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Bounds for Generalized Thrackles

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Abstract. A thrackle (resp. generalized thrackle) is a drawing of a graph in which each pair of edges meets precisely once (resp. an odd number of times). For a graph with n vertices and m edges, we show that, for drawings in the plane, $m \leq \frac{3}{2}(n-1)$ for thrackles, while $m \leq 2n-2$ for generalized thrackles. This improves theorems of Lovász, Pach, and Szegedy. The paper also examines thrackles in the more general setting of drawings on closed surfaces. The main result is: a bipartite graph G can be drawn as a generalized thrackle on a closed orientable connected surface if and only if G can be embedded in that surface.

Introduction

Let G be a finite graph with n vertices and m edges, and suppose that G is simple; that is, it has no loops or multiple edges. A *thrackle* of G is a drawing $\mathcal{T}(G)$ of G in the plane, where the edges are represented by Jordan arcs, such that each pair of edges meets precisely once, either at a vertex or at a proper crossing. (See [LPS] for definitions of *drawing* and *proper crossing*. Thrackles are mentioned in [CFG] and [PA].) Conway's celebrated thrackle conjecture is: $m \leq n$ (see [Wo1], [Wo2], [GR], [PRS], and [Ri]). A natural generalization of the notion of a thrackle is obtained by relaxing the condition that each pair of edges meets precisely once, and assuming only that each pair of edges meets an odd number of times. This gives rise to the notion of a *generalized thrackle* [Wo2]. Lovász et al. proved:

Theorem 1 [LPS].

(a) *for thrackles*, $m \leq 2n-3$,
(b) *for generalized thrackles*, $m \leq 3n-4$,
(c) *a bipartite graph can be drawn as a generalized thrackle if and only if it is planar.*

We give the following improvement:

Theorem 2.

(a) *for thrackles,* $m \leq \frac{3}{2}(n-1)$,
(b) *for generalized thrackles,* $m \leq 2n-2$.

We give examples below which show that the bound in Theorem 2(b) is sharp. In fact, our main focus in this paper is the study of *generalized thrackles* on surfaces of arbitrary genus, in the obvious sense. It was proved in [GR] that every finite graph can be thrackled on some surface. Our main result is:

Theorem 3. *A bipartite graph G can be drawn as a generalized thrackle on a closed orientable connected surface M_g of genus g if and only if G can be embedded in M_g.*

This has the corollary:

Corollary. *For a bipartite generalized thrackle on M_g, one has $m \leq 2n - 4 + 4g$.*

This bound is sharp: for example, the minimal genus embedding of the complete bipartite graph $K_{2p,2q}$ has $m = 2n - 4 + 4g$, by Ringel's theorem (see Theorem 4.5.3 of [GT]).

The strategy employed in the proof of our results is to use $\mathbb{Z}_2$-intersection forms, and to reduce the problem to that of bipartite generalized thrackles. Thus the arguments are entirely about generalized thrackles. The additional improvement for thrackles is due solely to the fact that in the plane, thrackles have no 4-cycles [Wo1]. The main ideas in this paper are most easily described for generalized thrackles in the plane; here we show that Conway doubling on an odd cycle produces a bipartite graph G' (see Lemma 2). Furthermore, G' can be embedded in the plane so that the even cycle, resulting from the Conway doubling, bounds a face in the associated cellular decomposition (see Lemma 4).

The paper is organized as follows. In Section 1 we recall some facts about $\mathbb{Z}_2$-intersection forms and the Conway doubling procedure. In Section 2 we prove Theorem 3, and we give examples which show that Theorem 3 does not extend to arbitrary graphs. In Section 3 we obtain Theorem 2 and the corollary as special cases of Theorem 4, which is a slightly stronger result, and we give examples which show that the bound of Theorem 2(b) is sharp. The paper concludes in Section 4 with some remarks.

In what follows, M_g denotes a closed oriented connected surface of genus g and G is a finite simple graph with n vertices and m edges.

1. Intersection Homology and Conway Doubling

First recall that the *intersection form* on M_g is the unique nondegenerate bilinear map

$$\Omega_{M_g}\colon\ H_1(M_g, \mathbb{Z}_2) \times H_1(M_g, \mathbb{Z}_2) \to \mathbb{Z}_2$$

having the following property: if γ_1 and γ_2 are closed curves in M_g which intersect in a finite number k of transverse crossings, then $\Omega_{M_g}(\gamma_1, \gamma_2) = k \bmod 2$. Clearly, Ω_{M_g} is symmetric and $\Omega_{M_g}(\gamma, \gamma) = 0$ for all closed curves γ. (See [ST] or [DFN] for further details. See [Fu] for an introductory account of intersection forms with values in $\mathbb{Z}$.)

Suppose that $\mathcal{T}\colon G \to M_g$ is a drawing of G, and that c_1 and c_2 are cycles in G. Recall that if v is a vertex of G, then the *rotation diagram* at $\mathcal{T}(v)$ is the cyclic order of the edges of G incident to v determined by $\mathcal{T}$ and the orientation of M_g. The *rotation system* of $\mathcal{T}$ is the set of rotation diagrams of the vertices of G (see [GT]). We want to relate $\Omega_{M_g}(\mathcal{T}(c_1), \mathcal{T}(c_2))$ to the rotation system of $\mathcal{T}$. The following notion depends only on the rotation system:

Definition 1. The *crossing* number $\sigma_{\mathcal{T}}(c_1, c_2)$ is defined as follows: choose an orientation for c_1 and c_2, and consider the set S of vertices in the boundary of $c_1 \cap c_2$. Let $v \in S$. If v is isolated in $c_1 \cap c_2$, set

$$\sigma_{\mathcal{T}}(v) = \begin{cases} 1 & \text{if } c_1 \text{ and } c_2 \text{ cross transversally at } v, \\ 0 & \text{otherwise.} \end{cases}$$

If v is not isolated in $c_1 \cap c_2$, set

$$\sigma_{\mathcal{T}}(v) = \begin{cases} \frac{1}{2} & \text{if } c_2 \text{ is positively oriented at } v \text{ with respect to } c_1, \\ -\frac{1}{2} & \text{if } c_2 \text{ is negatively oriented at } v \text{ with respect to } c_1, \end{cases}$$

where the sign convention is shown in Fig. 1. Then $\sigma_{\mathcal{T}}(c_1, c_2) = \sum_{v \in S} \sigma_{\mathcal{T}}(v) \bmod(2)$.

Let l denote the $\mathbb{Z}_2$-length function on the 1-chain complex of G; that is, given a path c in G, $l(c)$ is the number mod(2) of edges in c. The following result may be regarded as a generalization of Lemmas 2.2 and 2.3 of [LPS].

Lemma 1. *Suppose that* $\mathcal{T}\colon G \to M_g$ *is a generalized thrackle, and that* c_1 *and* c_2 *are cycles in* G. *Then* $\Omega_{M_g}(\mathcal{T}(c_1), \mathcal{T}(c_2)) = \sigma_{\mathcal{T}}(c_1, c_2) + l(c_1) \cdot l(c_2) + l(c_1 \cap c_2)$ (mod 2).

Proof. Divide the edges of c_1 into four disjoint subsets: (a) k_1 edges contained in $c_1 \cap c_2$, (b) k_2 edges which are not incident with $c_1 \cap c_2$, (c) k_3 edges not contained in $c_1 \cap c_2$

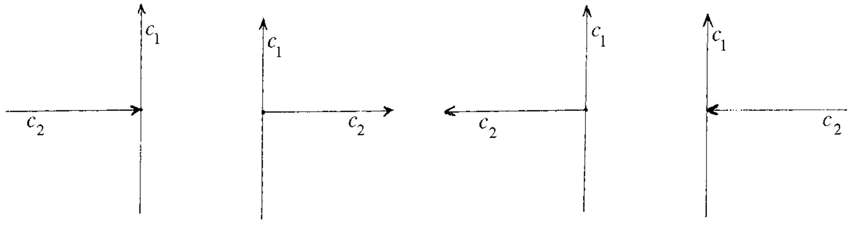

Fig. 1. Two $+\frac{1}{2}$ crossings and two $-\frac{1}{2}$ crossings.

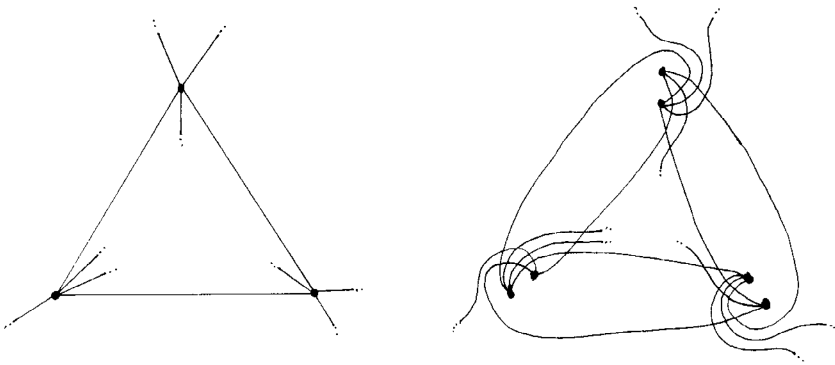

Fig. 2. Conway doubling on a 3-cycle: before and after.

which meet $c_1 \cap c_2$ at exactly one vertex, and (d) k_4 edges not contained in $c_1 \cap c_2$ which meet $c_1 \cap c_2$ at exactly two vertices. Modulo 2, one has

$$\begin{aligned}\Omega_{M_g}(\mathcal{T}(c_1), \mathcal{T}(c_2)) &= \sigma_{\mathcal{T}}(c_1, c_2) + k_1 \cdot (l(c_2) - 3) + k_2 \cdot l(c_2) \\ &\quad + k_3(l(c_2) - 2) + k_4(l(c_2) - 4) \\ &= \sigma_{\mathcal{T}}(c_1, c_2) + k_1 + (k_1 + k_2 + k_3 + k_4) \cdot l(c_2) \\ &= \sigma_{\mathcal{T}}(c_1, c_2) + l(c_1 \cap c_2) + l(c_1) \cdot l(c_2),\end{aligned}$$

as required. □

Conway's doubling procedure allows one to duplicate a thrackled cycle [Wo1]. If the original cycle is odd, one ends up with a thrackled even cycle which is twice as long, and if it is even, one obtains a pair of disjoint even cycles of the same length which still form a (disconnected) thrackle. This procedure can be carried out not only for a separate cycle, but also for a cycle within a thrackled graph, or within a generalized thrackle, and the procedure can be made on any surface (see Fig. 2).

Let $\mathcal{T}\colon G \to M_g$ be a generalized thrackle.

Lemma 2. *Suppose that c_1 is an odd cycle in G such that $\Omega_{M_g}(\mathcal{T}(c_1), \mathcal{T}(c_2)) = 0$ for all cycles c_2. Then Conway doubling on c_1 produces a bipartite graph.*

Proof. Let c_1 be as in the statement of the lemma. Perform the Conway doubling procedure on c_1: let G' be the resulting graph and let c_1' be the even cycle obtained from c_1. We claim that G' is bipartite. Suppose that G' has a cycle c_2'. By reversing the Conway doubling procedure, one sees that c_2' comes from a cycle c_2 say, in G. Obviously,

$$l(c_2') - l(c_1' \cap c_2') = l(c_2) - l(c_1 \cap c_2). \tag{1}$$

By Lemma 1, we have, modulo 2,

$$\sigma_{\mathcal{T}}(c_1, c_2) = l(c_1)l(c_2) + l(c_1 \cap c_2) = l(c_2) + l(c_1 \cap c_2)$$

and

$$\sigma_{\mathcal{T}}(c'_1, c'_2) = l(c'_1)l(c'_2) + l(c'_1 \cap c'_2) = l(c'_1 \cap c'_2).$$

Notice that $\sigma_{\mathcal{T}}(c_1, c_2) = \sigma_{\mathcal{T}}(c'_1, c'_2)$ and so

$$l(c_2) + l(c_1 \cap c_2) = l(c'_1 \cap c'_2). \tag{2}$$

So (1) and (2) give $l(c'_2) = 0 \pmod 2$, as required. □

We remark that the hypothesis in Lemma 2 is weaker than the assumption that $\mathcal{T}(c_1)$ is zero in $\mathbb{Z}_2$-homology. In particular, it holds in the plane. One of this paper's referees has informed us that in previous personal communication, Péter Hajnal had independently obtained Lemma 2 in the planar case, thus improving Theorem 1(a) to $m \leq 1.75n$.

2. Proof of Theorem 3

Let G be a bipartite graph and let $V(G) = V_1 \cup V_2$ be a splitting of the set of vertices of G such that all the edges join elements of V_1 with elements of V_2.

First suppose that there is an embedding $f\colon G \to M_g$. For convenience, we use f to identify G with $f(G)$, so that we may regard G as a subset of M_g. The following argument is similar to the one used in Theorem 1.4 of [LPS]. Choose a point x in the complement of G, and for each point $y \in V_1$, join x to y by a simple arc such that the set of arcs thus obtained is mutually disjoint outside of x. By deforming these arcs if necessary, we may assume that they avoid V_2, and cross the edges of G in proper crossings. Take a small closed ε-neighbourhood D of the union of these arcs (see Fig. 3). So D is homeomorphic to a disc, and the boundary of D is a simple closed curve γ which intersects every edge of G an odd number of times. Let γ' be a curve on $M_g \backslash D$ which is sufficiently close to γ that there are no vertices of G between γ and γ'. Cut out the disc D, flip it over, and attach it back to the surface joining the edges in the annulus between γ and γ' as shown on the Fig. 4. Notice that in the new drawing, any two non-incident edges e_i ($i = 1, 2$) intersecting γ q_i times respectively, intersect one another $q_1 q_2 \equiv 1 \pmod 2$ times, while every pair of incident edges meet an even number of times. Now, taking small circles around each of the vertices of G and performing the above procedure in each of the discs they bound, we obtain a new drawing in which

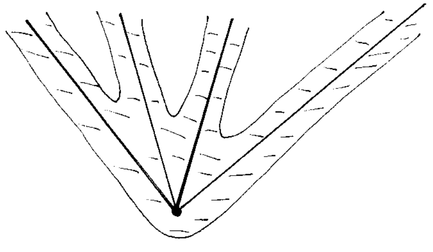

Fig. 3. The ε-neighbourhood D in the vicinity of x.

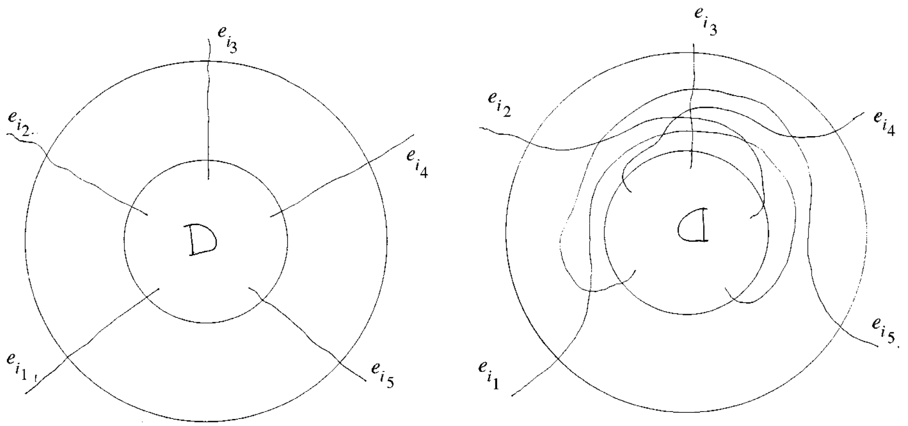

Fig. 4. Reattaching D: before and after.

every pair of edges meets an odd number of times. It remains to modify the drawing so that each edge becomes free of self-intersections. This can be achieved edge by edge: for each edge, it suffices to choose a Wiener switching at each of the self-intersections such that the resulting crossing-free curve is connected (see Fig. 5). This is easily done by induction. The resulting drawing is a generalized thrackle.

Conversely, suppose that we have a generalized thrackle $\mathcal{T}\colon G \to M_g$. Proceed as in the first part of this proof: choose a point x in the complement of G, and join x to V_1 by a set of arcs. Perform the procedure shown Fig. 4 in a small ε-neighbourhood of these arcs, and then choose small discs around each of vertices of G and repeat the procedure on each of these discs. One thus obtains a new drawing $\mathcal{D}\colon G \to M_g$, which is a $\mathbb{Z}_2$-embedding, in the following sense.

Definition 2. A $\mathbb{Z}_2$-*embedding* of a graph in M_g is a drawing of the graph such that every pair of edges meets an even number of times, outside the vertex set.

It remains to prove the following lemma (notice that we are not assuming here that G is bipartite).

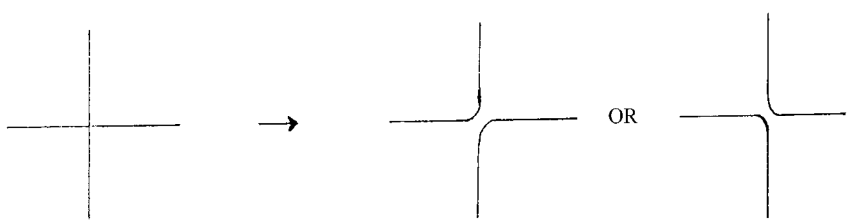

Fig. 5. Wiener switching: before and after.

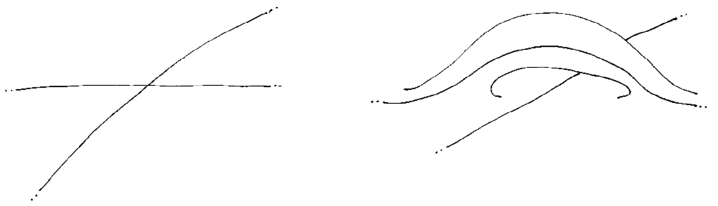

Fig. 6. Eliminating a crossing: before and after.

Lemma 3. *A graph G that can be $\mathbb{Z}_2$-embedded in M_g, can be embedded in M_g with the same rotation system.*

Proof of Lemma 3. Suppose that we have a $\mathbb{Z}_2$-embedding $\mathcal{D}$: $G \to M_g$. Obviously, as $\mathcal{D}$ is not a generalized thrackle, Lemma 1 does not apply. Instead one has

$$\Omega_{M_g}(\mathcal{D}(c_1), \mathcal{D}(c_2)) = \sigma_{\mathcal{D}}(c_1, c_2) \pmod 2, \tag{3}$$

for all cycles c_1 and c_2 in G. Now remove all the crossings in $\mathcal{D}(G)$ by attaching a handle at each crossing point (see Fig. 6), and let S be the resulting closed surface. So we have an embedding $\mathcal{I}(G)$ of G in S, but in general S has higher genus than M_g. Notice that since the surgery has been conducted in the complement of some neighbourhood of the vertex set, $\mathcal{I}(G)$ and $\mathcal{D}(G)$ have the same rotation systems; in particular, $\sigma_{\mathcal{I}} = \sigma_{\mathcal{D}}$. Take a closed ε-neighbourhood U of $\mathcal{I}(G)$; so $U \subset S$ is a compact surface, with boundary, containing $\mathcal{I}(G)$. Since $\mathcal{I}(G)$ is a deformation retract of U, we have $H_1(U, \mathbb{Z}_2) = H_1(\mathcal{I}(G), \mathbb{Z}_2)$ and $\Omega_U = \sigma_{\mathcal{I}}$. Attach discs to all the boundary components of U and let M' be the resulting surface. If G is not connected, then M' will not be connected; in this case, replace M' by the connected sum of its connected components. We now have an embedding $\mathcal{J}(G)$ of G in a connected closed oriented surface M', and it remains to show that the genus g' of M' is not greater than that of M_g. Notice that by construction, the map $H_1(U, \mathbb{Z}_2) \to H_1(M', \mathbb{Z}_2)$ is surjective. Hence

$$\begin{aligned} g' = \tfrac{1}{2}\operatorname{rank} H_1(M', \mathbb{Z}_2) &\le \operatorname{rank} \Omega_U \\ &= \operatorname{rank} \sigma_{\mathcal{I}} \\ &= \operatorname{rank} \sigma_{\mathcal{D}} \\ &\le \operatorname{rank} \Omega_{M_g} \qquad \text{by (3)} \\ &= g, \end{aligned}$$

as required. □

Examples. Figure 7 gives an example of a non-planar graph, homeomorphic to K_5, which can be drawn as a generalized thrackle in the plane.

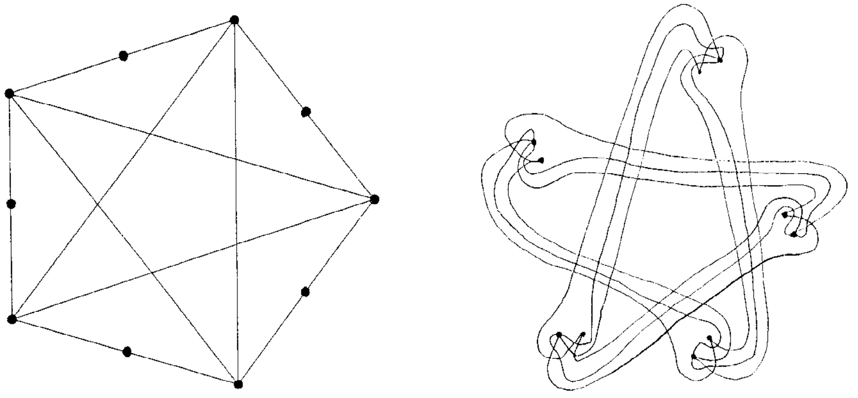

Fig. 7. A generalized thrackle homeomorphic to K_5.

On the other hand, there are planar graphs with $m \leq 2n - 2$ which cannot be drawn as generalized thrackles in the plane. For example, the wheel with four spokes, shown in Fig. 8, cannot be drawn as a generalized thrackle in the plane. Indeed, if it could be, then, by [LPS] or Lemma 1, the two 3-cycles to the left and right of the graph would necessarily cross each other transversally in a small neighbourhood of the vertex in the centre of the graph. However, the same reasoning applies to the two 3-cycles at the top and bottom of the graph. This leads to a contradiction. Similarly, the wheel with $2k$ spokes cannot be drawn as a generalized thrackle in the plane. Curiously, wheels with an odd number of spokes can be drawn as generalized thrackles in the plane (see Fig. 12 below).

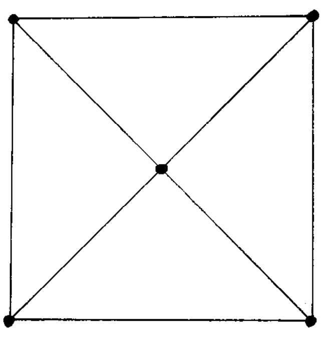

Fig. 8. Not a generalized thrackle in the plane.

3. Proof of Theorem 2 and the Corollary

Let k denote the number of connected components of G.

Theorem 4. *Suppose that* $\mathcal{T}\colon G \to M_g$ *is a generalized thrackle.*

(a) *If* G *is bipartite, then* $m \leq 2n - 4k + 4g$.
(b) *If* G *has an odd cycle* c_1 *such that* $\Omega_{M_g}(\mathcal{T}(c_1), \mathcal{T}(c_2)) = 0$ *for all cycles* c_2*, then* $m \leq 2n - 2k + 4g$.
(c) *If* G *is a thrackle in the plane, then* $m \leq \frac{3}{2}(n - k)$.

Proof. First suppose that G is bipartite. By Theorem 3, G can be embedded in M_g. So if G has connected components $G_1, \ldots, G_k$, then, for each i, we have a cellular embedding of G_i in a surface S_i of genus g_i, with $g_1 + \cdots + g_k \leq g$. Thus it suffices to treat the case where G is connected and cellularly embedded in M_g. In this case, part (a) is a direct consequence of Euler's formula, as employed in [LPS]. Indeed, $2 - 2g = f - m + n$, where f is the number of faces in the cellular decomposition of M_g determined by the embedding of G. As G has no 2-cycles or 3-cycles, one has $2m \geq 4f$. Hence

$$n = 2 - 2g - f + m \geq 2 - 2g - m/2 + m = 2 - 2g + m/2.$$

That is, $m \leq 2n - 4 + 4g$. This proves part (a).

Now suppose that G has an odd cycle c_1 such that $\Omega_{M_g}(\mathcal{T}(c_1), \mathcal{T}(c_2)) = 0$ for all cycles c_2. Perform Conway doubling on c_1: let G' be the resulting bipartite graph and let c_1' be the even cycle obtained from c_1. Let $\mathcal{J}\colon G' \to M_g$ be the embedding given by the construction of Theorem 3. Once again, it suffices to treat the case where G is connected and G' is cellularly embedded in M_g.

Lemma 4. $\mathcal{J}(c_1')$ *bounds a face in the cellular decomposition of* M_g *determined by the embedding of* G'.

Proof of Lemma 4. Since M_g is oriented, it makes sense to talk of the "left" and "right" sides of a closed curve, at least locally. It suffices to show that the image in M_g of the edges of $G' \backslash c_1'$ which are incident with c_1' all lie on the same side of $\mathcal{J}(c_1')$. Let $\mathcal{C}\colon G' \to M_g$ be the generalized thrackle drawing of G' obtained from $\mathcal{T}(G)$ by Conway doubling on c_1. Notice that the edges of $G' \backslash c_1'$ which are incident with c_1' at some given vertex v, all lie on the same side of $\mathcal{C}(c_1')$, either to the left or to the right (see Fig. 2). Moreover, the position of the incident edges alternates, left-right-left, etc., as one moves from vertex to vertex along $\mathcal{C}(c_1')$ (see Fig. 2). By Lemma 2, G' is bipartite: let $V(G) = V_1 \cup V_2$ be a splitting of the set of vertices of G such that all the edges join elements of V_1 with elements of V_2. Now notice that when one constructs the embedding $\mathcal{J}(G')$, using the method employed in the proof of Theorem 3, one effectively reverses the orientation in some neighbourhood of V_2, while maintaining the orientation in some neighbourhood of V_1. Consequently, as one travels along $\mathcal{J}(c_1')$, the edges incident with $\mathcal{J}(c_1')$ all lie on the same side of $\mathcal{J}(c_1')$. □

It remains to see how the above lemma gives the required result. Let f, m', n' be respectively the number of faces, edges and vertices in the cellular decomposition of M_g determined by the embedding of G'. Suppose that c_1 is a p-cycle. So c_1' is a $2p$-cycle, $n' = n + p$ and $m' = m + p$. As G has no 2-cycles or 3-cycles, one has $2m' \geq 4(f-1) + 2p$. Hence

$$n' = 2 - 2g - f + m' \geq 1 - 2g + \frac{m' + p}{2}.$$

That is, $m = m' - p \leq 2n' - 2p - 2 + 4g = 2n - 2 + 4g$. This proves part (b). To prove part (c), just repeat the calculation using the additional fact that G has no 4-cycles. This completes the proof of Theorem 4. □

Notice that Theorem 2 follows immediately from Theorem 4, since in the plane every cycle is $\mathbb{Z}_2$-null homologous. The corollary follows immediately from Theorem 4(a).

Examples. We first describe a useful construction. Let $\mathcal{T}\colon G \to M_g$ be a generalized thrackle. We say that two edges e_1 and e_2 of G are *neighbouring* if they share a common vertex v and if e_1 and e_2 are consecutive in the cyclic order of edges of $\mathcal{T}(G)$ at $\mathcal{T}(v)$. Figure 9 shows how one can add a 2-path joining the endpoints of neighbouring edges so that the resulting drawing is still a generalized thrackle. Notice that Fig. 7 is obtained by adding five 2-paths to the standard pentagonal musquash [Wo1].

Figure 10 gives a drawing of K_4 as a generalized thrackle in the plane. Figure 11 shows that by adding 2-paths, one can construct a generalized thrackle in the plane having n vertices and $2n - 4$ edges, for any number $n \geq 4$. Another example showing that the bound in Theorem 2(b) is sharp is given by the wheel with $(2k + 1)$ spokes. Figure 12 shows the wheel with five spokes and its representation as a generalized thrackle; this diagram is to be understood as follows: five edges meet at a vertex at infinity, and each pair of these edges cross precisely once in a small neighbourhood of infinity. This example is built on the standard pentagonal musquash [Wo1]. This same construction can be effected using the standard $(2k + 1)$-gonal musquash, for any $k \geq 1$. Notice that Figure 10 is the case $k = 1$.

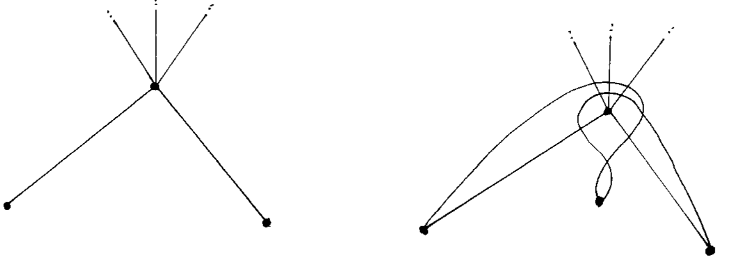

Fig. 9. Attaching a 2-path to a pair of neighbouring edges.

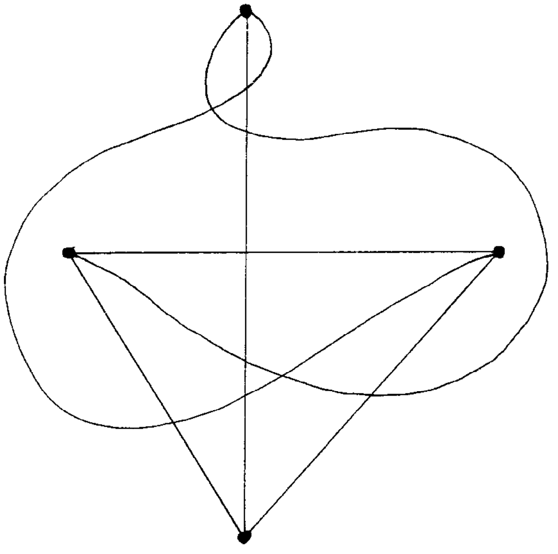

Fig. 10. Generalized thrackle of K_4 in the plane.

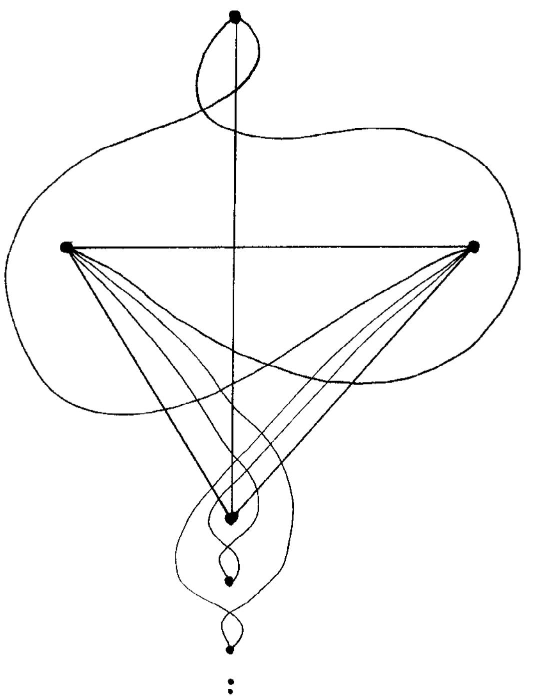

Fig. 11. Planar generalized thrackle with $m = 2n - 2$.

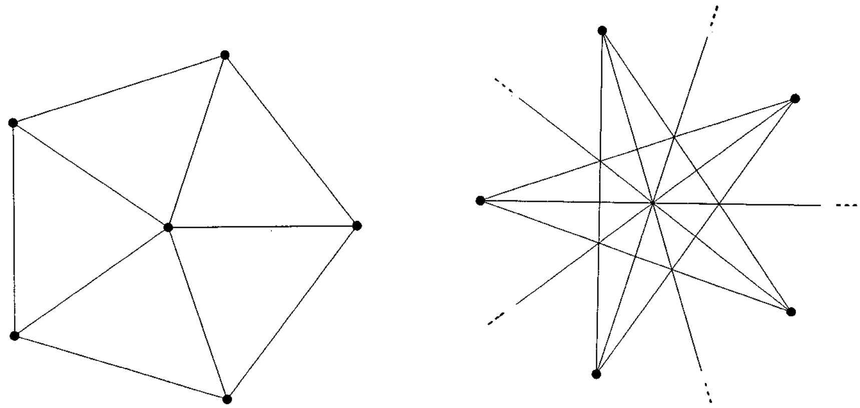

Fig. 12. Another planar generalized thrackle with $m = 2n - 2$.

4. Remarks

As we have seen above, Theorem 4 provides sharp bounds for generalized thrackles on M_g where G has no odd cycles, and where there is an odd cycle which is zero in $\mathbb{Z}_2$-homology. For arbitrary generalized thrackles, one has $m \leq 4n - 8 + 8g$, since every graph can be made bipartite by removing no more than half of its edges. However, this bound seems unduly coarse. We have found no counterexample to the following:

Conjecture 1. If $\mathcal{T}\colon\ G \to M_g$ is a generalized thrackle, then $m \leq 2n - 2 + 4g$.

In fact, it does not seem unreasonable to hope to obtain a complete classification of those graphs which can be drawn as generalized thrackles in the plane.

Our final remarks concern thrackles, as opposed to generalized thrackles. In [CFG], the authors remark: "We may consider analogous constructions on other surfaces, and presumably expect (with obvious notation) the appropriate conjecture to be that max$(m - n)$ depends on the genus of the surface." The following conjecture seems to be the obvious one, although as far as we are aware, it has not previously appeared explicitly in the literature:

Conjecture 2. If $\mathcal{T}\colon\ G \to M_g$ is a thrackle, then $m \leq n + 2g$.

Observe that, for any given genus g, there exists an example for which the bound $m = n + 2g$ is attained. This can be done inductively using a procedure similar to that employed in Fig. 14 of [Wo1]; one chooses an example for which the bound is attained on a surface of genus $g - 1$, and then adds a handle to the surface and replaces an edge by the system of five edges shown in Fig. 13. This increases the number of vertices by two and the number of edges by four.

Conjecture 2 can be verified for graphs with very few vertices. First, recall that thrackles in the plane have no 4-cycles [Wo1]. Moreover, they have at most one 3-cycle;

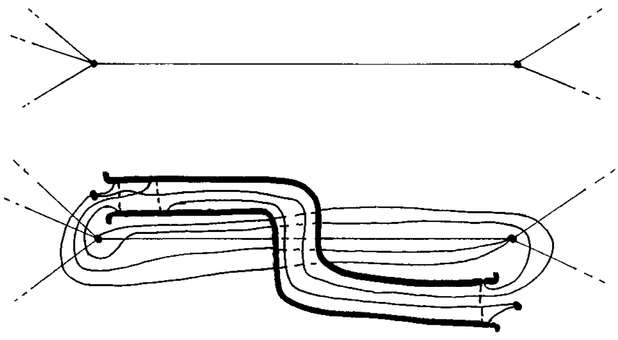

Fig. 13. Before and after.

indeed, if a thrackle in the plane had two 3-cycles, c_1 and c_2 say, then c_1 and c_2 must have nontrivial intersection (by [LPS]), but they cannot share a common edge (since otherwise there would be a 4-cycle) and it is easy to see that the case of a single common vertex is also impossible. In higher genus, one has:

Lemma 5. *Suppose that* $\mathcal{T}\colon G \to M_g$ *is a thrackle.*

(a) *If* $c \subset G$ *is a* 4*-cycle, then* $\mathcal{T}(c)$ *is nontrivial in* $\mathbb{Z}_2$*-homology.*
(b) *If* $c_1, c_2 \subset G$ *are* 3*-cycles, then* $\mathcal{T}(c_1)$ *and* $\mathcal{T}(c_2)$ *are not* $\mathbb{Z}_2$*-homologous.*

Proof. (a) Let $c = \{1234\}$ be a thrackled 4-cycle on M_g. Let $a = 12 \cap 34$, $b = 23 \cap 14$. Then the triangles $12b$ and $34b$ have exactly one point of the transversal crossing (namely, the point a). They cannot represent the same class in $H_1(M_g, \mathbb{Z}_2)$ and therefore their sum is nontrivial.

(b) Suppose that G consists of two 3-cycles c_1 and c_2, and that $\mathcal{T}(c_1)$ and $\mathcal{T}(c_2)$ are $\mathbb{Z}_2$-homologous. Since G is simple, c_1 and c_2 are either disjoint, share a single common edge, or share a single common vertex. First notice that if c_1 and c_2 shared a single common edge, then their sum would be a 4-cycle whose image in M_g would be trivial in $\mathbb{Z}_2$-homology. This would contradict part (a). So c_1 and c_2 are either disjoint, or share a single common vertex. As $\mathcal{T}(c_1)$ and $\mathcal{T}(c_2)$ are $\mathbb{Z}_2$-homologous, $\mathcal{T}(c_1)$ and $\mathcal{T}(c_2)$ must have zero intersection number. So Lemma 1 gives

$$0 = \Omega_{M_g}(\mathcal{T}(c_1), \mathcal{T}(c_2)) = \sigma_{\mathcal{T}}(c_1, c_2) + 1 + l(c_1 \cap c_2). \qquad (4)$$

If c_1 and c_2 were disjoint, then one would have $\sigma_{\mathcal{T}}(c_1, c_2) = 0$ and $l(c_1 \cap c_2) = 0$, which contradicts (4). So c_1 and c_2 share a single common vertex. Hence (4) gives $\sigma_{\mathcal{T}}(c_1, c_2) = 1$. Label the vertices of $\mathcal{T}(c_1)$ and $\mathcal{T}(c_2)$ respectively 123 and 145. As $\sigma_{\mathcal{T}}(c_1, c_2) = 1$, $\mathcal{T}(c_1)$ and $\mathcal{T}(c_2)$ cross transversally at the vertex 1. Let $a = 23 \cap 14$. By relabelling the vertices if necessary, one may assume that the arc $2a$ contains no crossing points. The curves $1a3 = 123 + 12a$ and $12a45 = 145 + 12a$ are still $\mathbb{Z}_2$-homologous and so they must have zero intersection number. However, they touch at the points 1 and a, and intersect transversally three times, which is impossible. □

Proposition. Conjecture 2 is true for all graphs G with $n \leq 5$ and $m \leq 9$.

Proof. The cases $n \leq 4$ are given directly by Theorem 2(a). Suppose that $\mathcal{T}\colon G \to M_g$ is a thrackle with $n = 5$. Since G is simple, G is a subgraph of K_5. Without loss of generality, we may assume that G has no vertices of index 1. We are required to show that:

(a) if $m \geq 6$, then G cannot be thrackled in the plane,
(b) if $m \geq 8$, then G cannot be thrackled on the torus,

To treat case (a), it suffices to note that if $m \geq 6$, then G either has a 4-cycle or at least two 3-cycles. To deal with case (b), note that on the torus, $H_1(\mathbb{T}^2, \mathbb{Z}_2) = \mathbb{Z}_2^2$, and so there are precisely four distinct $\mathbb{Z}_2$-homology classes:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So by Lemma 5, a thrackle on the torus can have at most four 3-cycles. Suppose that $m = 8$. So G is obtained from K_5 by deleting two edges. There are only two such graphs, according to whether or not the deleted edges share a common vertex. In the first case,

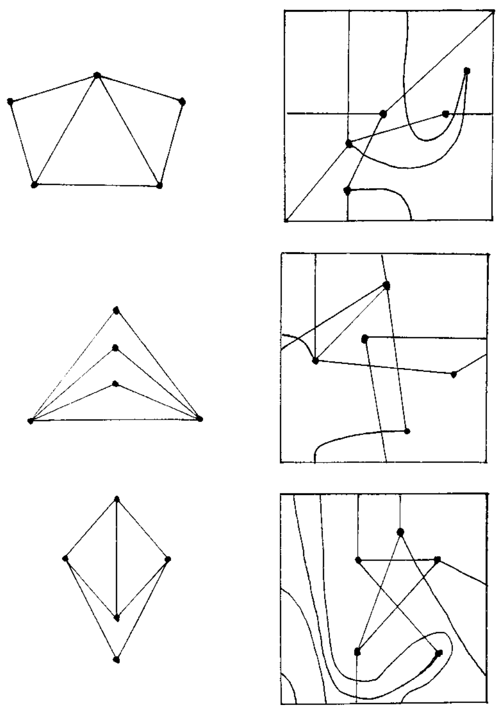

Fig. 14. Thrackles on the torus.

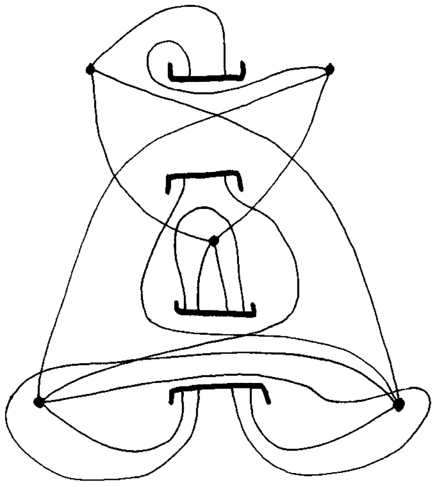

Fig. 15. $K_5\backslash\{\text{one edge}\}$ thrackled on the 2-torus.

G has five 3-cycles, and so G cannot be thrackled on the torus. In the second case, G has four 3-cycles whose sum is a 4-cycle. Suppose that G can be thrackled on the torus. By Lemma 5, G must have precisely one 3-cycle of each of the four $\mathbb{Z}_2$-homology types. So the sum of the 3-cycles is a 4-cycle which is zero in $\mathbb{Z}_2$-homology. This contradicts Lemma 5(a). □

Remark. In support of Conjecture 2, we remark that all graphs with five vertices and seven edges can be thrackled on the torus (see Fig. 14) and the connected graph with five vertices and nine edges, $K_5\backslash\{\text{one edge}\}$, can be thrackled on the 2-torus (see Fig. 15). It would be interesting to show that K_5 cannot be thrackled on the 2-torus.

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