Discrete Comput Geom 22:231–248 (1999)

Discrete & Computational Geometry

The Translative Kissing Number of Tetrahedra Is 18*

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Abstract. We show that the maximum number of mutually nonoverlapping translates of any tetrahedron T which touch T is 18. Moreover, in the case of 18 touching translates the arrangement turns out to be unique. We also give a description of all possible arrangements of 17 touching translates. Finally, we apply these results to determine the minimum and maximum densities of 17^+-neighbor translative packings of tetrahedra.

1. Introduction

First we recall some standard definitions. By a *d-dimensional convex body* we mean a compact convex subset of R^d with nonempty interior. Two subsets of R^d with nonempty interiors are *nonoverlapping* if they have no common interior point, and we say that they *touch each other* if they are nonoverlapping but their intersection is nonempty. Denote by $H(K)$ the *translative kissing number* of a d-dimensional convex body K, which is defined as the maximum number of mutually nonoverlapping translates of K that can be arranged so that all touch K. $H(K)$ is often called the *Hadwiger number* of K as well.

It was proved by Swinnerton-Dyer [17] that $H(K) \geq d^2 + d$ holds for every d-dimensional convex body K ($d \geq 1$). A recent result of Talata [18] improves on this bound for sufficiently large values of d, showing that there exists an absolute constant $c > 0$ such that $H(K) \geq 2^{cd}$ for every d-dimensional convex body K. Combining this result with the inequality $H(K) \leq 3^d - 1$, which was proved by Hadwiger [9], it turns out that the order of magnitude of $H(K)$ is exponential in the dimension of K for every convex body K.

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* This work was partially supported by the Hungarian National Science Foundation under Grant No. A-221/95.

A natural problem is the determination of $H(K)$ when K belongs to some well-known classes of convex bodies. Groemer [6] proved that $H(K) = 3^d - 1$ if and only if K is a paralleletope. In the case of Euclidean balls, the exponential lower bound $H(K) \geq (2/\sqrt{3} + o(1))^d$ was found by Shannon [15] and Wyner [19].

Grünbaum [7] proved that if a two-dimensional convex body K is different from a parallelogram, then $H(K) = 6$. However, the exact determination of $H(K)$ can be a very hard problem even for some three-dimensional convex bodies. For example, when K is a three-dimensional Euclidean ball, then this problem leads to the well-known Newton-Gregory problem, posed in 1694, which was first solved by Hoppe [10] in 1874, showing that in this case $H(K) = 12$. In the present paper we determine the translative kissing number of tetrahedra.

Let T be a three-dimensional tetrahedron. Zong [21] proved that $18 \leq H(T) \leq 19$ and conjectured that $H(T) = 18$. He also conjectured that there exists a unique lattice arrangement of nonoverlapping translates of T in which T has 18 touching members. In the following theorem we not only verify Zong's conjectures, but we are able to give a stronger uniqueness property.

Theorem 1. *Let T be a tetrahedron in R^3. Then*

$$H(T) = 18.$$

Moreover, the arrangement of 18 *mutually nonoverlapping translates of T in which all the translates touch T is unique.*

In the next section we show that the problem of the determination of the translative kissing number $H(K)$ of a convex body K can be reformulated by using the notion of 1-discrete sets. Thus we can characterize the translative kissing numbers in another way, with which we are not only able to reformulate Theorem 1, but we can also give a complete description of the arrangements where there are exactly 17 mutually nonoverlapping translates of the tetrahedron T which touch T. This characterization will be useful in the proofs of Section 3 as well. Finally, in Section 4 we apply the obtained description of arrangements of 17 touching translates to get the minimum and maximum densities of 17^+-neighbor translative packings of tetrahedra. Namely, there we prove that the inequalities

$$\tfrac{19}{60} \leq d^-(\mathcal{T}') \leq d^+(\mathcal{T}') \leq \tfrac{1}{3}$$

hold for any 17^+-neighbor translative packing $\mathcal{T}'$ of a tetrahedron, and these bounds are sharp (see Proposition 4.2). (Here we denote by $d^-(\mathcal{T}')$ and $d^+(\mathcal{T}')$ the lower and the upper densities of $\mathcal{T}'$, respectively.)

For additional related results and references on this topic, see [1], [2], [20], [22], and the survey papers [4] and [5].

2. Reformulation of the Problem

First we introduce notation and recall some facts, which will help us to reformulate the problem of the determination of translative kissing numbers.

Let $A, B \subset R^d$. We define $\alpha A + \beta B$ as the set $\{\alpha a + \beta b \mid a \in A, b \in B\}$ for any $\alpha, \beta \in R$. We denote by $A - B$ the set $A + (-1)B$. If $v \in R^d$, then we write $A + v$ instead of $A + \{v\}$. A subset A' of R^d is called a *homothetic copy* of A if $A' = \alpha A + v$ for a suitable $\alpha > 0$ and $v \in R^d$. We denote by ∂A the set of boundary points of A. If A is finite, then we use the notation $|A|$ for the cardinality of A. A map $f\colon R^3 \to R^3$ is a *congruence* if $\|f(x) - f(y)\| = \|x - y\|$ for every $x, y \in R^3$, where $\|\cdot\|$ denotes the usual Euclidean norm.

From now on, K denotes an arbitrary d-dimensional convex body.

We recall a well-known observation, the so-called "difference body method" of Minkowski [13] (also see [14] for a description of the method). Let $v_1, v_2 \in R^d$. Then the two translates $K + v_1$ and $K + v_2$ of K are mutually nonoverlapping (resp. touching) if and only if $\frac{1}{2}(K - K) + v_1$ and $\frac{1}{2}(K - K) + v_2$ are mutually nonoverlapping (resp. touching).

A simple consequence of this observation is the following. Let $\{v_1, v_2, \ldots, v_n\} \subset R^d$. The translates $\{K + v_i\}_{i=1}^n$ of K are mutually nonoverlapping (resp. touch K) if and only if the elements of the arrangement $\{\frac{1}{2}(K - K) + v_i\}_{i=1}^n$ are mutually nonoverlapping (resp. touch $\frac{1}{2}(K - K)$). From this fact it follows that $H(K) = H(K - K)$, and that it would be enough to prove Theorem 1 for $T - T$ instead of the tetrahedron T. However, we choose another way for the proof. We consider the set $\{v_1, v_2, \ldots, v_n\}$ of centers of the translates of $\frac{1}{2}(K - K)$ to get another description for $H(K)$.

We use the notation $\|\cdot\|_{K-K}$ for the *Minkowski norm with unit ball* $K - K$, i.e., if $v \in R^d$, then $\|v\|_{K-K} = \|v\|/\|w\|$, where $\|\cdot\|$ denotes the usual Euclidean norm, and w is a vector parallel to v and having its endpoint on the boundary of $K - K$.

Let r be a positive real number. A set $S \subset R^d$ is *r-discrete in the metric determined by* $K - K$ (or simply *r-discrete*, when $K - K$ is fixed) if $\|p - q\|_{K-K} \geq r$ for every $p, q \in S$. We note that every bounded r-discrete set is finite for any $r > 0$.

It is easily seen by the previous arguments that $H(K)$ is equal to the maximum cardinality of 1-discrete subsets of $\partial(K - K)$ in the metric determined by $K - K$. Moreover, there exists a unique arrangement of $H(K)$ mutually nonoverlapping translates of K which all touch K if and only if there exists a unique 1-discrete subset of $\partial(K - K)$ in the metric determined by $K - K$ with maximum cardinality.

On the other hand, the translative kissing numbers are affine invariant quantities, thus it is enough to prove Theorem 1 in the case when T is a fixed regular tetrahedron. Then it is easy to see that $C = T - T$ is a cuboctahedron (i.e., the convex hull of the midpoints of the edges of some cube). Denote by $v(C)$ and $z(C)$ the set of vertices and face centers of C, respectively. Let p_1, p_2, p_3, p_4 denote the consecutive vertices of a square face of C. Then $o = \frac{1}{2}(p_1 + p_3) \in z(C)$. Define $S_0 = v(C) \cup z(C)$. Let α, β be real numbers, $0 \leq \alpha \leq \beta - \frac{1}{2} \leq \frac{1}{2}$, $q_1 = \alpha p_1 + (1 - \alpha)p_3$, and $q_2 = \beta p_1 + (1 - \beta)p_3$. Define $S_{\alpha,\beta} = \bigl(S_0 \backslash \{p_1, p_2, o\}\bigr) \cup \{q_1, q_2\}$. Let $0 < \gamma < 1$, $t_i = \gamma p_i + (1 - \gamma)p_{i+1}$, $1 \leq i \leq 4$, with the notation $p_5 = p_1$. Define $S_\gamma = \bigl(S_0 \backslash \{p_1, p_2, p_3, p_4, o\}\bigr) \cup \{t_1, t_2, t_3, t_4\}$. Then $S_0, S_{\alpha,\beta}, S_\gamma$ are 1-discrete subsets of ∂C, $|S_0| = 18$ and $|S_{\alpha,\beta}| = |S_\gamma| = 17$.

This way we have that the following theorem is the reformulation of Theorem 1 with an additional description of the sets formed by the translation vectors of the arrangements where there are exactly 17 mutually nonoverlapping translates of the tetrahedron T which all touch T.

Theorem 2. *Let C be a cuboctahedron, and let $S \subset \partial C$ be a 1-discrete set in the metric determined by C. Then $|S| \leq 18$. Moreover, if $S_0, S_{\alpha,\beta}, S_\gamma$ denote the finite subsets of ∂C as defined above, then we have the following characterization.*

If $|S| = 18$, then $S = S_0$. If $|S| = 17$, then there exists a congruence $f: R^3 \to R^3$ with $f(\partial C) = \partial C$, such that one of the following holds:

(1) *$f(S) = S_{\alpha,\beta}$ for suitable $\alpha, \beta \in R$, $0 \leq \alpha \leq \beta - \frac{1}{2} \leq \frac{1}{2}$;*
(2) *$f(S) = S_\gamma$ for a suitable $\gamma \in R$, $0 < \gamma < 1$.*

3. Proof of Theorem 2

The main ideas of the proof of this theorem can be described as follows. In Section 3.1 we define several subsets of ∂C and certain weighted counting measures on them. In Section 3.2 we make a complete list of arrangements, showing where the elements of a 1-discrete set in a "region" of ∂C can be situated. Then in Section 3.3 we give upper bounds for the weighted counting measure of a 1-discrete set S of ∂C in a "belt", which is the union of three regions. We also prove that $|S| \leq 16$ for a special configuration. Finally, in Section 3.4, using that the sum of some weighted counting measures of S is equal to $|S|$, we prove three propositions, which collectively imply Theorem 2.

3.1. *Notation and Terminology*

Let $A \subseteq R^d$, $a, b \in R^d$. We denote by $\mathrm{conv}(A)$ the convex hull of A. We use the notation ab or $[ab]$ for $\mathrm{conv}(a, b)$, $[ab)$ or $(ba]$ for $\mathrm{conv}(a, b)\setminus\{b\}$, and (ab) for $\mathrm{conv}(a, b)\setminus\{a, b\}$. Assume now that $A \subseteq R^d$ is homeomorphic to a k-dimensional convex body, $0 \leq k \leq d$. Then we denote by $\mathrm{ri}(A)$ and by $\mathrm{rb}(A)$ the relative interior of A and the relative boundary of A, respectively. The set of vertices of a polytope P is denoted by $\mathrm{vert}(P)$.

We denote by D the cube $[-1, 1]^3$ of R^3. Denote the vertices of D by $v_1, v_2, \ldots, v_8$ in such a way that $v_1 v_i$ is an edge of D for any $2 \leq i \leq 4$, and $v_j = -v_{9-j}$ for any $1 \leq j \leq 8$. Let $I = \{(i, j) \in N \times N \mid v_i v_j \text{ is an edge of } D\}$.

If $(i, j) \in I$, then denote by p_{ij} the midpoint of the edge $v_i v_j$ of D (thus $p_{ij} = p_{ji}$). Let T be a regular tetrahedron with vertices from the vertex set of $\frac{1}{2}D$. Let $C = T - T$. Then C is a cuboctahedron, and $C = \mathrm{conv}(\{p_{ij} \in R^3 \mid (i, j) \in I\})$.

Consider an arbitrary $(i, j) \in I$. Denote by h_{ij} the homothety with center p_{ij} and with coefficient $\frac{1}{2}$. That is, $h_{ij}(x) = \frac{1}{2}(x + p_{ij})$ for any $x \in R^3$. Let $R_{ij} = h_{ij}(C) \cap \partial C$. R_{ij} is called a *region* of ∂C. From the definition of R_{ij} it follows that $R_{ij} = R_{ji}$ and that $\{R_{ij} \mid (i, j) \in I\}$ is a collection of 12 congruent and mutually nonoverlapping regions of ∂C (i.e., $\mathrm{ri}(R_{ij}) \cap \mathrm{ri}(R_{kl}) = \emptyset$ for every $(i, j), (k, l) \in I$ with $R_{ij} \neq R_{kl}$).

For $1 \leq i \leq 8$, we set $B_i = \bigcup\{R_{ij} \mid (i, j) \in I, 1 \leq j \leq 8\}$. Then B_i is the union of three regions of ∂C. B_i is called a *belt* of ∂C. Let $T_i = \mathrm{conv}(\{p_{ij} \mid (i, j) \in I, 1 \leq j \leq 8\})$. Then T_i is a triangle. Denote by M_i the smaller triangle determined by the midpoints of the sides of T_i. Let $Q_i = B_i \cup M_i$. Q_i is called a *quarter* of ∂C. The reason for this name is that if $V \subset \mathrm{vert}(D)$ and $\mathrm{conv}(V)$ forms a regular tetrahedron, then $\{Q_i \mid i \in V\}$ is a collection of four congruent and mutually nonoverlapping quarters, whose union "almost" covers ∂C (more precisely, it covers $\partial C \setminus \bigcup\{M_i \mid i \notin V\}$).

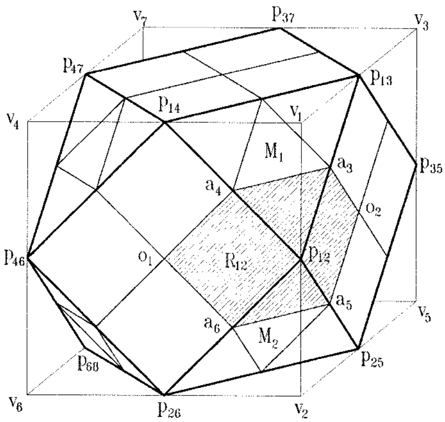

Fig. 1. The introduced notation on ∂C.

In the proof we define several nonnegative real-valued functions on ∂C. They correspond to certain subsets of ∂C. We call these functions weight functions, because they are related to the counting measure, so we are able to consider them as some kind of weighted counting measures. Our notation for a weight function corresponding to a set $A \subseteq \partial C$ is $w(A, \cdot)$. If $F \subset \partial C$ is a finite set, we use the notation $w(A, F)$ for $\sum_{x \in F} w(A, x)$. We note that the method of "weight" functions, or "cost" functions, which are usually piecewise constant, is often used in combinatorics and discrete geometry. For some nice examples of this method, see [8] (see also [3]), [21], and [22]. Our method is a refinement of the one used by Zong [21].

Consider the region R_{12}. Let $o_1 = \frac{1}{2}(p_{46} + p_{12})$ and $o_2 = \frac{1}{2}(p_{35} + p_{12})$. There are four edges of C containing p_{12}. We introduce notation for the midpoints of these edges. Let $a_i = \frac{1}{2}(p_{12} + p_{1i})$ for $i = 3, 4$, and $a_j = \frac{1}{2}(p_{12} + p_{2j})$ for $j = 5, 6$. With this notation, the relative boundary of R_{12} is formed by the closed (nonplanar) hexagon $o_1a_4a_3o_2a_5a_6$ (see Fig. 1).

We define the weight function $w(R_{12}, \cdot)\colon\ \partial C \to R$ in the following way:

$$w(R_{12}, x) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{for} \quad x \in \operatorname{ri}(R_{12}) \cup (a_3a_4) \cup (a_5a_6), \\ \frac{1}{2}, & \text{for} \quad x \in (o_1a_4] \cup [a_3o_2) \cup (o_2a_5] \cup [a_6o_1), \\ \frac{1}{4}, & \text{for} \quad x \in \{o_1, o_2\}, \\ 0, & \text{for} \quad x \in \partial C \backslash R_{12}. \end{cases}$$

Let us define the weight function $w(R_{ij}, \cdot)$ of an arbitrary region R_{ij} similarly as we defined the weight function $w(R_{12}, \cdot)$. More precisely, consider a congruence $g\colon\ R^3 \to R^3$ for which $g(\partial C) = \partial C$ and $g(R_{ij}) = R_{12}$ hold, and define $w(R_{ij}, x) = w(R_{12}, g(x))$ for every $x \in \partial C$. It is easy to see that then $w(R_{ij}, \cdot)$ is well-defined, i.e., its value does not depend on the selected congruence g.

Let $1 \le i \le 8$. We define the weight function corresponding to B_i as $w(B_i, x) = \sum\{w(R_{ij}, x) \mid (i, j) \in I, 1 \le j \le 8\}$ for every $x \in \partial C$. We also define the weight function of the triangle M_i as

$$w(M_i, x) \stackrel{\text{def}}{=} \begin{cases} 2, & \text{for} \quad x \in \text{ri}(M_i), \\ 0, & \text{for} \quad x \in \partial C \setminus \text{ri}(M_i). \end{cases}$$

Define the weight function of the quarter Q_i as $w(Q_i, x) = w(B_i, x) + w(M_i, x)$ for every $x \in \partial C$.

Let us define $M = \bigcup_{i=1}^{8} \text{ri}(M_i)$. Let F be an arbitrary finite subset of ∂C. It follows immediately by the definitions of the weight functions introduced above that

$$2|F| = \sum_{i=1}^{8} w(Q_i, F) = \sum_{i=1}^{8} w(B_i, F) + 2|M \cap F|.$$

3.2. *Local Case Analysis*

First we recall two simple statements which will help us to decide whether the distance between two points on ∂C is less than or equal to 1 in the metric determined by C. These statements are immediate consequences of the triangle inequality in the metric determined by $K - K$. We leave their proofs to the reader.

Proposition 3.1. *Let K be a d-dimensional convex body, let Z_1 and Z_2 be finite subsets of R^d, $p \in \text{conv}(Z_1)$, $q \in \text{conv}(Z_2)$. Then there exist points $z_1 \in Z_1$ and $z_2 \in Z_2$ such that*

$$\|p - q\|_{K-K} \leq \|z_1 - z_2\|_{K-K}.$$

Proposition 3.2. *Let K be a d-dimensional convex body, let L be a convex set in R^d, $p, q \in R^d$. Assume that $q \in \text{ri}(L)$, and that there exists a point $w \in L$ such that $\|p - w\|_{K-K} < \max_{z \in L} \|p - z\|_{K-K}$. Then*

$$\|p - q\|_{K-K} < \max_{z \in L} \|p - z\|_{K-K}.$$

From now on, we fix the metric determined by C, i.e., we consider R^3 equipped with the metric determined by the Minkowski norm with the unit ball C.

Let $S = \{s_1, s_2, \ldots, s_m\}$ be a 1-discrete subset of R_{12}. If $|S| = 1$, then we have $w(R_{12}, S) \leq 1$. We now make a complete list (up to a congruence of R_{12}) of eight cases, showing where the elements of S can be situated in R_{12} if $|S| \geq 2$. That is, this list will have the property that for any 1-discrete subset F of R_{12}, $|F| \geq 2$, there exists a congruence $f\colon R^3 \to R^3$ with $f(R_{12}) = R_{12}$ (and then with $f(\partial C) = \partial C$) such that one of the following eight cases holds for $S = f(F)$. Moreover, since for every $(i, j) \in I$ there exists a congruence $g\colon R^3 \to R^3$ with $g(R_{ij}) = R_{12}$ and $f(\partial C) = \partial C$, therefore this list will have the property that for any $(i, j) \in I$, any 1-discrete subset F of R_{ij}, $|F| \geq 2$, there exists a congruence $f\colon R^3 \to R^3$ with $f(R_{ij}) = R_{12}$ and $f(\partial C) = \partial C$, such that one of the following eight cases holds for $S = f(F)$.

Starting to make the list, we assume that $|S| \geq 2$. Since $R_{12} \subseteq h_{12}(C)$, so the distance between any two points of R_{12} is at most 1 (in the metric determined by C). First we consider the case when $p_{12} \in S$. Then it is easy to see that $S \subseteq \{p_{12}, o_1, o_2\}$. Thus we have two cases.

Case 1. $S = \{p_{12}, o_1\}$. Then $w(R_{12}, S) = 1.25$.
Case 2. $S = \{p_{12}, o_1, o_2\}$. Then $w(R_{12}, S) = 1.5$.

We now consider the case when $S \cap \left(\mathrm{ri}(R_{12}) \backslash \{p_{12}\}\right) \neq \emptyset$, say $s_1 \in \mathrm{ri}(R_{12})$, $s_1 \neq p_{12}$. Then there exists a segment $[p_{12}b] \subset R_{12}$ such that $s_1 \in (p_{12}b)$. Applying Proposition 3.2, we have $\|p_{12} - s_2\|_C = 1$, so $s_2 \in \{o_1, o_2\}$. However, a simple argument shows that $\{o_1, o_2\} \nsubseteq S$ and $s_1 \notin \mathrm{ri}(T_1)$. Thus we can reduce the situation to the following.

Case 3. $|S| = 2$, $s_1 \in \mathrm{ri}(R_{12}) \cap \mathrm{conv}(o_1, a_4, p_{12}, a_6)$, $s_2 = o_2$. Then $w(R_{12}, S) = 1.25$.

In the further cases we may assume that $S \cap \mathrm{ri}(R_{12}) = \emptyset$. Considering the case when $S \cap ((a_3a_4) \cup (a_5a_6)) \neq \emptyset$ we may assume by symmetry reasons that $s_1 \in (a_3a_4)$. Then by Proposition 3.2 we get that $S \backslash \{s_1\} \subset [a_5a_6]$. This way we can reduce the situation for one of the following two cases:

Case 4. $|S| = 2$, $s_1 \in (a_3a_4)$, $s_2 \in (a_5a_6)$. Then $w(R_{12}, S) = 2$.
Case 5. $|S| = 2$, $s_1 \in (a_3a_4)$, $s_2 = a_5$. Then $w(R_{12}, S) = 1.5$.

In the remaining cases we may assume that $S \subset [a_6o_1] \cup [o_1a_4] \cup [a_3o_2] \cup [o_2a_5]$. Then it is clear that either $|S \cap ([a_6o_1] \cup [o_1a_4])| \leq 1$ or $S \cap ([a_6o_1] \cup [o_1a_4]) = \{a_4, a_6\}$. By symmetry of R_{12} a similar result can be obtained for $S \cap ([a_3o_2] \cup [o_2a_4])$. However, by $\|a_3 - a_4\|_C = \frac{1}{2}$ and by Proposition 3.2, it is easy to see that $|S \cap ((o_1a_4] \cup (o_2a_3])| \leq 1$, and $|S \cap ((o_1a_6] \cup (o_2a_5])| \leq 1$. Then the reader can easily verify that these conditions reduce the situation to the following three, essentially different cases:

Case 6. $|S| = 2$, $s_1 = a_4$, $s_2 = a_6$. Then $w(R_{12}, S) = 1$.
Case 7. $|S| = 3$, $s_1 = a_4$, $s_2 = a_6$, and $s_3 = o_2$. Then $w(R_{12}, S) = 1.25$.
Case 8. $|S| = 2$, $s_1 \in (o_1a_4]$, $s_2 \in (o_2a_5]$. Then $w(R_{12}, S) \leq 1$.

This completes the list of cases.

3.3. *Lemmas*

We define the *subregion* R_{12}^1 of the region R_{12} as the relatively closed subset of R_{12} bounded by the closed (nonplanar) pentagon $o_1p_{12}o_2a_3a_4$. In general, we define two *subregions* R_{ij}^i and R_{ij}^j of the region R_{ij}. Let $R_{ij}^i = f(R_{12}^1)$, where $f\colon R^3 \to R^3$ is the unique congruence for which $f(R_{12}) = R_{ij}$, $f(M_1) = M_i$, and $\det(f) = 1$. However, $R_{ij} = R_{ji}$, so we assigned two subregions to each region R_{ij} by the above definition. It is easy to see that $\mathrm{ri}(R_{ij}^i) \cap \mathrm{ri}(R_{ij}^j) = \emptyset$ and $R_{ij} = R_{ij}^i \cup R_{ij}^j$ for every $(i, j) \in I$.

Lemma 3.1. *If $S \subset \partial C$ is a 1-discrete set, $(i, j) \in I$, and $S \cap \mathrm{ri}(M_i) \neq \emptyset$ (resp. $S \cap M_i \neq \emptyset$), then $S \cap R_{ij}^i = \emptyset$ (resp. $S \cap \mathrm{ri}(R_{ij}^i) = \emptyset$), and $|S \cap R_{ij}^j| \leq 1$.*

Proof of Lemma 3.1. Without loss of generality we may assume that $i = 1$ and $j = 2$. Suppose that $S \cap \mathrm{ri}(M_1) \neq \emptyset$. Then $|S \cap T_1| = 1$. Let $\{s_1\} = S \cap T_1$. We have $R_{12}^1 \backslash T_1 \subseteq \mathrm{conv}(p_{12}, o_1, a_4) \cup \mathrm{conv}(p_{12}, o_2, a_3)$. We prove that $S \cap R_{12}^1 = \emptyset$. By symmetry it is enough to show that $S \cap \mathrm{conv}(p_{12}, o_1, a_4) = \emptyset$. Using Proposition 3.1, it is easy to see that the diameter of $\mathrm{conv}(p_{12}, o_1, a_4) \cup M_1$ is equal to 1 (in the metric determined by C), and applying Proposition 3.2 for $p \in \mathrm{conv}(p_{12}, o_1, a_4)$, $q = s_1$, and $w = a_4$ we get the wanted inequality $\|p - s_1\|_C < 1$. A similar argument shows that in the case $S \cap M_1 \neq \emptyset$ we have $S \cap \mathrm{ri}(R_{12}^1) = \emptyset$. Finally we sketch how to prove that if $S \cap R_{12}^1 = \emptyset$, then $|S \cap R_{12}^2| \leq 1$. Assuming the contrary, i.e., $|S \cap R_{12}^2| \geq 2$, we get that there exists a congruence h: $R^3 \to R^3$ such that $h(R_{12}) = R_{12}$ and one of Cases 1–8 holds for $S' = h(S)$. Then either $S' \cap R_{12}^1 = \emptyset$ or $S' \cap R_{12}^2 = \emptyset$. However, both cases turn out to be impossible after taking a closer look at each of the eight local cases for S'. $\square$

Let $S \subset \partial C$ be a 1-discrete set, let i be an integer, $1 \leq i \leq 8$. We introduce the notation $n_i(S) = |\{j \mid (i, j) \in I, S \cap \mathrm{ri}(M_j) \neq \emptyset\}|$. Thus $0 \leq n_i(S) \leq 3$. The following lemma is an immediate consequence of Lemma 3.1.

Lemma 3.2. *Let* $S \subset \partial C$ *be a* 1*-discrete set,* $1 \leq i \leq 8$, $S \cap \mathrm{ri}(M_i) \neq \emptyset$. *Then* $w(Q_i, S) = 2 + w(B_i, S) \leq 5 - n_i(S)$.

Let $q_1 = \frac{1}{2}(p_{14} + p_{46})$ and $q_2 = \frac{1}{2}(p_{13} + p_{35})$. Let $\mathcal{S}_*$ be defined as the collection of all 1-discrete sets of ∂C for which $S \in \mathcal{S}_*$ if and only if S is 1-discrete and $S \cap B_1 = \{s_1, s_2, s_3, s_4, s_5\}$ with $s_1 \in (a_3 a_4)$, $s_2 \in (a_5 a_6)$, $s_3 \in \mathrm{ri}(\mathrm{conv}(o_1, p_{14}, q_1))$, $s_4 \in \mathrm{ri}(\mathrm{conv}(o_2, p_{13}, q_2))$, and $s_5 = \frac{1}{2}(p_{14} + p_{37})$. The reader can see a representation of $\{s_i\}_{i=1}^5$ in Fig. 2, which shows a planar projection of ∂C.

Let $\mathcal{H} = \{h \mid h \text{ is a congruence of } R^3, h(\partial C) = \partial C\}$. We note that for every $h \in \mathcal{H}$ there exists an $(i, j) \in I$ such that $h(R_{12}) = R_{ij}$. We introduce the notation $\mathcal{S}_\# = \{h(S) \mid S \in \mathcal{S}_*, h \in \mathcal{H}\}$.

Lemma 3.3. *Let* $S \subset \partial C$ *be a* 1*-discrete set,* $1 \leq i \leq 8$. *Assume that* $S \cap \mathrm{ri}(M_i) = \emptyset$. *Then the inequality* $w(B_i, S) \leq 4.5 - n_i(S)/2$ *holds. Moreover, if* $w(B_i, S) = 4.5$, *then either* $S \cap B_i \in \mathcal{S}_\#$ *or* $S \cap B_i = S_0 \cap B_i$.

We remark that $2w(B_i, S) \in Z$ since $w(B_i, x) \in \{\frac{1}{2}, 1\}$ for every $x \in B_i$. Thus if $w(B_i, S) < 4.5$, then $w(B_i, S) \leq 4$. We also keep in mind that if $S \cap \mathrm{ri}(M_i) = \emptyset$, then $w(Q_i, S) = w(B_i, S)$.

Proof of Lemma 3.3. Without loss of generality we may assume that $i = 1$ and that $w(R_{12}, S) \geq w(R_{13}, S) \geq w(R_{14}, S)$. We note that $B_1 = R_{12} \cup R_{13} \cup R_{14}$. Assume that $w(B_1, S) > 4$. By Cases 1–8 we have that if $w(R_{ij}, S) \geq 1$, then $w(R_{ij}, S) \in \{1.25, 1.5, 2\}$. A simple analysis of the eight cases combining with applications of Lemma 3.1 shows that if $w(R_{12}, S) = 2$, then $w(R_{13}, S)$ and $w(R_{14}, S)$ are at most 1.25, $\frac{1}{2}(p_{14} + p_{37}) \in S$, and finally we get $S \cap B_1 \in \mathcal{S}_*$. Otherwise we have $w(R_{12}, S) \leq 1.5$. Then, by the assumption $w(B_1, S) > 4$, it is clear that $w(R_{12}, S) = w(R_{13}, S) = 1.5$. Furthermore, considering the region R_{14}, we get that $p_{14} \in S$ must hold. Thus we have

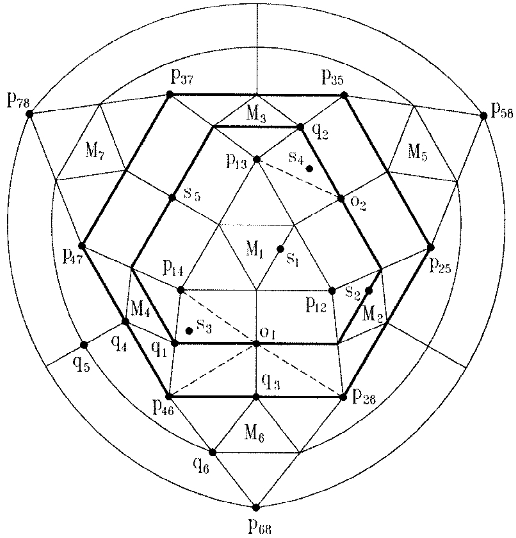

Fig. 2. A planar projection of ∂C.

$S \cap B_1 = S_0 \cap B_1$. It can be easily seen that in the two cases obtained for $w(B_1, S) > 4$ the equality $n_1(S) = 0$ must hold. Now assume that $n_1(S) = 2$. Suppose the contrary of the conclusion, i.e., that $w(B_1, S) \geq 4$. We may assume by symmetry that $S \cap \text{ri}(M_j) \neq \emptyset$ for $j = 3, 4$. Then, by Lemma 3.1, we have $w(R_{1j}, S) \leq 1$ for $j = 3$ and 4, so $w(R_{12}, S) = 2$ holds. Thus $S \cap M_1 \neq \emptyset$, therefore $S \cap \text{ri}(R^1_{1j}) = \emptyset$ for $j = 3, 4$ by Lemma 3.1. However, $S \cap \text{ri}(R_{13} \cap R_{14}) = \emptyset$ by Propositions 3.1 and 3.2, so we get $S \cap B_1 = S \cap R_{12}$. Hence $w(B_1, S) = 2$, which is a contradiction. Finally we assume that $n_1(S) = 3$. Then, by Lemma 3.1, we have $w(R_{1j}, S) \leq 1$ for $j = 1, 2$, and 3. Thus we get $w(B_1, S) \leq 3$. □

Lemma 3.4. *Let $S \subset \partial C$ be a* 1*-discrete set. If $S \cap B_i \in \mathcal{S}_\#$ for a suitable i*, $1 \leq i \leq 8$, *then* $|S| \leq 16$.

Proof of Lemma 3.4. By symmetry, we may assume that $S \cap R_{12} \in \mathcal{S}_*$. Thus we can use the notation $S = \{s_i \mid 1 \leq i \leq m\}$ where each s_i has the same property as in the definition of S_* for $1 \leq i \leq 5$ (see Fig. 2). Considering the position of s_2, we have by Lemma 3.1 that if $S \cap \text{ri}(M_6) \neq \emptyset$, then $w(Q_6, S) \leq 4$. If $S \cap \text{ri}(M_6) = \emptyset$, then considering R_{26} and applying Lemma 3.3 we have $w(Q_6, S) \leq 4$. Thus we get $w(Q_6, S) \leq 4$ for any S. Similarly, $w(Q_5, S) \leq 4$. Considering the points of $S \cap B_1$, we get by Lemma 3.1 that $S \cap \text{ri}(M_i) = \emptyset$ for $i = 1, 2, 3, 4$, and 7, so $w(Q_j, S) \leq 4.5$ for $j = 2, 7$ and $w(Q_j, S) \leq 4$ for $j = 3, 4$ by applying Lemma 3.3. We have two cases. If $w(Q_8, S) = 5$, then it is easy to see, by Lemma 3.3, that $w(Q_j, S) \leq 4$ for $j = 2, 7$. From this it follows that $2|S| \leq 33.5$, i.e., $|S| \leq 16$. In the second case $w(Q_8, S) \leq 4.5$, so we get the estimate $2|S| \leq 4 \cdot 4 + 4 \cdot 4.5 = 34$. Suppose to the contrary that $|S| \geq 17$. Then

$|S| = 17$, and each estimate for $w(Q_j, S)$ is sharp. In particular, we have $w(Q_2, S) = 4.5$ and $w(Q_6, S) = 4$. However, $w(R_{12}, S) = 2$, so we get $S \cap Q_2 \in S_\#$ by Lemma 3.3. This implies $w(R_{26}, S) = 1.25$. Recall $q_1 = \frac{1}{2}(p_{14} + p_{46})$. Let $q_3 = \frac{1}{2}(p_{26} + p_{46})$. Define q_j $(4 \le j \le 6)$ so that $q_1, q_4, q_5, q_6, q_3, o_1$ denote consecutive vertices of R_{46}. Then we have $S \cap \text{ri}(\text{conv}(o_1, q_3, p_{26})) \ne \emptyset$, so considering the position of s_3 also, we get, by applying Propositions 3.1 and 3.2, that $S \cap R_{46} = S \cap (\text{conv}(q_4, q_5, q_6, p_{46}) \setminus ([q_4, p_{46}] \cup [q_6, p_{46}]))$. Then $w(R_{46}, S) \le 1$. Thus $w(R_{68}, S) \ge 4 - 1.25 - 1 = 1.75$, that is, $w(R_{68}, S) = 2$ by Cases 1–8. However, $S \cap R_{68}$ contains a square face center of C by $w(Q_2, S) = 4.5$, which implies (by Cases 1–8) that $w(R_{68}, S) < 2$. This is a contradiction. □

3.4. *Global Case Analysis*

Finished with the preparations, we now turn to the actual proof of Theorem 2 which we carry out by the following three propositions. The reader should notice that collectively they imply Theorem 2.

Proposition 3.3. *Let $S \subset \partial C$ be a 1-discrete set. If $S \cap M \ne \emptyset$, then $|S| \le 16$.*

Proof of Proposition 3.3. By Lemma 3.4 we may assume that $S \cap B_i \notin \mathcal{S}_\#$ for each i, $1 \le i \le 8$. Then $w(B_i, S) = 4.5$ if and only if $S \cap B_i = S_0 \cap B_i$. First we prove that if $|S \cap M| \ge 2$, then $S \cap \text{ri}(M_i) = \emptyset$ implies that $w(Q_i, S) \le 4$. This follows from the fact that if $S \cap \text{ri}(M_1) \ne \emptyset$, then $\{o_1, o_2, o_3\} \cap S = \emptyset$, where $o_3 = \frac{1}{2}(p_{14} + p_{37})$, but $\{o_1, o_2, o_3\} \cap B_j \ne \emptyset$ for any $j \ne 8$, and $o_i \in S_0$ for $1 \le i \le 3$. Let $n(S) = |\{(i, j) \in I \mid i < j, S \cap \text{ri}(M_i) \ne \emptyset, S \cap \text{ri}(M_j) \ne \emptyset\}|$. Then, by Lemma 3.2, we have $2|S| = \sum_{i=1}^{8} w(Q_i, S) \le 32 + |S \cap M| - 2n(S)$ if $|S \cap M| \ge 2$. Using the trivial bound $n(S) \ge 3(|S \cap M| - 4)$, we get $2|S| \le 31 - 5(|S \cap M| - 5)$, which means that $|S| < 16$ if $|S \cap M| \ge 5$.

If $|S \cap M| = 1$, then without loss of generality we may assume that $S \cap \text{ri}(M_1) \ne \emptyset$. By the first part of this proof we now have $w(Q_i, S) \le 4$ for any $i \ne 1, 8$. Thus $2|S| \le 5 + 6 \cdot 4 + 4.5 = 33.5$, i.e., $|S| \le 16$.

If $|S \cap M| = 2$, then $2|S| \le 32 + 2 - 2n(S)$. So if $n(S) > 0$, then $|S| \le 16$. If there exists a square face of the cube D such that v_i and v_j are opposite vertices of that face, $|S \cap \text{ri}(M_i)| = |S \cap \text{ri}(M_j)| = 1$, then, denoting the indices of the other two vertices of that face by k_1 and k_2, we get $n_{k_t}(S) \ge 2$ for $t = 1, 2$. By Lemma 3.3, we obtain $w(B_{k_t}, S) \le 3.5$ $(t = 1, 2)$, so $2|S| \le 2 \cdot 5 + 2 \cdot 3.5 + 4 \cdot 4 = 33$, i.e., $|S| \le 16$. It is easy to see that the only remaining case is when $|S \cap \text{ri}(M_i)| = |S \cap \text{ri}(M_j)| = 1$ for $i + j = 9$, i.e., v_i and v_j are opposite vertices of D. Without loss of generality we may suppose that $i = 1$. It is enough to show that there exists an index z such that either $S \cap \text{ri}(M_z) = \emptyset$ and $w(B_z, S) < 4$, or $S \cap \text{ri}(M_z) \ne \emptyset$ and $w(Q_z, S) < 5$, because then we have $2|S| < 2 \cdot 5 + 6 \cdot 4 = 34$, which implies that $|S| \le 16$. Assume the contrary. Then we have $w(Q_1, S) = w(Q_8, S) = 5$ and $w(B_2, S) = 4$. Thus there exists an index $k \in \{1, 5, 6\}$ such that $w(R_{2k}, S) \ge 1.5$, since $4w(R_{z_1 z_2}, x) \in Z$ for each $x \in R_{z_1 z_2}$, $(z_1, z_2) \in I$. Observing the fact that the square face centers of C cannot belong to S, and analyzing Cases 1–8, we get that $k \ne 1$ and $S \cap M_k \ne \emptyset$. By Proposition 3.2 we obtain $S \cap \mathrm{rb}(R_{k8}) = \emptyset$. So, by Lemma 3.1, we have $S \cap R_{k8} = \emptyset$. However, this is a contradiction, because we assumed that $w(Q_8, S) = 5$, which implies $w(R_{k8}, S) = 1$.

If $|S \cap M| = 3$, then $2|S| \leq 32 + 3 - 2n(S)$, so $|S| \leq 16$ if $n(S) > 0$. Otherwise, when $n(S) = 0$, it is easy to see that there exists an index j, $1 \leq j \leq 8$, such that $S \cap \mathrm{ri}(M_i) \neq \emptyset$ for any i with $(i, j) \in I$. Then we have that there exists a set J of indices such that $|J| = 4$, $S \cap \mathrm{ri}(M_k) = \emptyset$, and $n_k(S) \geq 2$ for any $k \in J$. Thus, by Lemma 3.3, we get $w(B_k, S) \leq 3.5$ for any $k \in J$. This implies that $2|S| \leq 3 \cdot 5 + 4 \cdot 3.5 + 4 = 33$, that is $|S| \leq 16$.

If $|S \cap M| = 4$, then $2|S| \leq 32 + 4 - 2n(S)$, so $|S| \leq 16$ if $n(S) \geq 2$. It is easily seen that if $n(S) < 2$, then $n(S) = 0$ and $V_1 = \{v_i \mid S \cap \mathrm{ri}(M_i) \neq \emptyset\}$ is the vertex set of a regular tetrahedron. Then $n_j(S) = 3$ for each $j \in \mathrm{vert}(D)\backslash V_1$. Applying Lemma 3.3, we get $w(B_j, S) \leq 3$ for any $j \in \mathrm{vert}(D)\backslash V_1$. Thus we have $2|S| \leq 4 \cdot 5 + 4 \cdot 3 = 32$, i.e., $|S| \leq 16$. □

Proposition 3.4. *Let $S \subset \partial C$ be a 1-discrete set, $S \cap M = \emptyset$. Then $|S| \leq 18$ holds. Moreover, if $|S| = 18$, then $S = S_0$. If $|S| = 17$, then there exists a square face F of C, such that $S \cap (\partial C \backslash F) = S_0 \cap (\partial C \backslash F)$.*

Proof of Proposition 3.4. By Lemma 3.3, we have $2|S| \leq 8 \cdot 4.5 = 36$, so $|S| \leq 18$. Furthermore, if $|S| = 18$, then $w(B_i, S) = 4.5$ for each i $(1 \leq i \leq 8)$. By Lemmas 3.3 and 3.4, we then have $S \cap Q_i = S \cap S_0$ $(1 \leq i \leq 8)$. However, $\bigcup_{i=1}^{8} Q_i = \partial C$, so we get $S = S_0$.

Let $|S| = 17$. Then it can be easily seen that the proof can be reduced to showing the existence of a face F' of D with the property $w(B_i, S) = 4.5$ for any $i \in \mathrm{vert}(F')$. Then the square face F of C which is opposite to the square face $F' \cap C$ of C will have the property required by the proposition. Denote by V_1 a subset of vertices of D whose elements form the vertices of a regular tetrahedron. Let $V_2 = \mathrm{vert}(D)\backslash V_1$, $J_i = \{j \mid v_j \in V_i\}$ for $i = 1, 2$. Then $\sum\{w(B_j, S) \mid j \in J_i\} = |S| = 17$ for $i = 1, 2$. By $w(B_j, S) \leq 4.5$ and $2w(B_j, S) \in Z$ there are at least two elements j_1, j_2 of J_1 for which $w(B_{j_i}, S) = 4.5$ $(i = 1, 2)$. Then $[v_{j_1} v_{j_2}]$ is not an edge of D, and there exists a square face F' of D such that $v_{j_1}, v_{j_2} \in \mathrm{vert}(F')$. Without loss of generality we may assume that $j_1 = 2$, $j_2 = 8$, and $F' = \mathrm{conv}(v_2, v_5, v_8, v_6)$. Assume that the face F' of D does not have the required property. Then either $w(R_{46}, S) < 1.5$ or $w(R_{35}, S) < 1.5$. By symmetry we may suppose that $w(R_{46}, S) < 1.5$. However, $S \cap R_{46}$ contains the two points of R_{46} with weight 0.25, so we get (by Cases 1–8 or directly) that $w(R_{46}, S) = 0.5$. Then $w(B_i, S) = 3.5$ for $i = 4, 6$. From this it follows that $w(B_i, S) = 4.5$ for any $i \neq 4, 6$. Let $F'' = \mathrm{conv}(v_3, v_5, v_8, v_7)$. Then F'' is a face of D which has the required property. □

Let $F \subset R^2$ be a square with consecutive vertices c_1, c_2, c_3, c_4. Let $\alpha, \beta, \gamma \in [0, 1]$, $b_{i,t} = tc_i + (1 - t)c_{i+2}$, and $a_{i,t} = tc_i + (1 - t)c_{i+1}$ for every integer i, $1 \leq i \leq 4$, and $t \in [0, 1]$ (if $5 \leq k \leq 8$, we use the notation c_k for c_{k-4} as well). Let $F^1_{\alpha,\beta} = \{b_{1,\alpha}, b_{1,\beta}, c_2, c_4\}$, $F^2_{\alpha,\beta} = \{b_{2,\alpha}, b_{2,\beta}, c_1, c_3\}$, and $\mathcal{F}_{\alpha,\beta} = \{F^1_{\alpha,\beta}, F^2_{\alpha,\beta}\}$. Furthermore, let $F_\gamma = \{a_{i,\gamma} \mid 1 \leq i \leq 4\}$.

Proposition 3.5. *Let $N \subset R^2$ be a square with center o, where o is the origin of R^2. Let F be the square determined by the midpoints of the sides of N. Let S be a 1-discrete subset of F in the metric determined by N. Then $|S| \leq 5$. In particular, $|S| = 5$ if and only if $S = \text{vert}(F) \cup \{o\}$. Furthermore, $|S| = 4$ if and only if either $S \in \mathcal{F}_{\alpha,\beta}$ for suitable α and β, $0 \leq \alpha \leq \beta - \frac{1}{2} \leq \frac{1}{2}$, or $S = F_\gamma$ for a suitable γ, $0 < \gamma < 1$.*

Proof of Proposition 3.5. Obviously $\text{vert}(F) \cup \{o\}$, $F^i_{\alpha,\beta}$ $(i = 1, 2)$, and F_γ are 1-discrete subsets of F for any values of α, β, and γ described in Proposition 3.5. So we need to prove only the other direction of the proposition. Let $|S| \geq 4$. If $o \in S$, then $S \subseteq \text{vert}(F) \cup \{o\}$, so $|S| \leq 5$. Furthermore, then in the case of $|S| = 5$ we get $S = \text{vert}(F) \cup \{o\}$, while in the case of $|S| = 4$ we obtain $S \in \mathcal{F}_{0,\frac{1}{2}} \cup \mathcal{F}_{\frac{1}{2},1} \cup \mathcal{F}_{0,1}$. From now on we may assume that $o \notin S$. Let $S = \{s_1, s_2, \ldots, s_m\}$. For each i, define q_i as the unique point on the boundary of F for which $s_i \in [oq_i]$ $(1 \leq i \leq m)$. Then $s_i = \alpha_i q_i$ for a suitable real number α_i, $0 < \alpha_i \leq 1$ $(1 \leq i \leq m)$. Let $S' = \{q_1, q_2, \ldots, q_m\}$. S' is a 1-discrete set since, for any $i \neq j$, $\alpha_i \leq \alpha_j$, we have

$$1 \leq \frac{1}{\alpha_j} \leq \frac{1}{\alpha_j}\|s_i - s_j\|_N \leq \frac{\alpha_i}{\alpha_j}\|q_i - q_j\|_N + \left(1 - \frac{\alpha_i}{\alpha_j}\right)\|q_j\|_N \leq 1 + \frac{\alpha_i}{\alpha_j}(\|q_i - q_j\|_N - 1).$$

Moreover, if $\|q_i - q_j\|_N = 1$, then we have equalities in each of the preceding inequalities, thus $\alpha_j = 1$ and $\|o - q_j\|_N = \|q_i - q_j\|_N = \|s_i - q_j\| = 1$. This means that if $s_i \neq q_i$, then $[oq_i] \subset \partial(N - q_j)$, i.e., N has a side parallel to $[oq_i]$. In this special case we get that q_i and q_j $(= s_j)$ are vertices of F.

Denote by c_1, c_2, c_3, c_4 the consecutive vertices of F. Then $|S' \cap [c_i c_{i+1})| \leq 1$ $(1 \leq i \leq 4)$, so $|S| = |S'| = 4$ (we use the notation c_{j+4} for c_j as well). Without loss of generality we may assume that $q_i \in [c_i c_{i+1})$ for each i. Using the notation q_{j+4} for q_j as well, it can be shown by a simple argument that

$$\frac{\|q_i - c_i\|_N}{\|c_{i+1} - c_i\|_N} \leq \frac{\|q_{i+1} - c_{i+1}\|_N}{\|c_{i+2} - c_{i+1}\|_N} \qquad \text{for every } i, \quad 1 \leq i \leq 4.$$

Thus

$$\frac{\|q_1 - c_1\|_N}{\|c_2 - c_1\|_N} = \frac{\|q_i - c_i\|_N}{\|c_{i+1} - c_i\|_N} \qquad \text{for every } i, \quad 1 \leq i \leq 4.$$

Let $\gamma = \|q_1 - c_1\|_N / \|c_2 - c_1\|_N$. If $\gamma \neq 0$, then we get $S' = F_\gamma$, and $\|q_{i+1} - q_i\|_N = 1$ $(1 \leq i \leq 4)$. By $\gamma \neq 0$ we have that $[oq_1]$ is not parallel to any sides of N. This implies that $S = S' = F_\gamma$. If $\gamma = 0$, then we get that $q_i = s_i = c_i$ and $q_{i+2} = s_{i+2} = c_{i+2}$ for a suitable i, $1 \leq i \leq 2$. Then it is clear that $\{s_{i+1}, s_{i+3}\} \subset [c_{i+1} c_{i+3}]$. Hence $S \in \mathcal{F}_{\alpha,\beta}$ for suitable reals $\alpha, \beta \in [0, 1]$. We may assume that $\alpha < \beta$. However, S is a 1-discrete set, so $\alpha \leq \beta - \frac{1}{2}$. This completes the proof of Proposition 3.5. □

4. Applications

In this section we apply Theorem 2 to determine the minimum and maximum densities of 17^+-neighbor translative packings of tetrahedra. The main idea we use is to reduce the proof to a problem on packings with Z-translates of a certain cluster of R^3. Finally, in Proposition 4.3 we consider a d-dimensional generalization of this problem, but in this case we can only prove a somewhat weaker result than in the case of the original problem (Lemma 4.1). Thus Proposition 4.3 is not so closely connected to the other propositions and lemmas, but it has an interest of its own.

We recall some notions from the theory of packings (see [4] and [5]). Let $A, B \subseteq R^d$. B is called a *Z-translate* of A if $B = A + v$ for a suitable $v \in Z^d$. A collection $\mathcal{P}$ of mutually nonoverlapping subsets of R^d with nonempty interiors is called a *packing*. If all the members of the packing $\mathcal{P}$ are translates (resp. Z-translates) of a fixed set P, then $\mathcal{P}$ is called a *translative packing* (resp. *Z-translative packing*) of P. A packing (resp. Z-translative packing) $\mathcal{P}$ of R^d is called a *tiling* (resp. *Z-tiling*) if $\bigcup \mathcal{P} = R^d$. The *neighbors* of an element P of a packing $\mathcal{P}$ are the elements of the set $\{P' \in \mathcal{P} \mid P' \cap P \neq \emptyset\}$. A *$k$-neighbor packing* (resp. *k^+-neighbor packing*) is a packing where each element has exactly (resp. at least) k neighbors. A packing $\mathcal{P}$ is called *connected* if $\bigcup \mathcal{P}$ is a connected subset of R^d. The *connected components* of a packing $\mathcal{P}$ are the subcollections of $\mathcal{P}$ for which their unions form the connected components of the set $\bigcup \mathcal{P}$.

Let $\mathcal{P}$ be a packing in R^d, and $C_d(r) = [-r/2, r/2]^d$ for every $r > 0$. The *upper density* $d^+(\mathcal{P})$ of the packing $\mathcal{P}$ is defined as

$$d^+(\mathcal{P}) = \limsup_{r\to\infty}(V((\cup\mathcal{P}) \cap C_d(r))/r^d),$$

where $V(\cdot)$ denotes the volume in R^d. Similarly, the *lower density* $d^-(\mathcal{P})$ of the packing $\mathcal{P}$ is defined as

$$d^-(\mathcal{P}) = \liminf_{r\to\infty}(V((\cup\mathcal{P}) \cap C_d(r))/r^d).$$

If $d^+(\mathcal{P}) = d^-(\mathcal{P})$, then we denote their common value by $\delta(\mathcal{P})$. This $\delta(\mathcal{P})$ is called the *density* of the packing $\mathcal{P}$.

We now start to investigate the 17^+-neighbor translative packings of tetrahedra. Our main purpose is to determine the minimum and maximum densities of these packings. For analogous results of this kind for other convex bodies in two and three dimensions, see [12] and the survey paper [5].

In the following proposition we prove that any 17^+-neighbor translative packing of a tetrahedron can be obtained from its unique 18^+-neighbor translative packing by omitting some elements of that packing.

Proposition 4.1. *Let $T \subset R^3$ be a tetrahedron. Then there exists a unique* 18*-neighbor translative packing $\mathcal{T}$ of T such that $T \in \mathcal{T}$. If $\mathcal{T}'$ is a 17^+-neighbor translative packing of T with $T \in \mathcal{T}'$, then $\mathcal{T}' \subseteq \mathcal{T}$, and for every $T_1 \in \mathcal{T}\backslash\mathcal{T}'$ all the neighbors of T_1 in the packing $\mathcal{T}$ are elements of $\mathcal{T}'$ as well.*

Proof of Proposition 4.1. Using the same argument as in the proof of Theorem 2, we obtain that it is enough to prove the analogue of this proposition for the cuboctahedron $C_0 = \frac{1}{2}C$, where C is the cuboctahedron defined in Section 3.1. Let $\mathcal{C} = \{C_0 + v \mid v \in Z^3\}$. Then $\mathcal{C}$ is an 18-neighbor translative packing. Let $\mathcal{C}_1$ be an arbitrary 18-neighbor translative packing of C_0 such that $C_0 \in \mathcal{C}_1$. Then by Theorem 1 and the fact that $\mathcal{C}$ is connected, we get $\mathcal{C} \subseteq \mathcal{C}_1$. We show that $\mathcal{C}_1 = \mathcal{C}$. Otherwise there exists an element

$C_1 \in \mathcal{C}_1 \backslash \mathcal{C}$. Then we can consider a translate C_2 of C_1 for which $\mathcal{C} \cup \{C_2\}$ is a packing and C_2 touches at least one element of $\mathcal{C}$. However, then this element of $\mathcal{C}$ would have at least 19 neighbors in $\mathcal{C} \cup \{C_2\}$, which contradicts Theorem 1. This proves the uniqueness part of Proposition 4.1.

Let $\mathcal{C}'$ be a 17^+-neighbor translative packing of C_0 such that $C_0 \in \mathcal{C}'$. We denote by $\mathcal{C}'_0$ the connected component of $\mathcal{C}'$ which contains C_0. We prove that $\mathcal{C}'_0 \subseteq \mathcal{C}$. Assume the contrary. Then there exist $C_1 = C_0 + v_1 \in \mathcal{C}'_0 \backslash \mathcal{C}$ and $C_2 \in \mathcal{C}'_0 \cap \mathcal{C}$ such that C_1 and C_2 are neighbors. If C_1 has 18 neighbors in $\mathcal{C}'$, then $C_1 \in \mathcal{C}$ by Theorem 1, which is a contradiction. So, we may assume that C_1 has 17 neighbors in $\mathcal{C}'_0$. Then, by Theorem 2, for $C_1 - v_1$ we get that either $v_1 \in S_\gamma \backslash S_0$ for a suitable γ, $0 < \gamma < 1$, or $v_1 \in S_{\alpha,\beta} \backslash S_0$ for suitable α and β, $(\alpha, \beta) \notin \{(0, \frac{1}{2}), (\frac{1}{2}, 1), (0, 1)\}$. However, in this case it is easy to see that $C_2 - v_1$ would have less than 17 neighbors in $\mathcal{C}'_0 - v_1$, which is a contradiction. Thus we obtain that $\mathcal{C}'_0 \subseteq \mathcal{C}$. Therefore, if $C' \in \mathcal{C}'_0$, C' has 17 neighbors, then there exists a neighbor C'' of C' in $\mathcal{C}$ such that $C'' \notin \mathcal{C}'_0$, and $\mathcal{C}'_0 \cup \{C''\}$ is also a packing. Let $C^\#$ be an arbitrary neighbor of such a C' and C'' in $\mathcal{C}$. Then it is easy to see that $C^\# \in \mathcal{C}'_0$. This way we get that each neighbor of C'' in $\mathcal{C}$ is an element of $\mathcal{C}'_0$ as well. Let $\mathcal{C}''$ be the collection of elements of $\mathcal{C}$ which have some neighbors in $\mathcal{C}$ belonging to $\mathcal{C}'_0$ also. Then it is clear that $\mathcal{C}'_0 \cup \mathcal{C}''$ is an 18-neighbor packing of C_0 which contains C_0. Thus, by the already proven first part of Proposition 4.1, we have $\mathcal{C}'_0 \cup \mathcal{C}'' = \mathcal{C}$. Applying the argument used in the proof of the first part of this proposition, we get $\mathcal{C}'_0 = \mathcal{C}'$, and hence $\mathcal{C}' \cup \mathcal{C}'' = \mathcal{C}$. Consequently $\mathcal{C}' \subseteq \mathcal{C}$, and for every $C'' \in \mathcal{C} \backslash \mathcal{C}'$ all the neighbors of C'' in the packing $\mathcal{C}$ are elements of $\mathcal{C}'$ as well. □

Let $U = \frac{1}{2}D = [-\frac{1}{2}, \frac{1}{2}]^3$. Then $\mathcal{U} = \{U + v \mid v \in Z^3\}$ is a tiling of R^3. A finite subset of $\mathcal{U}$ is called a *cluster*. Let $E = \{v \in Z^3 \mid v \in [-1, 1]^3 \backslash \mathrm{vert}([-1, 1]^3)\}$. Define G to be the set $U + E = \bigcup\{U + v \mid v \in E\}$. Then G is a cluster, which can be obtained from a cube formed by the union of 27 translates of U, by leaving the 8 translates of U out at the vertices of the cube. Let us denote by L_* the sublattice of Z^3 generated by the vectors $(3, 1, 0)$, $(-1, 3, 0)$, $(1, 2, 2)$. Let $\mathcal{G}^* = \{G + v \mid v \in L_*\}$. Then $\mathcal{G}^*$ is a packing with Z-translates of G.

By Proposition 4.1, it is easy to see that there is a one-to-one correspondence between the 17^+-neighbor translative packings of a tetrahedron and the packings of R^3 with Z-translates of G. So, first we determine the maximum density of the packings mentioned last, and then we apply this result to get the minimum density bound for 17^+-neighbor translative packings of tetrahedra.

Lemma 4.1. *Let $\mathcal{G}$ be an arbitrary packing with Z-translates of G. Then $d^+(\mathcal{G}) \leq \delta(\mathcal{G}^*) = 0.95$.*

Proof of Lemma 4.1. It is easy to find a Z-translate U' of U which touches G and which is not covered by any elements of $\mathcal{G}^*$. Then observe that $\{(G \cup U') + v \mid v \in L_*\}$ is a tiling of R^3. From this it follows that $\delta(\mathcal{G}^*) = \frac{19}{20} = 0.95$.

Consider now $\mathcal{G}$. Without loss of generality, we may assume that $G \in \mathcal{G}$. We will find a vector $v \in Z^3$ with $\|v\| \leq 2$ for which $U + v$ is not covered by $\bigcup \mathcal{G}$. If $U + (0, 0, 2)$ is not covered by $\mathcal{G}$, then let $v = (0, 0, 2)$. Otherwise $U + (0, 0, 2) \subset G + w$ for a suitable $w \in Z^3$. It is clear that the third coordinate of w has to be 3. This way there are five choices for w. If $w = (0, 0, 3)$, then a simple argument shows that either $U + (1, 1, 1)$ or $U + (1, 1, 2)$ is not covered by $\mathcal{G}$. Choose $v = (1, 1, 1)$ if $U + (1, 1, 1)$ is not covered by $\mathcal{G}$, otherwise choose $v = (1, 1, 2)$. In the remaining four cases we may assume by symmetry that $w = (1, 0, 3)$. (If $w \neq (1, 0, 3)$, then we can apply a rotation $r\colon R^3 \to R^3$ around the third coordinate axis, which sends w to $(1, 0, 3)$. Then $r^{-1}(v)$ will correspond to $\mathcal{G}$, where v is the vector defined for $\{r(G') \mid G' \in \mathcal{G}\}$ in the following process.) Now, if $U + (1, -1, 1)$ is not covered by $\mathcal{G}$, then let $v = (1, -1, 1)$. Otherwise, considering $U + (1, 1, 1)$, if it is not covered by $\mathcal{G}$, then let $v = (1, 1, 1)$. Finally, it can be easily seen that if $U + (1, -1, 1)$ and $U + (1, 1, 1)$ are covered by $\mathcal{G}$, then $G + (2, -2, 1)$, $G + (2, 2, 1)$ are contained in $\mathcal{G}$, and $U + (2, 0, 1)$ cannot be covered by $\mathcal{G}$. Then let $v = (2, 0, 1)$. This way we assigned a vector v corresponding to the packing $\mathcal{G}$ in each case, for which $U + v$ is not covered by $\mathcal{G}$. Let $G' = G + w'$ be an arbitrary element of $\mathcal{G}$. Then $G = G' - w' \in \mathcal{G} - w'$, so by applying the above process for $\mathcal{G} - w'$, we can define a vector $v = v(\mathcal{G} - w')$ which corresponds to the packing $\mathcal{G} - w'$. Define a function $f\colon \mathcal{G} \to Z^3$ as $f(G') = v(\mathcal{G} - w')$ for any $G' \in \mathcal{G}$, where $w' \in Z^3$ is the unique vector for which $G' = G + w'$. A simple case analysis, which we leave to the reader, shows that if $G_1, G_2 \in \mathcal{G}$, $G_1 \neq G_2$, then $f(G_1) \neq f(G_2)$. This way we get that $\{G' \cup \{U + f(G')\} \mid G' \in \mathcal{G}\}$ is a packing of R^3, so $d^+(\mathcal{G}) \leq \frac{19}{20}$. □

Proposition 4.2. *Let $\mathcal{T}'$ be a 17^+-neighbor translative packing of a tetrahedron. Then*

$$\frac{19}{60} = \delta(\mathcal{T}'') \leq d^-(\mathcal{T}') \leq d^+(\mathcal{T}') \leq \delta(\mathcal{T}) = \frac{1}{3}$$

holds, where $\mathcal{T} = \{T_0 + v \mid v \in Z^3\}$ is an 18-neighbor packing, and $\mathcal{T}'' = \{T_0 + v \mid v \in Z^3 \backslash L_*\}$ is a 17^+-neighbor packing (here T_0 denotes a regular tetrahedron contained in the cube $U = [-\frac{1}{2}, \frac{1}{2}]^3$ with $\operatorname{vert}(T_0) \subset \operatorname{vert}(U)$).

Proof of Proposition 4.2. Obviously it is enough to consider the case when $\mathcal{T}'$ is a translative packing of T_0, and $T_0 \in \mathcal{T}'$. Then we have, by Proposition 4.1, that $\mathcal{T}' \subseteq \mathcal{T}$, so $d^+(\mathcal{T}') \leq \delta(\mathcal{T}) = \frac{1}{3}$. On the other hand, if $T'' \in \mathcal{T} \backslash \mathcal{T}'$, then the neighbors of T'' in $\mathcal{T}$, which form the set $\{T''\} + (E \backslash \{o\})$, are contained in $\mathcal{T}'$. However, $\mathcal{T}'$ is a 17^+-neighbor packing, so $T_1 + E$ and $T_2 + E$ are mutually nonoverlapping subsets of R^3 for any $T_1, T_2 \in \mathcal{T} \backslash \mathcal{T}'$. From this it follows that $\mathcal{G} = \{G + v \mid T_0 + v \in \mathcal{T} \backslash \mathcal{T}'\}$ is a Z-translative packing with $\frac{1}{3} \cdot \frac{1}{19} d^+(\mathcal{G}) = d^+(\mathcal{T} \backslash \mathcal{T}') = \frac{1}{3} - d^-(\mathcal{T}')$. By Lemma 4.1 we have $d^+(\mathcal{G}) \leq \frac{19}{20}$, thus $d^-(\mathcal{T}') \geq \frac{19}{60}$. Furthermore, if $\mathcal{T}' = T_0 + L_*$, then $\mathcal{G} = \mathcal{G}^*$, and therefore $d^-(\mathcal{T}') = \delta(\mathcal{T}') = \frac{19}{60}$. □

The following corollary shows an interesting property of 17^+-neighbor translative packings of tetrahedra.

Corollary 4.1. *If $\mathcal{T}'$ is a 17^+-neighbor translative packing of tetrahedra, then there exists a tetrahedron $T' \in \mathcal{T}'$ which has 18 neighbors.*

Proof of Corollary 4.1. We use the notation of the proof of Proposition 4.2. It was shown there that $\mathcal{T}_\# = \{T + (E \backslash \{o\}) \mid T \in \mathcal{T} \backslash \mathcal{T}'\}$ is a packing. We have $\bigcup \mathcal{T}_\# \subseteq \bigcup \mathcal{T}'$

and $\bigcup((\mathcal{T}\backslash\mathcal{T}')\cup\mathcal{T}_{\#})\subsetneq\bigcup\mathcal{T}$ since $\bigcup\mathcal{G}\neq R^3=U+Z^3$. Thus $\bigcup\mathcal{T}_{\#}\subsetneq\bigcup\mathcal{T}'$, and it is clear that for each element T' of $\mathcal{T}'$ with $T'\subsetneq\bigcup\mathcal{T}_{\#}$, T' has 18 neighbors in $\mathcal{T}'$. □

We formulate the analogue of the maximum density bound of Proposition 4.2 for 15^+-neighbor translative packings of tetrahedra, as a conjecture.

Conjecture 4.1. *Let $\mathcal{T}$ be a 15^+-neighbor translative packing of a tetrahedron. Then*

$$d^+(\mathcal{T})\leq\tfrac{1}{3}.$$

We also conjecture that the upper bound for the densities of lattice packings of tetrahedra, proved by Hoylman [11], remains valid for all translative packings of tetrahedra. We note that in this bound equality holds for a certain 14-neighbor lattice packing of tetrahedra (that explains why we consider only 15^+-neighbor translative packings in Conjecture 4.1).

Conjecture 4.2. *Let $\mathcal{T}$ be an arbitrary translative packing of a tetrahedron. Then*

$$d^+(\mathcal{T})\leq\tfrac{18}{49}.$$

Finally we show a generalization of the fact (which can be easily derived from Lemma 4.1) that there is no Z-tiling of R^3 with translates of G.

Proposition 4.3. *Let $E_{k,d}=\{v\in Z^d\mid v\in[0,k]^d\backslash\mathrm{vert}([0,k]^d)\}$ and $G_{k,d}=\bigcup\{[-\frac{1}{2},\frac{1}{2}]^d+v\mid v\in E_{k,d}\}$ for arbitrary integers $k,d\geq 2$. Then there is a tiling of R^d with translates of $G_{k,d}$ if and only if $d=2$ and either $k=2$ or $k=3$.*

Proof of Proposition 4.3. Let $L_1\subset R^2$ be the lattice generated by the vectors $(2,1)$ and $(-1,2)$. Similarly, let $L_2\subset R^2$ be the lattice generated by the vectors $(3,2)$ and $(0,4)$. Then it is easy to see that $\{G_{3,2}\}+L_1$ and $\{G_{2,2}\}+L_2$ are tilings of R^2.

In the following we prove the other direction of the proposition. By Theorem 4 on p. 36 of the book by Stein and Szabó [16], we may restrict ourselves to examining the existence of Z-tilings for a given k and d. Dealing with Z-tilings only, we prove the nonexistence of some more general tilings. Let $U_{k,d}=U+(Z^3\cap[0,k]^d)$. Consider an arbitrary packing $\mathcal{P}$ of Z-translates of $G_{k,d}$ and $U_{k,d}$ with the assumption that $\mathcal{P}$ contains at least one translate of $G_{k,d}$. We claim that $\mathcal{P}$ is not a tiling if $d\geq 3$, or $d=2$ and $k\geq 4$.

We use induction on d to prove this claim. For $d=3$ and $k=2$ we get by Lemma 4.1 that $\mathcal{P}$ is not a tiling. For $d=3$ and $k=3$ a similar argument can be applied as in the proof of Lemma 4.1 (where we found a vector v corresponding to $\mathcal{G}$ such that $U+v$ was not covered by $\mathcal{G}$), to find an uncovered translate of U neighborly to an element $G_{k,d}+v'\in\mathcal{P}$. For $d=2$ and $k\geq 4$ a similar argument can be used, but everything is much simpler in this planar case. We leave the details to the reader. We now prove that if $\mathcal{P}$ is a Z-tiling with the required properties for some k and d, then there exists a Z-tiling $\mathcal{P}'$ with the required properties for k and $d-1$. Obviously this will complete the proof of Proposition 4.3. Let $G_{k,d}+v'\in\mathcal{P}$. Denote by e_1 the vector $(1,0,0,\ldots,0)$ of R^d.

Consider $\mathcal{P}' = \{(P - v') \cap H \mid P \in \mathcal{P}\}$, where H is the hyperplane which is orthogonal to e_1 and contains the origin of R^d (thus $H \cong R^{d-1}$ and $Z^d \cap H \cong Z^{d-1}$). Then it is clear that $\mathcal{P}'$ has the required properties, since if $P \in \mathcal{P}'$, then $(P - v') \cap H$ is either a translate of $G_{k,d-1}$ or a translate of $U_{k,d-1}$, and we have $G_{k,d-1} \in \mathcal{P}'$ by construction. We also get that $\mathcal{P}'$ is a Z-tiling. This completes the proof. □

The determination of the maximum density for translative packings of $E_{k,d}$ (the analogue of Lemma 4.1) remains unsolved in general. So we pose this in the following.

Problem 4.1. Determine the maximum density of translative packings of $E_{k,d}$ in R^d for arbitrary integers $k, d \geq 2$.

Acknowledgments

For helpful discussions, I am obliged to Dr. E. Daróczy Kiss. For valuable comments and encouragement, I am grateful to Professors A. Bezdek and W. Kuperberg.

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Received November 4, 1997, *and in revised form February* 5, 1998.