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Art Galleries with Interior Walls

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Abstract. Consider an art gallery formed by a polygon on n vertices with m pairs of vertices joined by interior diagonals, the interior walls. Each interior wall has an arbitrarily placed, arbitrarily small doorway. We show that the minimum number of guards that suffice to guard all art galleries with n vertices and m interior walls is $\min\{\lfloor(2n-3)/3\rfloor, \lfloor(2n+m-2)/4\rfloor, \lfloor(2m+n)/3\rfloor\}$. If we restrict ourselves to galleries with convex rooms of size at least r, the answer improves to $\min\{m, \lfloor(n+m)/r\rfloor\}$. The proofs lead to linear time guard placement algorithms in most cases.

Introduction

The original art gallery problem, posed by Klee and solved by Chvátal [6], is to find the smallest number of guards necessary to cover any simple polygon, the art gallery, not necessarily convex, on n vertices. Here a covering by g guards means that one can find g points in the interior of the polygon such that every point in the interior is covered by some guard, that is, for each point in the interior the line segment between it and some guard does not intersect the polygon. The *comb polygons* in Fig. 1 show that $\lfloor n/3\rfloor$ guards are sometimes necessary—if n is not divisible by 3 simply take a comb on $3\lfloor n/3\rfloor$ vertices and subdivide one or two of its edges. Chvátal also showed that $\lfloor n/3\rfloor$ guards always suffice. For more information on the history of this problem and related problems, see [15] and [17].

Hutchinson [11] generalized the basic art gallery problem by allowing interior walls. Throughout this paper an *art gallery* (*with interior walls*) will be a simple polygon on n vertices with some pairs of vertices joined by nonintersecting interior diagonals, the *interior walls*. Also suppose that in the interior of each of the walls there is an arbitrarily placed, arbitrarily small opening, the *doorway*. Figure 2 is an example of an art gallery on $n = 15$ vertices and 8 interior walls that requires 9 guards. Hutchinson now asked to

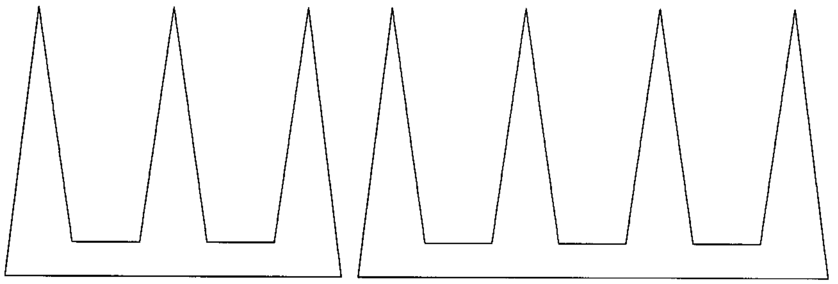

Fig. 1. Comb polygons.

determine the minimum number of guards that suffice to cover any such art gallery on n vertices.

To motivate our proofs we give Fisk's [9] elegant proof of Chvátal's result:

First triangulate the polygon. The resulting plane graph, with vertex set the corner points of the polygon, has all its vertices on the outside face. Graphs that can be embedded in the plane in such a way are called *outerplanar*. It is well known that outerplanar graphs are 3-colorable, which can be easily seen by cutting along a chord and applying induction. Since each triangle in the triangulation must have vertices of all three colors, putting a guard at each vertex in the smallest color class produces a covering set of $\lfloor n/3 \rfloor$ guards.

We now answer Hutchinson's question with an argument in the spirit of Fisk's proof (see also [13]).

Lemma 1. $\lfloor (2n-3)/3 \rfloor$ *guards suffice to cover any art gallery on n vertices and there are galleries with* $\lfloor 2n/3 \rfloor - 2$ *interior walls where this many guards are required*.

Proof. We may assume that the interior walls triangulate the art gallery, since adding extra interior walls cannot make it easier to guard the gallery. This outerplanar graph can now be 3-colored. From such a coloring, we get a labeling of the edges of the graph by assigning to each edge the color *not used* on its endpoints. Now each triangle has each color appearing on one of its incident edges. Placing a guard in the doorway, for interior walls, or just next to the wall, for exterior walls, we can see that each set of labels corresponds to a set of guards that covers the whole art gallery. Since n-vertex

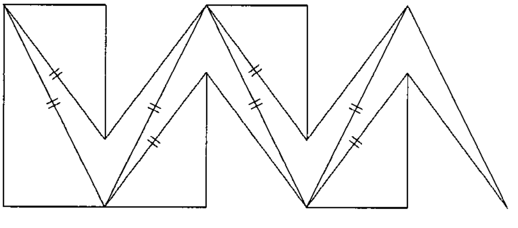

Fig. 2. Art gallery with interior walls

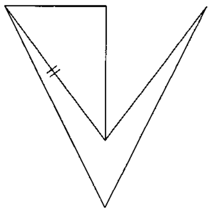

Fig. 3. V.

outerplanar triangulations have $2n - 3$ edges, which can be seen by induction or by Euler's formula, taking the least frequent color suffices.

Art galleries of the type in Fig. 2 achieve this bound. They are obtained by starting with a small gallery, and attaching $k = \lfloor n/3 \rfloor - 1$ of the V-shaped galleries in Fig. 3. If we attach a V with the leftmost wall to an already existing art gallery and put the doorway exactly in the center we increase the number of vertices by three, the number of interior walls by two, and the number of guards needed by two. Note that even if a guard from the smaller gallery can be placed in the doorway of the interior wall connecting the V to the smaller gallery that guard still cannot see the other branch of the V or the triangle, so that those will require one additional guard each.

When $n = 3k + 3$ our starting gallery is a triangle (as in Fig. 2), when $n = 3k + 4$ any quadrilateral, and when $n = 3k + 5$ a V. This produces art galleries with $\lfloor 2n/3 \rfloor - 2$ (that is $2k$, $2k$, and $2k + 1$, respectively) interior walls. The galleries also require exactly one more guard than they have interior walls. □

This settles the problem when the number of interior walls is unspecified. However, what happens when we have a specified number of interior walls, say m? This question was suggested by J. Griggs. If $m \geq \lfloor 2n/3 \rfloor - 2$, then Lemma 1 shows that the answer is still $\lfloor (2n - 3)/3 \rfloor$, since adding additional interior walls in the art galleries provided does not make guarding any easier.

Theorem 2. *The minimum number of guards that suffice to cover all art galleries with n vertices and m interior walls, $g(n, m)$, is*

$$\min\left\{\left\lfloor\frac{2n-3}{3}\right\rfloor, \left\lfloor\frac{2m+n}{3}\right\rfloor, \left\lfloor\frac{2n+m-2}{4}\right\rfloor\right\}$$

or, more precisely,

$$g(n,m) = \begin{cases} \left\lfloor\dfrac{2n-3}{3}\right\rfloor & \textit{for} \quad m \geq \lfloor\frac{2}{3}n\rfloor - 2, \\ \left\lfloor\dfrac{2m+n}{3}\right\rfloor & \textit{for} \quad m < \lfloor\frac{2}{5}n\rfloor, \\ \left\lfloor\dfrac{2n+m-2}{4}\right\rfloor & \textit{otherwise.} \end{cases}$$

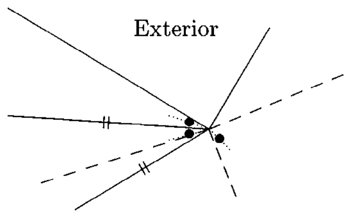

Fig. 4. Guard placement.

Lemma 1 proves the first part of the statement, and the other two parts are Lemmas 3 and 4.

Lemma 3. *Always $g(n, m) \le \lfloor (2m + n)/3 \rfloor$, and for $m < \lfloor 2n/5 \rfloor$ there are galleries with m interior walls where this many guards are required.*

Proof. The bound can be easily established by induction, but there is also a Fisk-type argument. Before triangulating the art gallery, assign each vertex v a weight of $d(v) - 1$, where $d(v)$ is the degree of v in the outerplanar graph determined by the gallery, i.e., the number of walls meeting at v. For example in the art gallery without interior walls each vertex has weight 1. Now triangulate the gallery, also using the interior walls that are already present, 3-color the triangulation, and find the color class of smallest total weight W. It will suffice to find a guard set with W guards, since the total weight on all vertices is

$$\sum_{v \in V(G)} (d(v) - 1) = 2|E(G)| - n = 2m + n.$$

To do this simply put $d(v) - 1$ guards at each vertex v in the color class of total weight W by putting one guard on every interior angular bisector of walls that meet v, as shown in Fig. 4, close to v. Note that when we place the guards we ignore the chords that are not walls and were introduced in the triangulation step. Since every triangle has a vertex of our chosen color, and each triangle is covered by a guard at the vertex associated with it, we are done.

Call the art gallery in Fig. 5 an E. Notice that an E has 7 vertices, 1 interior wall, and requires 3 guards since no guard can cover more than one of the three alcoves. If we

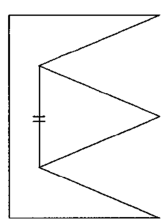

Fig. 5. E.

attach an E with the vertical wall to an already existing art gallery and put the doorway exactly in the center we increase the number of vertices by five, the number of interior walls by two and the number of guards necessary by three. We put the doorway in the middle of the vertical wall, so that a guard there cannot cover either alcove.

To achieve the bound, let $m = 2k + \varepsilon$ with $\varepsilon \in \{0, 1\}$.

Start the construction with a comb on $n - 5k - \varepsilon$ vertices. Since $n - 5k - \varepsilon \geq 3 + \varepsilon$, such a comb exists. If m is odd, attach one triangular room to a wall parallel to the long horizontal wall in the comb. This adds 1 vertex, 1 interior wall, and 1 guard, since the guard that is needed to cover the triangle cannot cover any prong of the comb. (If $n - 5k - 1 \in \{4, 5\}$, then start with a V and subdivide any wall if necessary.) In either case add k E's, starting from an end of the comb. The resulting art gallery has n vertices and m interior walls. The number of guards required is

for $m = 2k$:

$$\left\lfloor \frac{n - 5k}{3} \right\rfloor + 3k = \left\lfloor \frac{n + 4k}{3} \right\rfloor = \left\lfloor \frac{2m + n}{3} \right\rfloor;$$

for $m = 2k + 1$:

$$\left\lfloor \frac{n - 5k - 1}{3} \right\rfloor + 1 + 3k \; = \; \left\lfloor \frac{n + 4k + 2}{3} \right\rfloor = \left\lfloor \frac{2m + n}{3} \right\rfloor.$$

□

Lemma 4. *Always* $g(n, m) \leq \lfloor (2n + m - 2)/4 \rfloor$, *and for* $\lfloor (2n - 4)/5 \rfloor \leq m \leq \lfloor (2n - 5)/3 \rfloor$ *there are galleries with* m *interior walls where this many guards are required*.

Proof. For the construction let $k = \lfloor (2n - 3m - 5)/4 \rfloor \geq 0$. Start with the art gallery with $n - 5k \geq 3$ vertices and $m - 2k$ interior walls constructed in Lemma 1. It is a straightforward computation to check that indeed $\lfloor 2(n - 5k)/3 \rfloor - 2 \leq m - 2k$ (use the fact that, for every integer x, $x \geq \lfloor y \rfloor \iff x + 1 > y$), so that the construction from Lemma 1 applies. Now add k E's. The number of guards needed for this art gallery is

$$\begin{aligned} 3k + \left\lfloor \frac{2(n - 5k) - 3}{3} \right\rfloor &= \left\lfloor \frac{2n - \lfloor (2n - 3m - 5)/4 \rfloor - 3}{3} \right\rfloor \\ &\geq \left\lfloor \frac{8n - 2n + 3m + 5 - 12}{12} \right\rfloor \\ &= \left\lfloor \frac{2n + m - 2}{4} \right\rfloor, \end{aligned}$$

unless $2n + m - 2$ is divisible by 4. However, in that case our inequality was not sharp and we can gain an additional $\frac{1}{12}$ from that term.

We now prove the upper bound by induction on n, with the added requirement that when $2n + m - 2$ is divisible by 4 we can place one of the guards arbitrarily at an exterior wall.

The bound holds for $m = 0$, since $g(n, 0) = \lfloor n/3 \rfloor \leq \lfloor (n - 1)/2 \rfloor$ for $n \geq 3$. This is the only feasible value for the base case $n = 3$. The added requirement also holds when $n = 3$: we can place the guard where we want since triangles are convex.

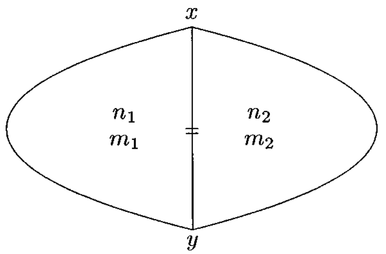

Fig. 6.

Consider an art gallery A with $n > 3$ vertices and $m > 0$ walls, and let $g(A)$ be the number of guards needed to cover A. Cut the art gallery along an interior wall xy (see Fig. 6). This splits the gallery into two parts sharing two vertices. For $i \in \{1, 2\}$, let n_i and m_i be the number of vertices and interior walls in the ith part. Hence $n_1 + n_2 = n + 2$ and $m_1 + m_2 = m - 1$, so that

$$\begin{aligned} g(A) &\le g(n_1, m_1) + g(n_2, m_2) \le \left\lfloor \frac{2n_1 + m_1 - 2}{4} \right\rfloor + \left\lfloor \frac{2n_2 + m_2 - 2}{4} \right\rfloor \\ &\le \frac{2(n_1 + n_2) + (m_1 + m_2) - 4}{4} = \frac{2n + m - 1}{4}. \end{aligned}$$

Therefore $g(A) \le \lfloor (2n+m-2)/4 \rfloor$, unless equality holds everywhere, which implies that $(2n_1 + m_1 - 2)$ and $(2n_2 + m_2 - 2)$ are divisible by 4. However, in that case we can invoke the stronger hypothesis and require guards to be on either side of the separating wall xy right next to the doorway. Now replacing these two guards by a single guard in the doorway yields the claim.

To show the additional requirement now suppose that $2n + m - 2$ is divisible by 4 and that a guard must be placed near exterior wall uv. Triangulate the gallery, also using the interior walls that are already present, and let w be the third vertex in the triangle containing uv. Removing the triangle uvw, which is already covered, splits the gallery

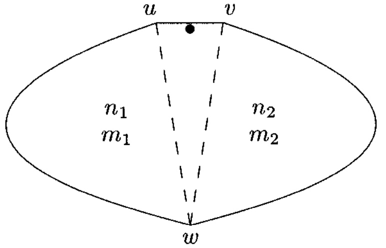

Fig. 7.

into two parts (see Fig. 7), satisfying $n_1 + n_2 = n + 1$ and $m_1 + m_2 \leq m$. Thus

$$\begin{aligned} g(A) &\leq g(n_1, m_1) + g(n_2, m_2) + 1 \leq \left\lfloor \frac{2n_1 + m_1 - 2}{4} \right\rfloor + \left\lfloor \frac{2n_2 + m_2 - 2}{4} \right\rfloor + 1 \\ &\leq \frac{2(n_1 + n_2) + (m_1 + m_2)}{4} \leq \frac{2n + m + 2}{4}. \end{aligned}$$

This also works in the degenerate case when one of the $n_i = 2$. So we are done unless equality holds everywhere, since $2n + m - 2$ is divisible by 4. This implies that $(2n_1 + m_1 - 2)$ and $(2n_2 + m_2 - 2)$ are divisible by 4. Also $m_1 + m_2 = m$ so that neither of the chords uw or vw is an interior wall. Again we invoke the stronger induction hypothesis and require guards near the chords uw and vw close to w. Replacing these two guards by a single guard right at w finishes the proof. □

Art Galleries with Convex Rooms

In [7] Czyzowicz et al. study art galleries that consist of polygons on n vertices that are subdivided, not necessarily along chords, into k *convex* regions and show that these can be covered with $\lfloor 2(n+k)/3 \rfloor$ guards. This result is independent of Theorem 2, since our problem allows rooms of arbitrary shape but requires the interior walls to be chords.

It would be a common special case to study art galleries with n vertices and m interior walls such that all $k = m + 1$ interior rooms are convex. Notice that we are not requiring the polygon itself to be convex. However, doing so does not change the answer, since our construction achieving the upper bound can easily be built to meet this additional requirement. Since it will pose no additional difficulty, we also require each room to have at least $r \geq 3$ walls, with $r = 3$ being the general case.

Summing the sizes of the rooms yields $n + 2m$, since the interior walls are counted twice. Hence for art galleries such that all rooms have size at least r, we have $n + 2m \leq (m + 1)r$, or equivalently $n \geq m(r - 2) + r$.

Theorem 5. *The minimum number of guards that suffice to cover all art galleries with $m > 0$ interior walls and $n \geq m(r-2)+r$ vertices, such that all rooms are convex with at least r walls, $g_r^*(n, m)$, is*

$$\min\left\{m, \left\lfloor \frac{n+m}{r} \right\rfloor\right\}$$

or, more precisely,

$$g_r^*(n, m) = \begin{cases} m & \text{for} \quad m \leq \left\lfloor \dfrac{n}{r-1} \right\rfloor, \\[2ex] \left\lfloor \dfrac{n+m}{r} \right\rfloor, & \text{for} \quad m > \left\lfloor \dfrac{n}{r-1} \right\rfloor. \end{cases}$$

Proof. The bounds are straightforward. For the first bound, place a guard in each doorway. Since the rooms are convex every room can be covered by at least one of the guards.

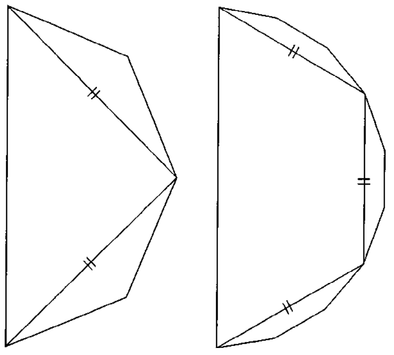

Fig. 8. D_3, D_4, and D_5.

For the second bound, one can first provide a labeling of the walls with r labels, such that each room has every label on one of its walls. We do so by induction on m with the result being trivial for $m = 0$. For $m > 0$ cut along an interior wall and apply induction on both parts. To combine both labelings into one it may be necessary to swap labels in one of the parts if they disagree on the separating wall. Now placing one guard at each wall suggested by the label used least frequently establishes the bound.

For the construction, let D_r be an art gallery with one central room and $r - 1$ side rooms of size r. The side rooms share a wall only with the central room. Figure 8 shows D_3, D_4, and D_5. Each D_r has $r - 1$ interior walls, $r(r - 2) + 2$ vertices, and requires $r - 1$ guards.

If $m \leq \lfloor n/(r - 1) \rfloor$, our gallery will be similar to a D_r:

Take a center room of size $n - m(r - 2) \geq r$, and attach m side rooms of size r each. This is possible, since when $m \leq \lfloor n/(r - 1) \rfloor$ we have $n - m(r - 2) \geq m$, and this gallery requires m guards.

If on the other hand $m > \lfloor n/(r - 1) \rfloor$, then let $k = \lceil (m(r - 1) - n)/r \rceil > 0$. We form our gallery by taking a gallery on $n' = n - kr(r - 2)$ vertices and $m' = m - kr \geq 0$ interior walls and then attaching k D_r's to it. Always attaching the next D_r with the free wall of its center room to any other room we obtain a gallery on n vertices and m walls.

It can be readily checked that $n' \geq m'(r - 2) + r$ and $n' \geq m'(r - 1)$ so that we can take a gallery from the first case to start out with. So we need

$$m' + k(r - 1) = m - k = \left\lfloor m - \frac{m(r - 1) - n}{r} \right\rfloor = \left\lfloor \frac{m + n}{r} \right\rfloor$$

guards to cover this gallery. □

Complexity

In implementing Fisk's proof, Avis and Toussaint [2] obtained an $O(n \log n)$ algorithm to place $\lfloor n/3 \rfloor$ guards to guard an n-vertex art gallery. This approach can be improved to obtain a linear time algorithm, since Chazelle [3], [4] showed that an n-vertex polygon can be triangulated, at least theoretically, in time $O(n)$.

First triangulate the polygon in time $O(n)$. Since outerplanar triangulations are chordal graphs, one can find a vertex elimination scheme (see, for example, [16]) and then use this to obtain a 3-coloring of the vertices [10], both in $O(n + e) = O(n)$ time. Now placing the guards just requires $O(n)$ time and the algorithm is linear.

From here it is easy to see how the upper bound arguments in Lemmas 1 and 3 can be used to find linear time algorithms for these problems too. The only problem could be that the existence of interior walls does not necessarily make triangulation easier. However, we can triangulate each room separately. This is still possible in linear time, since they have a total of $n + 2m < 3n$ vertices. Finding the rooms and therefore also the weak dual can be done in linear time. See [5] or [12] to find the rotation scheme from which this can be done.

A straightforward implementation of Lemma 4 results in an $O(n^2)$ algorithm, due to the stronger statement, even for $m = 0$. However, this case can be implemented in linear time even with the stronger statement. For $n = 3$ just add the guard where requested. For $n \geq 4$ apply the basic algorithm and just add the extra guard at the required place if necessary. This will work since $\lfloor n/3 \rfloor + \varepsilon \leq \lfloor (n-1)/2 \rfloor$, for $n \geq 4$ with $\varepsilon = 1$ when n is odd and 0 otherwise.

This makes a faster algorithm plausible, however, it is still an open question whether a linear time algorithm can be obtained in this case. A Fisk-type proof for the upper bound in Lemma 4 would certainly yield a fast algorithm.

In the case of convex rooms the situation is easier. The first bound trivially leads to an $O(m)$ algorithm. The second bound can be implemented in time $O(m) = O(n)$ as well, since the labeling can be found in linear time: Find the weak dual of the art gallery, then starting at any vertex conduct a Breadth-First Search on this dual tree to determine the order in which the rooms will be labeled. In the first room label the edges using each one of the r labels at least once. On each consecutive room one wall is already labeled, so label the remaining walls accordingly to assure that every label is being used.

It is important to note that although these algorithms give fast algorithms for guarding given classes of art galleries efficiently they do not necessarily give the best possible answer for a specific art gallery. This problem is known to be NP-hard even when the art gallery has no interior walls [1], [14].

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References

1. A. Aggarwal, The art gallery theorem: its variations, applications and algorithmic aspects, Ph.D. thesis, The Johns Hopkins University, Baltimore, MD, 1984.
2. D. Avis, G.T. Toussaint, An efficient algorithm for decomposing a polygon into star-shaped polygons, *Pattern Recognition* **13** (1981), 395–398.
3. B. Chazelle, Triangulating a simple polygon in linear time, *Proc.* 31*st Symp. Foundations of Computer Science*, vol. I (1990), pp. 220–230.

4. B. Chazelle, Triangulating a simple polygon in linear time, *Discrete Comput. Geom.* **6** (1991), 485–524.
5. N. Chiba, T. Nishizeki, S. Abe, T. Ozawa, A linear algorithm for embedding planar graphs using PQ-trees, *J. Comput. System Sci.* **30** (1985), 54–76.
6. V. Chvátal, A combinatorial theorem in plane geometry, *J. Combin. Theory Ser. B* **18** (1975), 39–41.
7. J. Czyzowicz, E. Rivera-Campo, N. Santoro, J. Urrutia, J. Zaks, *Tight Bounds for the Rectangular Art Gallery Problem*, Lecture Notes in Computer Science, vol. 570, Springer-Verlag, Berlin, 1992, pp. 105–112.
8. J. Czyzowicz, E. Rivera-Campo, N. Santoro, J. Urrutia, J. Zaks, Guarding rectangular art galleries, *Discrete Appl. Math.* **50** (1994), 149–157.
9. S. Fisk, A short proof of Chvátal's watchman theorem, *J. Combin. Theory Ser. B* **24** (1978), 374.
10. F. Gavril, Algorithms for minimum coloring, maximum clique, minimum covering by cliques, and maximum independent set of a chordal graph, *SIAM J. Comput.* **1** (1972), 180–187.
11. J. Hutchinson, Art galleries with walls, Problem #10478, *Amer. Math. Monthly* **102** (1995), 746.
12. R. Jayakumar, K. Thulasiraman, M. Swamy, Planar embedding: linear-time algorithms for vertex placement and edge ordering, *IEEE Trans. Circuits Systems* **35** (1988), 334–344.
13. A. Kündgen, Art galleries with walls, solution to Problem #10478, *Amer. Math. Monthly* **105** (1998), 247–248.
14. D. T. Lee, A. K. Lin, Computational complexity of art gallery problems, *IEEE Trans. Inform. Theory* **32** (1986), 276–282.
15. J. O'Rourke, *Art Gallery Theorems and Algorithms*, The International Series of Monographs on Computer Science, The Clarendon Press, New York, 1987.
16. D.J. Rose, R.E. Tarjan, G.S. Lueker, Algorithmic aspects of vertex elimination on graphs, *SIAM J. Comput.* **5** (1976), 266–283.
17. T. Shermer, Recent results in art galleries, *Proc. IEEE* **80** (1992), 1384–1399.

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