Discrete Comput Geom 23:157–169 (2000)
DOI: 10.1007/s004549910011

Discrete & Computational
Geometry


# On Random Sections of the Cube*

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**Abstract.** Let $f(j, k, n)$ denote the expected number of $j$-faces of a random $k$-section of the $n$-cube. A formula for $f(0, k, n)$ is presented, and, for $j \geq 1$, a lower bound for $f(j, k, n)$ is derived, which implies a precise asymptotic formula for $f(n-m, n-l, n)$ when $1 \leq l < m$ are fixed integers and $n \to \infty$.

## 1. Introduction

The principal object in this paper is the expected number of $j$-dimensional faces (in short, $j$-faces) of a random $k$-dimensional central section (in short, $k$-section) of the $n$-cube $B_\infty^n = [-1, 1]^n$ in $\mathbb{R}^n$. We denote this number by $f(j, k, n)$. The normalized rotation invariant measure on the set $G_{n,k}$ of all $k$-dimensional subspaces of $\mathbb{R}^n$ provides the probabilistic framework.

Section 2 contains a calculation of the expected number of vertices of a random $k$-section of the $n$-cube. The result is

$$f(0, k, n) = 2^k \binom{n}{k} \sqrt{\frac{2k}{\pi}} \int_0^\infty e^{-kt^2/2} \gamma_{n-k}(t B_\infty^{n-k})\, dt, \tag{1}$$

where $\gamma_{n-k}$ denotes the $(n-k)$-dimensional Gaussian probability measure.

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* This work was supported in part by the US–Israel Binational Science Foundation. This work forms part of a dissertation prepared under the direction of Professor Joram Lindenstrauss, to whom I am greatly indebted for much valuable advice.

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In Section 3 we derive a lower bound for $f(j,k,n)$ for every $1 \le j < k < n$. The main result is

$$\frac{f(0,k-j,n)}{f(j,k,n)} < \sqrt{\frac{2}{\pi}}\left(\frac{j(k-j)}{n-k+j}\right)^{1/2}\int_0^\infty \exp\left(-\frac{j(k-j)}{n-k+j}t^2/2\right)\gamma_j(tB_\infty^j)\,dt.$$

The lower bound for $f(j,k,n)$ derived from this inequality, combined with (1), leads in some cases to asymptotically best possible results. For example, in Section 3 we deduce from it the following asymptotic formula, for fixed integers $1 \le l < m$:

$$f(n-m, n-l, n) \backsim \frac{(2n)^{m-l}}{(m-l)!} \qquad \text{as} \quad n \to \infty. \tag{2}$$

The notation $a_n \sim b_n$ means $a_n/b_n \to 1$ as $n \to \infty$. Formula (2) can be interpreted as follows: the probability that a random fixed-codimensional subspace of $\mathbb{R}^n$ intersects a fixed-codimensional face of the $n$-cube, tends to 1 as $n \to \infty$. Formula (2) itself follows also from the work of Affentranger and Schneider. (See Remark 1 of Section 3 below.) In [1], they found a formula for the expected number $\mathbf{E}(f_j(\Pi_k P))$ of $j$-faces of an orthogonal projection of an $n$-polytope $P$ onto a $k$-dimensional random subspace. Formula (5) of [1] reads as follows:

$$\mathbf{E}(f_j(\Pi_k P)) = f_j(P) - 2\sum_{s\ge 0}\sum_{F\in\mathcal{F}_j(P)}\sum_{G\in\mathcal{F}_{k+1+2s}(P)}\beta(F,G)\gamma(G,P). \tag{3}$$

Here $\mathcal{F}_j(P)$ denotes the set of $k$-faces of $P$, and $f_j(P) = \operatorname{card}\mathcal{F}_j(P)$. $\beta(F,G)$ denotes the internal angle [8, p. 297], of the face $G$ at its face $F$, and $\gamma(G,P)$—the external angle [8, p. 308], of $P$ at its face $G$. It is shown in [1] that (3) implies that if $0 \le j < k$ are fixed integers, then, as $n \to \infty$,

$$\mathbf{E}(f_j(\Pi_k T^n)) \sim \frac{2^k}{\sqrt{k}}\binom{k}{j+1}\beta(T^j, T^{k-1})(\pi\log n)^{(k-1)/2}. \tag{4}$$

Here $T^n$ stands for the regular $n$-simplex.

In a very recent work [5], Böröczky, Jr., and Henk showed that (3) implies the same asymptotic formula (4) also for $\mathbf{E}(f_j(\Pi_k B_1^n))$, where $B_1^n$ is the regular cross-polytope. In addition, they found an asymptotic formula for the internal angles $\beta(T^j, T^{k-1})$, when $k/j^2 \to \infty$. Therefore if $j$ is fixed, $k$ is much larger than $j^2$ and $n$ much larger than $k$, then explicit estimates for $\mathbf{E}(f_j(\Pi_k B_1^n))$ are available. See [5] for more details. Explicit asymptotic formulas for $\mathbf{E}(f_j(\Pi_k T^n))$ were established independently by Vershik and Sporyshev [10], when $j, k$ are both proportional to $n$ and $n \to \infty$.

A simple duality argument shows that

$$\mathbf{E}(f_j(\Pi_k B_1^n)) = f(k-j-1, k, n).$$

Choose $j = k-1$ in (4). Applying the result for $B_1^n$, one has

$$f(0,k,n) = \mathbf{E}(f_{k-1}(\Pi_k B_1^n)) \sim \frac{2^k}{\sqrt{k}}(\pi\log n)^{(k-1)/2} \qquad \text{as} \quad n \to \infty. \tag{5}$$

The last asymptotic formula follows also from (1). In fact, if $\{g_i\}_{i=1}^m$ are independent $N(0,1)$ (that is, with mean 0 and variance 1) Gaussian variables, then $\gamma_m(tB_\infty^m)$ coincides with the probability of the event $\{\max_{1\le i\le m} |g_i| \le t\}$. This probabilistic interpretation allows a straightforward evaluation of the asymptotic behavior of the integral in (1), when $k$ is fixed and $n \to \infty$.

Formula (1) also yields information about $f(0, k, n)$ for $k$ not necessarily fixed. For example, if $k = n - 1$, then the integral in (1) can be computed and the result is

$$f(0, n-1, n) = \frac{2^n n}{\pi} \arctan \frac{1}{\sqrt{n-1}} \sim \frac{2^n \sqrt{n}}{\pi}. \tag{6}$$

Particular values of the last formula were computed numerically in Table 2 of [5]. For the expected number of vertices of random sections of fixed codimension, we have the following inequality, which is a consequence of (1):

$$f(0, n-d, n) \ge \binom{n}{d} 2^n \left( \frac{1}{\pi} \arctan \frac{1}{\sqrt{n-d}} \right)^d \qquad (d \ge 1).$$

Equality holds for $d = 1$.

To obtain a lower bound for $f(j, k, n)$, it turns out that it is useful to know an estimate for the Gaussian measure of a cone generated by a section of a face of a cube. In Section 3 we find such an estimate, by modifying Ball's calculation of the maximal volume of a cube-section, based on Brascamp–Lieb's inequality [2].

Dvoretzky's theorem on almost Euclidean sections asserts that there exists a function $k(\varepsilon, n) \ge 1$, tending to infinity as $n \to \infty$ for each fixed $\varepsilon > 0$, such that if $K$ is an $n$-dimensional centrally symmetric convex body (that is, a convex compact set in $\mathbb{R}^n$ with nonempty interior, satisfying $K = -K$), and $\varepsilon > 0$, then for each $1 \le k \le k(\varepsilon, n)$ there exists a $k$-dimensional subspace $X$, and a linear automorphism $T$ of $X$ for which

$$X \cap B_2^n \subset T(X \cap K) \subset (1 + \varepsilon)(X \cap B_2^n), \tag{7}$$

where $B_2^n$ denotes the Euclidean unit ball. The proof of Dvoretzky's theorem in [6] shows that $k(\varepsilon, n) \ge c\varepsilon^2 |\log \varepsilon|^{-1} \log n$, for some absolute constant $c > 0$. That proof determined the best possible dependence of $k$ on $n$. The dependence of $k$ on $\varepsilon$ was improved by Gordon [7], who discovered another proof of Dvoretzky's theorem with $k(\varepsilon, n) \ge c\varepsilon^2 \log n$. Both proofs are probabilistic; they show that not only do there exist almost Euclidean sections, but actually most sections are such. More precisely, if $X$ is a random subspace whose dimension does not exceed $k(\varepsilon, n)$, then the probability that the section $X \cap K$ is $(1+\varepsilon)$-Euclidean (common terminology for expressing that (7) holds) tends to 1 as $n \to \infty$. These facts motivate an investigation of the random $f$-vector $\{f(j, k, n)\}_{j=0}^{k-1}$, especially since it is well known that every $k$-dimensional symmetric polytope that has $2n$ facets is affinely equivalent to a $k$-section of an $n$-cube.

## 2. Vertices

Let $G_{n,k}$ denote the set of $k$-dimensional subspaces of $\mathbb{R}^n$. We denote its normalized rotation invariant measure by "Prob." Recall that this measure is related to the normalized Haar measure $H$ of the orthogonal group $\mathrm{O}(n)$ by the equality

$$\mathrm{Prob}\{X \in B\} = H\{g \in \mathrm{O}(n)\colon\ g[e_i]_{i=1}^k \in B\},$$

where $B$ is a Borel subset of $G_{n,k}$ and $[e_i]_{i=1}^k$ is the $k$-dimensional subspace spanned by the first $k$ unit vectors in $\mathbb{R}^n$. Fix $X \in G_{n,k}$. For each $0 \le j \le k-1$, the set of $j$-faces of the polytope $X \cap B_\infty^n$ coincides with the set of intersections of $(n-k+j)$-faces of $B_\infty^n$ with $X$. Every $(n-k+j)$-face of $B_\infty^n$ has the same probability to be intersected. Therefore if one particular $(n-k+j)$-face $F_{n-k+j}$ is fixed, then the expected number of $j$-faces of the section $X \cap B_\infty^n$ is equal to

$$2^{k-j}\binom{n}{k-j}\mathrm{Prob}\{X \cap F_{n-k+j} \neq \emptyset\}.$$

Let $C(F_{n-k+j})$ denote the cone generated by $F_{n-k+j}$:

$$C(F_{n-k+j}) = \bigcup_{x \in F_{n-k+j}} \{tx\colon\ t \ge 0\}.$$

Put $C_1(F_{n-k+j}) = C(F_{n-k+j}) \cap \mathbb{S}^{n-1}$. For every subspace $X$,

$$X \cap F_{n-k+j} \neq \emptyset \quad \Longleftrightarrow \quad (X \cap \mathbb{S}^{n-1}) \cap C_1(F_{n-k+j}) \neq \emptyset.$$

For $n = 0, 1, \ldots$ we denote by $\sigma_n$ the normalized rotation-invariant measure on the unit-sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. The next lemma will prove useful for dealing with intersections of subsets of the sphere with random subspaces.

**Lemma 2.1.** *Let $l, m, n$ be positive integers satisfying $l + m \ge n - 1$. Suppose that $A \subset \mathbb{S}^m$ and $B \subset \mathbb{S}^l$ are Borel subsets. Then, for $p = l + m - n + 1$,*

$$\int_{\mathrm{O}(n)} \sigma_p(gB \cap A)\, dH(g) = \sigma_l(B)\sigma_m(A). \tag{8}$$

To prove the lemma one observes that for fixed $A$ (resp. $B$) the integral defines an invariant measure on $\mathbb{S}^l$ (resp. $\mathbb{S}^m$); the conclusion follows from that.

Lemma 2.1 is now applied to $B = X \cap \mathbb{S}^{n-1}$, which we denote by $\mathbb{S}^{k-1}$, and to $A = C_1(F_{n-k+j})$. For $l = k-1$ and $m = n-k+j$ equality (8) becomes

$$\int_{\mathrm{O}(n)} \sigma_j(g\mathbb{S}^{k-1} \cap A)\, dH(g) = \sigma_{n-k+j}(A). \tag{9}$$

We are ready to compute the expected number of vertices. The Gaussian measure in $\mathbb{R}^m$ whose density is $(2\pi)^{-m/2}\exp(-\sum_1^m x_i^2/2)$ is denoted by $\gamma_m$.

**Proposition 2.2.** *The expected number of vertices of a random $k$-dimensional central section of the $n$-cube is given by the formula*

$$f(0,k,n) = 2^k\binom{n}{k}\sqrt{\frac{2k}{\pi}}\int_0^\infty e^{-kt^2/2}\gamma_{n-k}(tB_\infty^{n-k})\, dt.$$

*Proof.* For each $g \in \mathrm{O}(n)$ we have

$$g\mathbb{S}^{k-1} \cap C_1(F_{n-k}) = (\mathrm{span}(g\mathbb{S}^{k-1}) \cap C(F_{n-k})) \cap \mathbb{S}^{n-1}.$$

For almost every $g$ the intersection $\operatorname{span}(g\mathbb{S}^{k-1}) \cap C(F_{n-k})$ is either the origin itself, or else a ray emanating from the origin. Therefore the intersection $g\mathbb{S}^{k-1} \cap C_1(F_{n-k})$ is either empty or a singleton, for almost every $g$. Choose $j = 0$ in (9), with $A = C_1(F_{n-k})$. Since the measure $\sigma_0$ is concentrated on two points giving mass $\frac{1}{2}$ to each, we deduce from (9) that

$$\operatorname{Prob}\{X \cap F_{n-k} \neq \emptyset\} = 2\sigma_{n-k}(C_1(F_{n-k})). \tag{10}$$

To compute the right-hand side of (10), consider an $(n-k)$-dimensional cube of edge-length 1 inside $\mathbb{R}^{n-k+1}$, at a distance $\sqrt{k}$ from the origin, form the cone it generates, and compute the measure of its intersection with the sphere $\mathbb{S}^{n-k}$. Invoking polar coordinates we see that

$$\sigma_{n-k}(C_1(F_{n-k})) = \gamma_{n-k+1}(C(F_{n-k})).$$

By rotational symmetry of the Gaussian measure we may assume that $F_{n-k}$ is specifically the set $\{x\colon |x_i| \leq 1,\ 1 \leq i \leq n-k,\ x_{n-k+1} = \sqrt{k}\}$. The intersection of the hyperplane $\{x_{n-k+1} = t\}$ with $C(F_{n-k})$ is an $(n-k)$-dimensional cube of edge-length $t/\sqrt{k}$. Therefore by Fubini's theorem

$$\begin{aligned} \gamma_{n-k+1}(C(F_{n-k})) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t^2/2} \gamma_{n-k}\left(\frac{t}{\sqrt{k}} B_\infty^{n-k}\right) dt \\ &= \sqrt{\frac{k}{2\pi}} \int_0^\infty e^{-kt^2/2} \gamma_{n-k}(tB_\infty^{n-k})\, dt. \end{aligned}$$

The last equality, together with (10), implies the desired formula. □

The next lemma points out the precise asymptotic behavior of $f(0,k,n)$ when $k$ is fixed and $n \to \infty$, and also that of $f(n-m, n-l, n)$, when $l, m$ are fixed and $n \to \infty$. (To be used in Section 3.)

**Lemma 2.3.** *Suppose that $\{\alpha_n\}_{n=1}^\infty$ is a sequence of real numbers that has a positive limit $\alpha$. Then, as $n \to \infty$,*

$$\int_0^\infty e^{-\alpha_n t^2/2} \gamma_n(tB_\infty^n)\, dt \sim \Gamma(\alpha) \frac{\pi^{\alpha/2}}{\sqrt{2}} \frac{(\log n)^{(\alpha_n - 1)/2}}{n^{\alpha_n}}, \tag{11}$$

*where $\Gamma$ is the gamma function.*

*Proof.* Let $F_n(t) = \operatorname{Prob}\{\max_i |g_i| \leq t\}$, where $g_1, \ldots, g_n$ are independent $N(0,1)$-Gaussian variables. We have

$$\gamma_n(tB_\infty^n) = \left(\sqrt{\frac{2}{\pi}} \int_0^t e^{-x^2/2}\, dx\right)^n = F_n(t).$$

For $n > 1$, put

$$a_n = \frac{1}{\sqrt{2\log n}} \quad \text{and} \quad b_n = \sqrt{2\log n} - \frac{\log(\pi \log n)}{2\sqrt{2\log n}}.$$

The well known tail approximation

$$\sqrt{\frac{2}{\pi}}\int_t^\infty e^{-x^2/2}\,dx = \sqrt{\frac{2}{\pi}}\,\frac{1+o(1)}{t}e^{-t^2/2} \qquad \text{as} \quad t\to\infty, \tag{12}$$

combined with a simple calculation, implies that

$$\lim_{n\to\infty} F_n(a_n x + b_n) = \exp(-e^{-x}), \qquad \forall\, x \in \mathbb{R}. \tag{13}$$

A change of variables gives

$$\begin{aligned}\int_0^\infty e^{-\alpha_n t^2/2}\gamma_n(tB_\infty^n)\,dt \;&=\; a_n\int_{-b_n/a_n}^\infty e^{-\alpha_n(a_n x+b_n)^2/2}F_n(a_n x+b_n)\,dx\\ &=\; \frac{\pi^{\alpha_n/2}}{\sqrt{2}}\,\frac{(\log n)^{(\alpha_n-1)/2}}{n^{\alpha_n}}e^{-o(1)}\\ &\qquad \cdot\int_{-\infty}^\infty e^{-x^2\,o(1)}e^{-\alpha_n x(1-o(1))}F_n(a_n x+b_n)\chi_n(x)\,dx.\end{aligned}$$

Here $\chi_n$ stands for the characteristic function of the interval $[-b_n/a_n,\infty)$. All four terms of the integrand in the last integral are nonnegative for each $x$. For $x \geq 0$ and sufficiently large $n$ we have $e^{-\alpha_n x(1-o(1))} < e^{-\alpha x/2}$, while the rest of the terms are majorized by 1. For $x < 0$, the tail estimate (12) implies the existence of a constant $c = c(\alpha) > 0$, such that $F_n(a_n x + b_n) < e^{(\alpha+1)x}$ for every $x$ in $(-\log n, -c)$, and sufficiently large $n$. One more application of (12) shows that $\lim_{n\to\infty}\int_{-b_n/a_n}^{-\log n} e^{-\alpha_n x}F_n(a_n x+b_n)\,dx = 0$. By (13), the integrand converges pointwise to the function $e^{-\alpha x}\exp(-e^{-x})$; Lebesgue's bounded convergence theorem can be applied:

$$\begin{aligned}\lim_{n\to\infty}\int_{-b_n/a_n} e^{-x^2 o(1)}e^{-\alpha_n x(1-o(1))}F_n(a_n x+b_n)\,dx \;&=\; \int_{-\infty}^\infty e^{-\alpha x}\exp(-e^{-x})\,dx\\ &=\; \Gamma(\alpha).\end{aligned}$$

The proof of Lemma 2.3 is complete. □

Taking $\alpha_n \equiv k$ in Lemma 2.3 and bearing in mind Proposition 2.2 re-proves the following result, which was mentioned in the Introduction.

**Corollary 2.4.** *For fixed $k$,*

$$f(0,k,n) \sim \frac{2^k}{\sqrt{k}}(\pi\log n)^{(k-1)/2} \qquad \textit{as} \quad n\to\infty.$$

We turn now to the case of fixed codimension. The next result is deduced from Proposition 2.2.

**Proposition 2.5.** *For $d \geq 1$,*

$$f(0,n-d,n) \geq \binom{n}{d}2^n\left(\frac{1}{\pi}\arctan\frac{1}{\sqrt{n-d}}\right)^d \qquad (d\geq 1).$$

*Equality holds for* $d = 1$:

$$f(0, n-1, n) = \frac{2^n n}{\pi} \arctan \frac{1}{\sqrt{n-1}}. \tag{14}$$

*Proof.* Consider the probability measure $d\mu(t) = 2\sqrt{(k/\pi)}e^{-kt^2}dt$ on the half-line $[0, \infty)$. Put

$$\Phi(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2}\, dx.$$

Then

$$\gamma_{n-k}(tB_\infty^{n-k}) = \left(\sqrt{\frac{2}{\pi}} \int_0^t e^{-x^2/2}\, dx\right)^{n-k} = \Phi^{n-k}\left(\frac{t}{\sqrt{2}}\right).$$

Therefore

$$\begin{aligned}
\sqrt{\frac{2k}{\pi}} \int_0^\infty e^{-kt^2/2} \gamma_{n-k}(tB_\infty^{n-k})\, dt &= \sqrt{\frac{2k}{\pi}} \int_0^\infty e^{-kt^2/2} \Phi^{n-k}\left(\frac{t}{\sqrt{2}}\right) dt \\
&= \int_0^\infty \Phi^{n-k}(t)\, d\mu(t) \\
&\geq \left(\int_0^\infty \Phi(t)\, d\mu(t)\right)^{n-k}.
\end{aligned} \tag{15}$$

Elementary calculation shows that

$$\int_0^\infty e^{-kt^2} \Phi(t)\, dt = \frac{1}{\sqrt{\pi k}} \arctan \frac{1}{\sqrt{k}}.$$

A combination of (15) with Proposition 2.2 gives the desired inequality, after a replacement of $k$ by $n - d$. Observe that, for $k = n - 1$ (that is, $d = 1$), there is equality in the inequality of (15). □

**Remarks.** 1. For $n = 3$ we get, from (14), $f(0, 2, 3) = (24/\pi) \arctan(1/\sqrt{2}) \approx 4.7$. Therefore a random 2-section of the 3-cube is more likely to be a parallelogram than a hexagon.

2. Bárány and Lovász proved in [3] that (in particular) almost every $k$-section of the $n$-cube has at least $2^k$ vertices. Clearly this is a precise lower bound. For $k = n - 1$, our result shows that the expected value is asymptotically $\sqrt{n}/\pi$ times the minimal value.

3. The asymptotic behavior of the integral

$$\int_0^\infty e^{-kt^2/2} \gamma_{n-k}(tB_\infty^{n-k})\, dt$$

for fixed $k$ and $n \to \infty$ was determined in [5] (following [9]), and was used to prove formula (4) of the Introduction. See also [1]. The asymptotic result is basically a corollary of the classical tail approximation of a single $N(0, 1)$-Gaussian variable. Our approach to the proof of Lemma 2.3 seems to simplify the analysis.

4. As was indicated in the Introduction, we can choose $\varepsilon = c/\sqrt{\log n}$ for some constant $c > 0$, and then with high probability a random 2-section of the cube is $(1 + c/\sqrt{\log n})$-Euclidean. It is well known that among all centrally symmetric polygons having $2m$ vertices, the regular $2m$-gon minimizes the Banach–Mazur distance to the Euclidean disk; the minimal distance is $(\cos(\pi/2m))^{-1}$. Consequently with high probability we have

$$\left(\cos\frac{\pi}{2m}\right)^{-1} < 1 + \frac{c}{\sqrt{\log n}}.$$

Hence most 2-sections of the $n$-cube have at least $C(\log n)^{1/4}$ vertices, for some positive constant $C$. By Proposition 2.2 (after suitable rearrangement)

$$f(0, 2, n) = 2\sqrt{\pi}\,\mathbf{E}\left(\max_{1\le i\le n} |g_i|\right),$$

which is of the order of magnitude of $\sqrt{\log n}$. Summarizing these observations, we conclude: a typical 2-section of the $n$-cube is $(1 + c/\sqrt{\log n})$-Euclidean, hence it cannot have too few vertices—it has at least $C(\log n)^{1/4}$ vertices with probability that tends to 1 as $n \to \infty$. It does not however tend to be a regular polygon, because the expected number of its vertices is too high for that.

## 3. Other Faces

We now turn to the case $j \ge 1$, and prove the following result.

**Proposition 3.1.** *For $j \ge 1$, the following inequality holds*:

$$\frac{f(0, k-j, n)}{f(j, k, n)} < \sqrt{\frac{2}{\pi}}\left(\frac{j(k-j)}{n-k+j}\right)^{1/2}\int_0^\infty \exp\left(-\frac{j(k-j)}{n-k+j}t^2/2\right)\gamma_j(tB_\infty^j)\,dt.$$

The starting point in the proof of Proposition 3.1 is (9). Again, we choose $A = C_1(F_{n-k+j})$. The random variable $g \to \sigma_j(g\mathbb{S}^{k-1} \cap A)$, which is defined on $\mathrm{O}(n)$, has values in $[0, 1]$. Hence

$$\int_{\mathrm{O}(n)} \sigma_j(g\mathbb{S}^{k-1} \cap A)\,dH(g) = \int_0^1 H\{g\colon \sigma_j(g\mathbb{S}^{k-1} \cap A) \ge t\}\,dt. \tag{16}$$

The integrand is nonincreasing, and

$$H\{g\colon \sigma_j(g\mathbb{S}^{k-1} \cap A) \ge 0\} = \mathrm{Prob}\{X \cap F_{n-k+j} \ne \emptyset\}, \tag{17}$$

because the event $\{g\mathbb{S}^{k-1} \cap A \ne \emptyset$ and $\sigma_j(g\mathbb{S}^{k-1} \cap A) = 0\}$ has Haar measure zero. Therefore, by (9),

$$\begin{aligned}\sigma_{n-k+j}(A) &\le \mathrm{Prob}\{X \cap F_{n-k+j} \ne \emptyset\}\sup\{t\colon H\{g\colon \sigma_j(g\mathbb{S}^{k-1} \cap A) \ge t\} > 0\}\\ &\le \mathrm{Prob}\{X \cap F_{n-k+j} \ne \emptyset\}\sup\{\sigma_j(g\mathbb{S}^{k-1} \cap A)\colon g \in \mathrm{O}(n)\}.\end{aligned}$$

Let

$$t_{j,k,n} = \sup\{\sigma_j(g\mathbb{S}^{k-1} \cap A)\colon\ g \in \mathrm{O}(n)\}.$$

By (9), (15), and (16) we get

$$\mathrm{Prob}\{X \cap F_{n-k+j} \neq \emptyset\} \geq \frac{\sigma_{n-k+j}(A)}{t_{j,k,n}}.$$

Hence, by (10),

$$f(j,k,n) \geq 2^{k-j}\binom{n}{k-j}\frac{\sigma_{n-k+j}(A)}{t_{j,k,n}} = \frac{\frac{1}{2}f(0,k-j,n)}{t_{j,k,n}}.$$

We must bound $t_{j,k,n}$ from above. Since $A$ is contained in a half-space, a trivial bound is $t_{j,k,n} \leq \frac{1}{2}$. In some cases this bound can be significantly improved. The main lemma in this section is the following.

**Lemma 3.2.** *If* $1 \leq j < k < n$, *then*

$$t_{j,k,n} \leq \frac{1}{\sqrt{2\pi}}\left(\frac{j(k-j)}{n-k+j}\right)^{1/2}\int_0^\infty \exp\left(-\frac{j(k-j)}{n-k+j}t^2/2\right)\gamma_j(tB_\infty^j)\,dt.$$

The next lemma will be used in the proof of Lemma 3.2.

**Lemma 3.3.** *Given a positive number* $\tau > 0$, *a* $j$*-dimensional affine subspace* $Y$ *of* $\mathbb{R}^m$, *and a point* $y_0 \in Y$, *the following inequality holds*:

$$\gamma_j((Y \cap \tau B_\infty^m) - y_0) \leq \gamma_j\left(\tau\sqrt{\frac{m}{j}}B_\infty^j\right). \tag{18}$$

*Proof.* Let $Q$ denote the orthogonal projection onto $Y - y_0$. As usual, $\{e_i\}_{i=1}^m$ are the standard unit vectors in $\mathbb{R}^m$. Put $u_i = Qe_i/\|Qe_i\|$ if $Qe_i \neq 0$, and $u_i = 0$ otherwise; put $c_i = \|Qe_i\|^2$ and $\alpha_i = \langle y_0, e_i\rangle$ for $1 \leq i \leq m$. ($\langle\cdot,\cdot\rangle$ is the standard scalar product.) Then

$$\begin{aligned} Y \cap \tau B_\infty^m &= \{y \in Y\colon\ |\langle y, e_i\rangle| \leq \tau,\ \ \forall\, i\} \\ &= \{y \in Y\colon\ |\langle y - y_0, e_i\rangle + \langle y_0, e_i\rangle| \leq \tau,\ \ \forall\, i\} \\ &= \left\{y \in Y\colon\ \frac{-\alpha_i - \tau}{\sqrt{c_i}} \leq \langle y - y_0, u_i\rangle \leq \frac{-\alpha_i + \tau}{\sqrt{c_i}}\right\}. \end{aligned}$$

Therefore

$$(Y \cap \tau B_\infty^m) - y_0 = \left\{x \in Y - y_0\colon\ \frac{-\alpha_i - \tau}{\sqrt{c_i}} \leq \langle x, u_i\rangle \leq \frac{-\alpha_i + \tau}{\sqrt{c_i}}\right\}.$$

Now we can imitate Ball's argument from [2] concerning sections of maximal volume. Instead of the Lebesgue measure, we have to consider the Gaussian measure.

In $Y - y_0$, the identity operator can be written as $\sum_1^m c_i u_i \otimes u_i$. In particular,

$$\sum_{i=1}^m c_i = j \quad \text{and} \quad \|x\|^2 = \sum_{i=1}^m c_i \langle x, u_i\rangle^2, \qquad \forall x \in Y - y_0.$$

Therefore the Gaussian measure in $Y - y_0$ is equal to

$$(2\pi)^{-j/2} \exp\left(-\frac{1}{2}\sum_{i=1}^m c_i \langle x, u_i\rangle^2\right) dx.$$

Let $\chi_i$ denote the characteristic function of the interval $[(-\alpha_i - \tau)/\sqrt{c_i}, (-\alpha_i + \tau)/\sqrt{c_i}\,]$. Then, by the above,

$$\begin{aligned} \gamma_j((Y \cap \tau B_\infty^m) - y_0) &= (2\pi)^{-j/2} \int_{Y-y_0} \left(\prod_{i=1}^m \chi_i(\langle x, u_i\rangle) e^{-c_i\langle x, u_i\rangle^2/2}\right) dx \\ &= (2\pi)^{-j/2} \int_{Y-y_0} \prod_{i=1}^m (\chi_i(\langle x, u_i\rangle) e^{-\langle x, u_i\rangle^2/2})^{c_i}\, dx \\ &\le (2\pi)^{-j/2} \prod_{i=1}^m \left(\int_{(-\alpha_i-\tau)/\sqrt{c_i}}^{(-\alpha_i+\tau)/\sqrt{c_i}} e^{-s^2/2}\, ds\right)^{c_i}. \end{aligned} \tag{19}$$

The last inequality is a consequence of Brascamp–Lieb's inequality, which is stated in [2] as follows:

**Lemma.** *Let $(u_i)_1^m$ be a sequence of unit vectors in $\mathbb{R}^n$ and let $(c_i)_1^m$ be a sequence of positive numbers so that*

$$\sum_1^m c_i u_i \otimes u_i = I_n.$$

*For each $i$, let $f_i\colon \mathbb{R} \to [0, \infty)$ be integrable. Then*

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(\langle u_i, x\rangle)^{c_i}\, dx \le \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i\right)^{c_i}.$$

The $i$th integral in the product of (19) is not larger than $\int_{-\tau/\sqrt{c_i}}^{\tau/\sqrt{c_i}} e^{-s^2/2}\, ds$. Hence the last expression in (19) is bounded above by

$$(2\pi)^{-j/2} \prod_{i=1}^m \left(2\int_0^{\tau/\sqrt{c_i}} e^{-s^2/2}\, ds\right)^{c_i},$$

which is maximized when all the $c_i$'s are equal. Hence

$$\begin{aligned} \gamma_j((Y \cap \tau B_\infty^m) - y_0) &\le \left(\sqrt{\frac{2}{\pi}} \int_0^{\tau\sqrt{m/j}} e^{-s^2/2}\, ds\right)^j \\ &= \gamma_j\left(\tau\sqrt{\frac{m}{j}} B_\infty^j\right). \end{aligned} \tag{20}$$

The proof of Lemma 3.3 is complete. □

*Proof of Lemma* 3.2. For $g \in \mathrm{O}(n)$,

$$\begin{aligned}\sigma_j(g\mathbb{S}^{k-1} \cap A) &= \gamma_{j+1}\left(C(F_{n-k+j}) \cap \operatorname{span}(g\mathbb{S}^{k-1})\right) \\ &= \gamma_{j+1}\left(C[F_{n-k+j} \cap \operatorname{span}(g\mathbb{S}^{k-1})]\right).\end{aligned}$$

The second equality is a consequence of the identity $C(F_{n-k+j}) \cap X = C(F_{n-k+j} \cap X)$, which trivially holds for every subspace $X \subset \mathbb{R}^n$. Fix a subspace $X \in G_{n,k}$ for which the section $X \cap F_{n-k+j}$ is $j$-dimensional; almost every $X \in G_{n,k}$ has this property. Let $C$ denote the $(j+1)$-dimensional cone generated by $X \cap F_{n-k+j}$; put $X_0 = \operatorname{span} C$. By $M$ we denote the affine subspace spanned by $X \cap F_{n-k+j}$, and by $d$ its distance from the origin of $X$. The Gaussian measure of the cone $C$ is computed as follows. Take the unit vector $\xi \in X_0$ which is orthogonal to $M$, and for which $d\xi \in M$. For $t > 0$, put $W_t = \{x \in X_0\colon \langle x, \xi\rangle = t\}$. Observe that $C \cap W_t = (t/d)(X \cap F_{n-k+j})$. Let $P$ denote the orthogonal projection from $X_0$ onto $W_0$. By Fubini's theorem,

$$\begin{aligned}\gamma_{j+1}(C) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-t^2/2}\gamma_j(P(C \cap W_t))\,dt \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-t^2/2}\gamma_j\left(P\frac{t}{d}(X \cap F_{n-k+j})\right)dt. \qquad (21)\end{aligned}$$

Our task is to estimate the expression $\gamma_j(P\tau(X \cap F_{n-k+j}))$ for every $\tau > 0$. We need to discuss Gaussian measures in different subspaces. Whenever $M$ is an $m$-dimensional subspace of $\mathbb{R}^n$ and $q \in M$, let $\mathbb{G}_{M,q}$ denote the measure $(2\pi)^{-m/2}\exp(-\|x-q\|^2/2)\,dx$. In case $M$ is an $m$-dimensional linear subspace of $\mathbb{R}^n$ and $q = 0$ we simply write $\mathbb{G}_{M,0} = \gamma_m$. If $T$ is an isometry of $\mathbb{R}^n$, then for every Borel subset $S \subset M$ we have

$$\mathbb{G}_{M,q}(S) = \mathbb{G}_{TM,Tq}(TS). \qquad (22)$$

We momentarily assume that $\tau = 1$. Let $q$ denote the nearest point of $M$ to the origin of $X$. Both $M$ and the range of the projection $P$ are $j$-dimensional affine subspaces of $X_0$. We have

$$P(X \cap F_{n-k+j}) = (X \cap F_{n-k+j}) - q,$$

hence, by (22),

$$\mathbb{G}_{M,q}(X \cap F_{n-k+j}) = \gamma_j(P(X \cap F_{n-k+j})). \qquad (23)$$

Now let $L$ denote the affine subspace spanned by $F_{n-k+j}$, whose origin $O_L$ is taken as the center of the face $F_{n-k+j}$. (So if $X$ passes through the center of $F_{n-k+j}$, then $q = O_L$.) $M$ is also a $j$-dimensional affine subspace of $L$. By (22),

$$\mathbb{G}_{M,q}(X \cap F_{n-k+j}) = \mathbb{G}_{M-(q-O_L),O_L}\left((X \cap F_{n-k+j}) - (q - O_L)\right).$$

Applying the same argument for arbitrary $\tau > 0$ we conclude that

$$\gamma_j(P\tau(X \cap F_{n-k+j})) = \mathbb{G}_{\tau M-\tau(q-O_L),\tau O_L}\left(\tau(X \cap F_{n-k+j}) - \tau(q - O_L)\right). \qquad (24)$$

We may think of $\tau L$ as $\mathbb{R}^{n-k+j}$, of $\tau F_{n-k+j}$ as $\tau B_\infty^{n-k+j}$, and of $\tau(X \cap F_{n-k+j})$ as an affine $j$-dimensional section of $\tau B_\infty^{n-k+j}$. Thus for each $t > 0$ Lemma 3.3 can be used with $\tau = t/d$ and $m = n-k+j$. By the definition of $d$, we have $d \geq \sqrt{k-j}$. Combining (18), (21), and (24) we deduce that

$$\begin{aligned}\gamma_{j+1}(C) &\leq \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-t^2/2}\gamma_j\left(t\left(\frac{n-k+j}{j(k-j)}\right)^{1/2}B_\infty^j\right)dt\\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{j(k-j)}{n-k+j}\right)^{1/2}\int_0^\infty \exp\left(-\frac{j(k-j)}{n-k+j}t^2/2\right)\gamma_j(tB_\infty^j)\,dt.\end{aligned}$$

The proof of Lemma 3.2 and thus of Proposition 3.1 is complete. □

By using the asymptotic formulas of Section 2, namely Lemma 2.3 and Corollary 2.4, we can now prove the following result, which shows that the lower bound for $f(j,k,n)$ derived from Proposition 3.1 is, in some cases, asymptotically best possible.

**Corollary 3.4.** *For fixed integers* $1 \leq l < m$,

$$f(n-m, n-l, n) \backsim \frac{(2n)^{m-l}}{(m-l)!} \qquad as \quad n \to \infty. \tag{25}$$

*Proof.* Put $\alpha_n = (m-l)(n-m)/(n-m+l)$. By Proposition 3.1,

$$\frac{f(0, m-l, n)}{f(n-m, n-l, n)} < \sqrt{\frac{2\alpha_n}{\pi}}\int_0^\infty e^{-\alpha_n t^2/2}\gamma_{n-m}(tB_\infty^{n-m})\,dt. \tag{26}$$

Put $b_n = (\log(n-m))^{(\alpha_n-1)/2}/(n-m)^{\alpha_n}$ and $c_n = (\log n)^{(m-l-1)/2}$. Let $d_n$ denote the right-hand side of (26), from which we get

$$f(n-m, n-l, n)\frac{b_n}{c_n} > \frac{f(0, m-l, n)}{c_n}\frac{b_n}{d_n}.$$

Since $\lim_{n\to\infty}\alpha_n = (m-l)$, Lemma 2.3 implies that

$$\lim_{n\to\infty}\frac{b_n}{d_n} = \frac{1}{\pi^{(m-l-1)/2}\Gamma(m-l)\sqrt{m-l}}.$$

Moreover, by Corollary 2.4,

$$\lim_{n\to\infty}\frac{f(0, m-l, n)}{c_n} = \frac{2^{m-l}\pi^{(m-l-1)/2}}{\sqrt{m-l}}.$$

Thus, the sequence $f(n-m, n-l, n)(b_n/c_n)$ is larger than a sequence that tends to $2^{m-l}/(m-l)!$ as $n$ tends to infinity. On the other hand we have $f(n-m, n-l, n) < 2^{m-l}\binom{n}{m-l}$, so

$$f(n-m, n-l, n)\frac{b_n}{c_n} < 2^{m-l}\binom{n}{m-l}\frac{b_n}{c_n},$$

and since $b_n/c_n \backsim n^{l-m}$, the right-hand side here tends to $2^{m-l}/(m-l)!$. Consequently,

$$\lim_{n\to\infty} f(n-m, n-l, n)\frac{b_n}{c_n} = \frac{2^{m-l}}{(m-l)!}.$$

The required asymptotic formula follows immediately. The proof of Corollary 3.4 is complete. □

**Remarks.** 1. The previous corollary implies that the number of $(n-m)$-faces of a random $(n-l)$-section of the $n$-cube tends to concentrate near the value $2^{m-l}\binom{n}{m-l}$, which bounds it from above. So, for example, a typical 1-codimensional section of the $n$-cube will have $2n - o(n)$ facets as $n \to \infty$. This result can also be deduced from the identity (3). Indeed, by duality we have $f(n-m, n-l, n) = \mathbf{E}(f_{m-l-1}(\Pi_{n-l}(B_1^n)))$, and replacing $T^n$ by $B_1^n$ in the proof of Theorem 2 in [1] (the details of this replacement appear in [5]; see the proof of Theorem 1.1 there) we get the previous corollary.

2. According to a remark made in [5], the number $f(j, k, n)$ is equal to the expected number of $(k-j-1)$-faces of the convex hull of $\pm G_1, \dots, \pm G_n$, where the $G_i$'s are independent copies of a $k$-dimensional Gaussian vector. See also [4]. Hence, the results for $f(0, k, n)$ can be interpreted as results for the expected number of facets of the convex hull of $\{\pm G_i\}_1^n$ in $\mathbb{R}^k$. For example, we can translate the first remark at the end of Section 2 to the following statement:

*If three points in the plane are chosen at random, then their symmetric convex hull is more likely to be a parallelogram than a hexagon.*

## Acknowledgments

I thank Itai Benjamini for getting me interested in the subject of this paper, and for several stimulating discussions. I also thank the referees for their useful comments.

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*Received August* 1, 1998, *and in revised form December* 15, 1998.