbf{e_i} = s \sum_{j=k+1}^{m} \beta_j \mathbf{d_j} + (1-s) \sum_{j=k+1}^{m} \beta_j \mathbf{e_j}. \tag{1}$$

Let $u = 1 - 1/s$. We can rewrite (1) as

$$\sum_{i=1}^{k} \alpha_i (\mathbf{d_i} - u\mathbf{e_i}) - \sum_{j=k+1}^{m} \beta_j (\mathbf{d_j} - u\mathbf{e_j}) = 0 \tag{2}$$

(where $\sum_{j=k+1}^{m} \beta_j = 1$, since $\beta_{k+1}, \ldots, \beta_m$ are barycentric coordinates). Hence u is an eigenvalue of matrix

$$D = ((\mathbf{d_1})^\top, \ldots, (\mathbf{d_m})^\top)$$

and $(\alpha_1, \ldots, \alpha_k, -\beta_{k+1}, \ldots, -\beta_m)^\top$ is its eigenvector.

Since $\mathbf{b_1}, \ldots, \mathbf{b_N}, \mathbf{c_1}, \ldots, \mathbf{c_N}$ are generic, all eigenvalues of D are simple, and so for each real eigenvalue of D, the dimension of the corresponding eigenspace is equal to 1. Therefore, for each real eigenvalue of D there is at most one eigenvector $(\alpha_1, \ldots, \alpha_k, -\beta_{k+1}, \ldots, -\beta_m)^\top$ satisfying $\sum \beta_j = 1$. Since $x \in \sigma'$ and $y \in \tau'$ are uniquely determined by $\alpha_1, \ldots, \alpha_k$ and $\beta_{k+1}, \ldots, \beta_m$, respectively, and $s = t$ is uniquely determined by u, we obtain that the number of pairs $((x,s),(y,t))$ such that $h(x,s) = h(x,t)$ is not larger than the total number of eigenvalues of $m \times m$ matrix D, which is equal to m. $\square$

Notation.

1. Let $\tilde{D}(u) = D - uI_m$ (where I_m is the identity matrix and D is the same matrix as in Lemma 3.1). Let $P_D(u) = \det \tilde{D}(u)$ be the value of the characteristic polynomial of D at u, and let D_i^u $(i = 1, \ldots, m)$ be the ith column of $\tilde{D}(u)$.

2. Given matrix B, denote by B_{-i} the matrix obtained from B by deleting the ith row, and by $B_{-i,-j}$ the matrix obtained from B by deleting the ith row and the jth column.

Let σ', τ', and $A = A_{\sigma',\tau'}$ be as in Lemma 3.1 and let $u = 1 - 1/s$.

Lemma 3.2. *For a fixed orientation of* $\mathbf{R}^{m+1}$ *there exists* $\tilde{\varepsilon} = \varepsilon(\sigma', \tau') \in \{\pm 1\}$ *such that for each point* $((x, s), (y, s)) \in (\sigma' \times I) \times (\tau' \times I)$ *satisfying* $A \circ h(x, s) = A \circ h(y, s)$ *the index of intersection of* $\sigma' \times I$ *and* $\tau' \times I$ *at this point has the same sign as* $\tilde{\varepsilon} \cdot P_D'(u)$ (*where* $P_D'(u)$ *is the derivative of* P_D *at* $u = 1 - 1/s$).

Proof. If $(x, t) \in \sigma' \times I$ and $(\xi_0, \xi_1, \ldots, \xi_k)$ are the barycentric coordinates of x in σ', then (x, t) is uniquely determined by $\xi_1, \ldots, \xi_k, t$. In these coordinates, the restriction of the map $A \circ h$ to $\sigma' \times I$ (we denote this map by U), is given by

$$(\xi_1, \ldots, \xi_k, t) \overset{U}{\mapsto} \left(t \sum_{i=1}^{k} \xi_i \mathbf{d_i} + (1 - t) \sum_{i=1}^{k} \xi_i \mathbf{e_i},\ t \right).$$

Similarly, if $(y, t) \in \tau' \times I$ and $(\mu_{k+1}, \ldots, \mu_m)$ are the barycentric coordinates of y in τ', then (y, t) is uniquely determined by $\mu_{k+1}, \ldots, \mu_{m-1}, t$. In these coordinates, the restriction of the map $A \circ h$ to $\tau' \times I$ (we denote this map by V) is given by

$$(\mu_{k+1}, \ldots, \mu_{m-1}, t) \overset{V}{\mapsto} \left(t \left(\sum_{j=k+1}^{m-1} \mu_j (\mathbf{d_j} - \mathbf{d_m}) + \mathbf{d_m} \right) + (1 - t) \left(\sum_{j=k+1}^{m-1} \mu_j (\mathbf{e_j} - \mathbf{e_m}) + \mathbf{e_m} \right),\ t \right).$$

Since U and V are smooth maps, there exists $\varepsilon \in \{\pm 1\}$ such that for each point $((\alpha_1, \ldots, \alpha_k, s), (\beta_{k+1}, \ldots, \beta_{m-1}, s))$ satisfying $U(\overline{\alpha}, s) = V(\overline{\beta}, s)$, the sign of the index of intersection at this point is equal to the sign of

$$\varepsilon \cdot \det \left(\left(\frac{\partial U}{\partial \xi_1} \right)^{\top}, \ldots, \left(\frac{\partial U}{\partial \xi_k} \right)^{\top}, \left(\frac{\partial U}{\partial t} \right)^{\top}, \left(\frac{\partial V}{\partial \mu_{k+1}} \right)^{\top}, \ldots, \left(\frac{\partial V}{\partial \mu_{m-1}} \right)^{\top}, \left(\frac{\partial V}{\partial t} \right)^{\top} \right), \tag{3}$$

where the first $k + 1$ derivatives are calculated at $(\alpha_1, \ldots, \alpha_k, s)$ and the last $m - k$ are calculated at $(\beta_{k+1}, \ldots, \beta_{m-1}, s)$.

Let $\beta_m = 1 - \sum_{j=k+1}^{m-1} \beta_j$ and let $u = 1 - 1/s$. A straightforward calculation shows that

$$\left(\frac{\partial U}{\partial \xi_i} (\overline{\alpha}, s) \right)^{\top} = s \begin{pmatrix} D_i^u \\ 0 \end{pmatrix}, \qquad \left(\frac{\partial V}{\partial \mu_j} (\overline{\beta}, s) \right)^{\top} = s \begin{pmatrix} D_j^u - D_m^u \\ 0 \end{pmatrix}, \qquad \frac{\partial V_{m+1}}{\partial t} = 1,$$

and

$$\begin{aligned} \frac{\partial U}{\partial \tau} (\overline{\alpha}, s) - \frac{\partial V}{\partial \tau} (\overline{\beta}, s) &= \left(\sum_{i=1}^{k} \alpha_i (\mathbf{d_i} - \mathbf{e_i}) - \sum_{j=k+1}^{m} \beta_j (\mathbf{d_j} - \mathbf{e_j}), 0 \right) \\ &\overset{\text{by (1)}}{=} -\frac{1}{s} (\alpha_1, \ldots, \alpha_k, -\beta_{k+1}, \ldots, -\beta_m, 0). \end{aligned}$$

Substituting these results in (3), and using the properties of determinant (together with the fact that $s \in (0, 1)$, and so $s > 0$), we obtain that the sign of the index of intersection at $((\overline{\alpha}, s), (\overline{\beta}, s))$ is equal to the sign of

$$\begin{aligned}
&\varepsilon \cdot \det \left(\left(\frac{\partial U}{\partial \xi_1} \right)^\top, \ldots, \left(\frac{\partial U}{\partial \xi_k} \right)^\top, \left(\frac{\partial U}{\partial t} - \frac{\partial V}{\partial t} \right)^\top, \left(\frac{\partial V}{\partial \mu_{k+1}} \right)^\top, \ldots, \left(\frac{\partial V}{\partial \mu_{m-1}} \right)^\top \right)_{-(m+1)} \\
&\qquad = (-1)^k \varepsilon \left(\sum_{i=1}^{k} (-1)^i \alpha_i \det B_{-i} + \sum_{j=k+1}^{m} (-1)^j (-\beta_j) \det B_{-j} \right), \qquad (4)
\end{aligned}$$

where B is an $(m-1) \times m$ matrix $(D_1^u, \ldots, D_k^u, D_{k+1}^u - D_m^u, \ldots, D_{m-1}^u - D_m^u)$.

Since $U(\overline{\alpha}, s) = V(\overline{\beta}, s)$, the numbers $u, \alpha_1, \ldots, \alpha_k, \beta_{k+1}, \ldots, \beta_m$ satisfy (2). Using the fact that $\sum_{j=k+1}^m \beta_j = 1$, we can rewrite (2) as

$$\sum_{i=1}^{k} \alpha_i D_i^u + \sum_{i=k+1}^{m-1} (-\beta_i)(D_i^u - D_m^u) = D_m^u. \qquad (5)$$

Deleting the ith row from this system and solving the remaining system using Cramer's rule we conclude that

$$\begin{aligned}
&\alpha_i \det B_{-i} \\
&\qquad = \det \left(D_1^u, \ldots, D_{i-1}^u, D_m^u, D_{i+1}^u, \ldots, D_k^u, D_{k+1}^u - D_m^u, \ldots, D_{m-1}^u - D_m^u \right)_{-i} \\
&\qquad = (-1)^{m-i-1} \det \left(\tilde{D}(u)_{-i,-i} \right) \qquad \text{for} \quad i = 1, \ldots, k, \qquad (6)
\end{aligned}$$

$$-\beta_i \det B_{-i} = (-1)^{m-i-1} \det \left(\tilde{D}(u)_{-i,-i} \right) \qquad \text{for} \quad i = k+1, \ldots, m-1, \qquad (7)$$

and (deleting the mth row)

$$\begin{aligned}
&\beta_m \det B_{-m} \\
&\qquad = \left(1 - \sum_{i=k+1}^{m-1} \beta_i \right) \det B_{-m} = \det B_{-m} + \sum_{i=k+1}^{m-1} (-1)^{m-i-1} \det \left(\tilde{D}(u)_{-m,-i} \right) \\
&\qquad = \det \left(D_1^u, \ldots, D_k^u, D_{k+1}^u - D_m^u, \ldots, D_{m-1}^u - D_m^u \right)_{-m} \\
&\qquad\quad + \sum_{i=k+1}^{m-1} (-1)^{m-i-1} \det \left(D_1^u, \ldots, D_k^u, \ldots, D_{i-1}^u, D_{i+1}^u, \ldots, D_m^u \right)_{-m} \\
&\qquad = \det \left(D_1^u, \ldots, D_{m-1}^u \right)_{-m} = \det \left(\tilde{D}(u)_{-m,-m} \right). \qquad (8)
\end{aligned}$$

(The penultimate equality follows from the multilinearity of determinant). Substituting (6)–(8) in (4) we obtain that the sign of the index of intersection at $((\overline{\alpha}, s), (\overline{\beta}, s))$ is equal to the sign of

$$(-1)^{k+m-1} \varepsilon \cdot \sum_{i=1}^{m} \det \left(\tilde{D}(u)_{-i,-i} \right) = (-1)^{k+m} \varepsilon \cdot P'_D(u) = \tilde{\varepsilon} \cdot P'_D(u). \qquad \square$$

To complete the proof of Theorem 3.1 note that for any two consecutive real roots u', u'' of the polynomial $P_D(u)$, the numbers $P'_D(u')$ and $P'_D(u'')$ have opposite signs. Therefore, it follows from Lemmas 3.1 and 3.2 that among the points $\{((x, s), (y, t)) \in (\sigma' \times I) \times (\tau' \times I)\colon\ h(x, s) = h(y, t)\}$ there are at most $\lceil m/2\rceil$ points at which the index of intersection is equal to $+1$ and at most $\lceil m/2\rceil$ points at which the index of intersection is equal to -1. Thus, the (total) index of intersection

$$|\phi_h(\sigma' \times \tau')| = |h(\sigma' \times I) \cdot h(\tau' \times I)| \leq \left\lceil \frac{m}{2} \right\rceil. \qquad \square$$

We now present alternative statements of Theorems 2.2 and 3.1. Given a simplicial complex K on the vertex set $a_1, \ldots, a_N$ and an integer m, we can associate with K and m the following linear system of equations: for every $(m-1)$-cell $\sigma' \times \tau' \in K * K$ there is a variable $x_{\sigma',\tau'}$ and an equation

$$x_{\sigma',\tau'} = (-1)^{(\dim\sigma'+1)(\dim\tau'+1)} x_{\tau',\sigma'}, \tag{9}$$

and for every m-cell $\sigma \times \tau \in K * K$, $\sigma = (a_{i_0}, \ldots, a_{i_k})$, $\tau = (a_{j_0}, \ldots, a_{j_{m-k}})$ with $k < m-k$, or $k = m-k$ and $(i_0, \ldots, i_k) <_{lex} (j_0, \ldots, j_{m-k})$ there is an equation $A_{\sigma,\tau}$:

$$\sum_{l=0}^{k} (-1)^l x_{(a_{i_0},\ldots,\widehat{a_{i_l}},\ldots,a_{i_k}),\tau} + (-1)^k \sum_{l=0}^{m-k} (-1)^l x_{\sigma,(a_{j_0},\ldots,\widehat{a_{j_l}},\ldots,a_{j_{m-k}})}$$
$$= \begin{cases} 1 & \text{if} \quad m = 2k \quad \text{and} \quad i_0 < j_0 < \cdots < i_k < j_k, \\ 1 & \text{if} \quad m = 2k+1 \quad \text{and} \quad j_0 < i_0 < j_1 < \cdots < i_k < j_{k+1}, \\ 0 & \text{otherwise}. \end{cases} \tag{10}$$

In terms of this linear system Theorems 2.2 and 3.1 are equivalent to:

Corollary 3.1. *Let K be a simplicial complex.*

1. *If K is embeddable in $\mathbf{R}^m$, then the linear system associated with K and m has an integral solution.*
2. *If K is geometrically embeddable in $\mathbf{R}^m$, then the linear system associated with K and m has an integral solution with all variables having the absolute value of less than $\lceil m/2\rceil + 1$.*

Remarks. 1. Note that if we renumber the vertices of a simplicial complex, then in general we will obtain another system of linear equations. While the existence of an integral solution for one of these systems does imply the existence of an integral solution for the other (as follows from Proposition 2.1), we do not know whether the existence of a *small* integral solution for one of these systems implies the existence of a small integral solution for the other as well.

2. In order to prove that a certain complex K is geometrically nonembeddable in $\mathbf{R}^m$, using the criterion of Corollary 3.1, one has to show that the corresponding linear system has no solution in $\{0, \pm 1, \ldots, \pm\lceil m/2\rceil\}$. In particular, in the case of $m = 3$, one should check whether there are solutions in $\{0, \pm 1, \pm 2\}$. Lutz executed such a computer check for two complexes known to be geometrically nonembeddable in $\mathbf{R}^3$:

Brehm's Möbius strip [3] and the triangulated closed two-dimensional manifold of genus 6 [2]. Unfortunately, it turned out that in both of these cases the corresponding linear systems possess a solution in $\{0, \pm 1, \pm 2\}$. Thus our criterion fails to prove geometric nonembeddability in these cases (at least, using one particular numbering of vertices.)

3. It is interesting to note that the same proof as in Theorem 3.1 shows that if K is any simplicial complex (embeddable or not embeddable in $\mathbf{R}^m$), then there exists a cochain $\tilde{\lambda} \in C^{m-1}_{\rho_m}(K * K)$ such that $\delta(\tilde{\lambda}) = 2\Phi^m$ and

$$|\tilde{\lambda}(\sigma_1 \times \sigma_2)| \leq \left\lceil \frac{m}{2} \right\rceil \qquad \text{for any } (m-1)\text{-cell} \quad \sigma_1 \times \sigma_2 \in K * K.$$

In other words, if $A\mathbf{x} = \mathbf{b}$ is the linear system associated with K and m, then the system $A\mathbf{x} = 2\mathbf{b}$ does possess a small integral solution.

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