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# A Characterization of Homothetic Simplices

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**Abstract.** It is proved that two convex bodies $K_1, K_2 \subset E^d$ are homothetic simplices if and only if the $d$-dimensional intersections $K_1 \cap (z + K_2)$, $z \in E^d$, belong to at most countably many homothety equivalence classes of convex bodies in $E^d$.

## 1. Introduction

In 1956 Choquet [4] defined a *simplex* (afterwards called a *Choquet simplex*) as a convex set $S$ in linear space $E$ such that for any two homothetic copies of $S$ their intersection, if nonempty, is again a homothetic copy of $S$:

$$(z + \mu S) \cap (y + \nu S) = x + \lambda S, \qquad z, y, x \in E, \quad \mu, \nu, \lambda \geq 0. \tag{1}$$

By using the technique of representing measures (see, e.g., [19]), it was shown later that a finite-dimensional compact Choquet simplex is a simplex in the usual sense, i.e., it is the convex hull of finitely many affinely independent points.

Independently of Choquet, Rogers and Shephard [20] gave a geometric proof of the assertion that a convex body $K$ in the $d$-dimensional linear space $E^d$ is a simplex if and only if every nonempty intersection of $K$ and a translate of $K$ is homothetic to $K$:

$$K \cap (z + K) = x + \lambda K, \qquad z, x \in E^d, \quad \lambda \geq 0. \tag{2}$$

In fact, analysis of their proof shows that a convex body $K \subset E^d$ is a simplex provided all $d$-dimensional intersections $K \cap (z + K)$, $z \in E^d$, are homothetic copies of $K$. The original proof of Rogers and Shephard is rather long. A shorter proof for convex

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polytopes was given by Eggleston et al. [8], and for arbitrary convex bodies by Martini [17] (see also pp. 411–412 of [23]).

These two approaches were developed in various directions. Gruber [12] has shown that vectors $z$ from (2) can be chosen within an arbitrarily small neighborhood of the origin of $E^d$. He also proved that $K$ can be considered a priori as a compact set with nonempty interior, not necessarily convex (see [15] for additional results).

For further development of Rogers and Shephard's assertion, leading to characterizations of direct linear sums of simplices and simplicial cones, see [21], [22], [13], [14], [16], and [18]. We mention here the following result by Gruber [13], [16]: a convex body $K \subset E^d$ is the direct sum of simplices if and only if all $d$-dimensional intersections $K \cap (z + K)$, $z \in E^d$, are affine images of $K$.

In 1970 Simons [24] proved that a bounded Choquet simplex in $E^d$ is the intersection of $d + 1$ half-spaces, each of them being open or closed. Independently, this assertion was strengthened by Eggleston [7] who showed, confirming a conjecture of Gruber [12], that a bounded measurable set of positive measure in $E^d$ satisfying condition (2) is the interior of a $d$-simplex together with the relative interiors of some of its faces. Bair and Fourneau [2] proved that a closed, unbounded, and line-free Choquet simplex in $E^d$ is a convex cone whose base is a $k$-simplex ($k \leq d$). Fourneau [9]–[11] studied nonclosed unbounded Choquet simplices in $E^d$ (see also [1] and [3]).

In what follows, by a convex body in $E^d$ we mean a compact convex set with nonempty interior. The usual abbreviations int $K$ and exp $K$ are taken for the interior and the set of exposed points of a convex body $K$, respectively; rint $F$ serves for the relative interior of a convex set $F \subset E^d$, and $[v, w]$, $]v, w[$ stand for closed and open line segments, both with the endpoints $v, w$. By a convex $d$-polytope we mean a convex polytope of dimension $d$. A facet of a convex $d$-polytope is any of its $(d - 1)$-dimensional faces. We denote by $N(Q)$ the family of outward unit normals to the facets of a convex $d$-polytope $Q \subset E^d$.

Two convex bodies $K, L \subset E^d$ are called *homothetic* provided $K = x + \lambda L$ for a vector $x \in E^d$ and a real number $\lambda > 0$. Clearly, the relation of homothety is reflexive, symmetric, and transitive, i.e., it is a relation of equivalence. Hence the family of convex bodies in $E^d$ can be considered as the union of pairwise disjoint subfamilies such that any two convex bodies in $E^d$ belong to the same subfamily if and only if they are homothetic. We call these subfamilies *homothety classes*.

## 2. Main Result

The assertion by Rogers and Shephard can be formulated in terms of the homothety relation as follows: a convex body $K \subset E^d$ is a simplex if and only if the $d$-dimensional intersections $K \cap (z + K)$, $z \in E^d$, belong to a unique homothety class (namely, the class containing $K$). Based on this, we formulate the main result of the paper.

**Theorem.** *For convex bodies $K_1, K_2 \subset E^d$, $d \geq 1$, the following conditions are equivalent*:

(1) *$K_1$ and $K_2$ are homothetic $d$-simplices.*

(2) *The $d$-dimensional intersections $K_1 \cap (z + K_2)$, $z \in E^d$, belong to a unique homothety class of convex bodies.*
(3) *The $d$-dimensional intersections $K_1 \cap (z + K_2)$, $z \in E^d$, belong to at most countably many homothety classes of convex bodies.*

Based on the result by Gruber mentioned above we formulate the following problem, where affine equivalence classes are defined analogously to homothety classes.

**Problem.** Is it true that for convex bodies $K_1, K_2 \subset E^d$, $d \geq 1$, the following three conditions are equivalent?

(1) $K_1$ and $K_2$ can be represented as direct sums

$$K_1 = L_1 \oplus \cdots \oplus L_t, \qquad K_2 = M_1 \oplus \cdots \oplus M_t, \qquad t \geq 1,$$

such that $L_i$ and $M_i$ are homothetic simplices for all $i = 1, \ldots, t$.
(2) The $d$-dimensional intersections $K_1 \cap (z + K_2)$, $z \in E^d$, belong to a unique affine equivalence class of convex bodies.
(3) The $d$-dimensional intersections $K_1 \cap (z + K_2)$, $z \in E^d$, belong to at most countably many affine equivalence classes of convex bodies.

## 3. Proof of the Theorem

(1) $\Rightarrow$ (2) We can represent $K_1$ in a suitable coordinate system of $E^d$ as

$$K_1 = \{(\xi_1, \ldots, \xi_d) \in E^d : \xi_1 \geq 0, \ldots, \xi_d \geq 0, \ \xi_1 + \cdots + \xi_d \leq 1\}.$$

If $K_2 = z + \mu K_1$, with $z = (\eta_1, \ldots, \eta_d) \in E^d$ and $\mu > 0$, then

$$K_2 = \{(\xi_1, \ldots, \xi_d) \in E^d : \xi_1 \geq \eta_1, \ldots, \xi_d \geq \eta_d, \ \xi_1 + \cdots + \xi_d \leq \mu + \eta_1 + \cdots + \eta_d\}.$$

In this case

$$\begin{aligned} K_1 \cap K_2 \ = \ &\{(\xi_1, \ldots, \xi_d) \in E^d : \xi_1 \geq \max\{0, \eta_1\}, \ldots, \xi_d \geq \max\{0, \eta_d\}, \\ &\xi_1 + \cdots + \xi_d \leq \min\{1, \mu + \eta_1 + \cdots + \eta_d\}\}. \end{aligned}$$

Clearly, the intersection $K_1 \cap K_2$ is $d$-dimensional if and only if

$$\min\{1, \mu + \eta_1 + \cdots + \eta_d\} > (\max\{0, \eta_1\} + \cdots + \max\{0, \eta_d\}).$$

One has $K_1 \cap K_2 = x + \lambda K_1$, where $x = (\max\{0, \eta_1\}, \ldots, \max\{0, \eta_d\})$ and

$$\lambda = \min\{1, \mu + \eta_1 + \cdots + \eta_d\} - (\max\{0, \eta_1\} + \cdots + \max\{0, \eta_d\}).$$

(2) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (1) We prove this implication by induction on $d = \dim E^d$. The case $d = 1$ trivially holds. Assume that (3) $\Rightarrow$ (1) for all $d \leq n-1$, $n \geq 2$, and let $K_1, K_2$ be convex bodies in $E^n$, satisfying condition (3) with $n$ instead of $d$. We divide our consideration into a sequence of lemmas.

**Lemma 1.** *Both $K_1$ and $K_2$ are convex $n$- polytopes.*

*Proof.* Assume that $K_1$ is not a polytope (the case when $K_2$ is not a polytope is considered similarly). Then the set $\exp K_1$ is infinite and has at least one point, say $a$, of accumulation. Denote by $H_1$ and $H_1'$ parallel hyperplanes both supporting $K_1$ such that $a \in H_1 \cap K_1$. Choose any point $a' \in H_1' \cap K_1$.

Clearly, there is a translate $z + K_2$ such that $a \in \operatorname{int}(z + K_2)$ and one of the two hyperplanes supporting $z + K_2$ and parallel to $H_1$ intersects the open line segment $]a, a'[$. Denote this hyperplane by $H_2$. Choose two distinct points $b, c \in \exp K_1$ so close to $a$ that $b, c \in \operatorname{int}(z + K_2)$. By a continuity argument, there is a real number $\varphi > 0$ such that the set $\{a, b, c\}$ lies in the interior of the body $\varepsilon(a - a') + z + K_2$ for all $\varepsilon \in\ ]0, \varphi[$.

Now choose two closed half-spaces, say $P_b$ and $P_c$, satisfying $K_1 \cap P_b = \{b\}$ and $K_1 \cap P_c = \{c\}$. By the construction above, both points $b$ and $c$ are exposed for every convex body

$$M(\varepsilon) := K_1 \cap (\varepsilon(a - a') + z + K_2), \qquad \varepsilon \in\ ]0, \varphi[,$$

and $M(\varepsilon) \cap P_b = \{b\}$, $M(\varepsilon) \cap P_c = \{c\}$.

Due to condition (3), the bodies $M(\varepsilon)$, $\varepsilon \in\ ]0, \varphi[$, belong to at most countably many homothety classes. Hence there are at least two distinct (in fact, uncountably many) homothetic bodies $M(\varepsilon_1)$, $M(\varepsilon_2)$, $\varepsilon_1, \varepsilon_2 \in\ ]0, \varphi[$. Let $\varepsilon_1 < \varepsilon_2$. Then the width of $M(\varepsilon_1)$ in the direction $l$ orthogonal to $H_1$ is smaller than the width of $M(\varepsilon_2)$ in the same direction. Therefore $M(\varepsilon_1)$ is a smaller copy of $M(\varepsilon_2)$:

$$M(\varepsilon_1) = x + \lambda M(\varepsilon_2), \qquad x \in E^n, \quad 0 < \lambda < 1,$$

and from $M(\varepsilon_2) \cap P_b = \{b\}$ we deduce

$$M(\varepsilon_1) \cap (x + \lambda P_b) = (x + \lambda M(\varepsilon_2)) \cap (x + \lambda P_b) = x + \lambda(M(\varepsilon_2) \cap P_b) = \{x + \lambda b\}.$$

On the other hand, $M(\varepsilon_1) \cap P_b = \{b\}$. Since the half-space $x + \lambda P_b$ is a translate of $P_b$, both $P_b$ and $x + \lambda P_b$ support $M(\varepsilon_1)$ at the same exposed point, i.e., $b = x + \lambda b$.

In a similar way, $c = x + \lambda c$. Thus $b - c = \lambda(b - c)$, contradicting $b \neq c$ and $0 < \lambda < 1$. Hence $K_1$ is a convex $n$-polytope. □

**Lemma 2.** *The polytopes $K_1$ and $K_2$ have the same family of outward unit normals to their facets*: $N(K_1) = N(K_2)$.

*Proof.* By a symmetry argument, it is sufficient to prove the inclusion $N(K_2) \subset N(K_1)$. Let $e \in N(K_2)$ and $F$ be the facet of $K_2$ with the outward unit normal $e$. Choose a point $v \in \operatorname{rint} F$. Denote by $H_2$ the hyperplane containing $F$ and by $P_2$ the closed half-space bounded by $H_2$ and containing $K_2$.

Assume that $e \notin N(K_1)$, and let $P_1$ be the translate of $P_2$ containing $K_1$ such that the boundary hyperplane $H_1$ of $P_1$ supports $K_1$. By the assumption, the set $G = H_1 \cap K_1$ is a face of $K_1$ of dimension at most $n - 2$. Choose any vertex $w$ of $G$. Clearly, there are at least two distinct edges of $K_1$, say $[w, z]$ and $[w, z']$, both intersecting $\operatorname{int} P_1$.

Now consider the polytope $w - v + K_2$. Since $v \in \operatorname{rint} F$, the vertex $w$ belongs to the relative interior of the facet $w - v + F$. Hence both line segments $[w, z]$, $[w, z']$ intersect the interior of the polytope $w - v + K_2$. By a continuity argument, there is a real number $\varphi > 0$ such that both line segments $[w, z]$ and $[w, z']$ intersect the relative interior of the facet $\varepsilon(z - w) + (w - v + F)$, as well as the interior of the $n$-polytope $\varepsilon(z - w) + (w - v + K_2)$ for all $\varepsilon \in ]0, \varphi[$. It is easily seen that both points

$$a(\varepsilon) := [w, z] \cap (\varepsilon(z-w)+(w-v+F)), \qquad a'(\varepsilon) := [w, z'] \cap (\varepsilon(z-w)+(w-v+F))$$

are vertices of the convex $n$-polytope

$$M(\varepsilon) := K_1 \cap (\varepsilon(z - w) + (w - v + K_2)),$$

and the line segment

$$s(\varepsilon) := [w, z] \cap (\varepsilon(z - w) + (w - v + K_2))$$

is an edge of $M(\varepsilon)$.

Summing up, we have that the family $\{M(\varepsilon), \varepsilon \in ]0, \varphi[\}$ contains uncountably many pairwise nonhomothetic convex $n$-polytopes. Indeed, the distance between the vertices $a(\varepsilon)$ and $a'(\varepsilon)$ continuously tends to zero when $\varepsilon \to 0$, while the length of the edge $s(\varepsilon)$ does not decrease. The last contradicts condition (3). Hence $e \in N(K_1)$ and $N(K_2) \subset N(K_1)$. □

**Lemma 3.** *Any two facets of $K_1$ and $K_2$, respectively, with the same outward unit normal are homothetic $(n - 1)$-simplices.*

*Proof.* Let $F_1$ and $F_2$ be two facets of $K_1$ and $K_2$, respectively, having the same outward unit normal. Denote by $H$ the hyperplane containing $F_1$, and consider the family of translates $z+K_2$, $z \in E^n$, having the facets $z+F_2$ in $H$. Clearly, $\operatorname{rint} F_1 \cap \operatorname{rint}(z+F_2) \neq \emptyset$ implies $\operatorname{int} K_1 \cap \operatorname{int}(z + K_2) \neq \emptyset$. From condition (3) it follows that the family of convex $(n-1)$-polytopes $F_1 \cap (z + F_2)$, being facets of the $n$-polytopes $K_1 \cap (z + K_2)$, belongs to at most countably many homothety classes in the $(n - 1)$-space $H$. By the inductive hypothesis, $F_1$ and $F_2$ are homothetic $(n - 1)$-simplices. □

From Lemmas 2 and 3 we deduce the following corollary.

**Corollary 1.** *The polytopes $K_1$ and $K_2$ are homothetic.*

Without loss of generality, we may assume in what follows that $K_1$ is not larger than $K_2$: $K_1 = x + \lambda K_2$ with $0 < \lambda \leq 1$.

To formulate the next lemma, we need a definition. Let $F$ be a facet of an $n$-polytope $Q \subset E^n$, and let $H$, $H'$ be two parallel hyperplanes both supporting $Q$ such that $F \subset H$ (since $F$ is a facet, $H$ and $H'$ are uniquely determined). The face $Q \cap H'$ of $Q$ is called *antipodal* to $F$.

**Lemma 4.** *For any facet of $K_1$ its antipodal face consists of a single point.*

*Proof.* Assume that for a facet $F$ of $K_1$, its antipodal face $F'$ contains more than one point. Let $H$ be the hyperplane containing $F$, and let $H'$ be the hyperplane parallel to $H$ such that $H' \cap K_1 = F'$. Choose an edge $[v_1, w_1]$ of $F'$, and let $[v_2, w_2]$ be the corresponding edge of $K_2$ (under the homothety $K_1 = x + \lambda K_2$). The polytope $v_1 - v_2 + K_2$ contains $K_1$ entirely, and the edge $[v_1, w_1]$ lies in the corresponding edge $[v_1, v_1 - v_2 + w_2]$ of $v_1 - v_2 + K_2$. Let $\varphi > 0$ be such a small real number that rint $F$ has common points with every open set int $(\varepsilon(w_1 - v_1) + v_1 - v_2 + K_2)$, $\varepsilon \in ]0, \varphi[$, and the line segments

$$[v_1, w_1] \quad \text{and} \quad [\varphi(w_1 - v_1) + v_1, \varphi(w_1 - v_1) + v_1 - v_2 + w_2]$$

have more than one point in common. We claim that the family

$$\{M(\varepsilon) := K_1 \cap (\varepsilon(w_1 - v_1) + v_1 - v_2 + K_2),\ \varepsilon \in ]0, \varphi[\}$$

contains uncountably many pairwise nonhomothetic convex $n$-polytopes, contradicting condition (3). Indeed, it is easily seen that the line segment

$$[v_1, w_1] \cap [\varepsilon(w_1 - v_1) + v_1, \varepsilon(w_1 - v_1) + v_1 - v_2 + w_2]$$

is an edge of $M(\varepsilon)$, whose length continuously increases when $\varepsilon \to 0$. At the same time, the width of $M(\varepsilon)$ in the direction $l$ orthogonal to $H$ equals the distance between $H$ and $H'$, i.e., it is independent of $\varepsilon$. □

The next lemma and Corollary 1 give the final point in the proof of the theorem.

**Lemma 5.** *$K_1$ is a simplex.*

*Proof.* Assume that $K_1$ is not a simplex and choose a facet $F_1$ of $K_1$. By Lemma 3, $F$ is an $(n-1)$-simplex. Let $v_1$ be the vertex of $K_1$ antipodal to $F_1$ (see Lemma 4). Denote by $H$ the hyperplane containing $F_1$ and by $H'$ the hyperplane through $v_1$ parallel to $H$. Let $l$ be the direction orthogonal to $H$. By the assumption, $K_1$ is distinct from the $n$-simplex $\operatorname{conv}(v_1 \cup F_1)$. Hence there is at least one facet $G_1$ of $K_1$, having $(n-2)$-dimensional intersection with $F_1$ and not containing $v_1$. Put $L = F_1 \cap G_1$. Choose any point $w_1 \in \operatorname{rint} L$. Since $v_1 \notin F_1 \cup G_1$, we easily deduce that the open line segment $]v_1, w_1[$ lies in int $K_1$. Denote by $z_1$ the vertex of $F_1$ which is not in $L$.

Now let $v_2$, $w_2$, and $z_2$ be the respective vertices of $K_2$ (under the homothety $K_1 = x + \lambda K_2$). Clearly, $K_1 \subset K_2' := w_1 - w_2 + K_2$ and $v_1 \in ]w_1, w_1 - w_2 + v_2[ \subset \operatorname{int} K_2'$. By a continuity argument, there is a real number $\varphi > 0$ such that $v_1$ belongs to every translate $\varepsilon(z_1 - w_1) + K_2'$, $\varepsilon \in ]0, \varphi[$. The number $\varphi$ can be also chosen so small that the intersection

$$F(\varepsilon) := F_1 \cap (\varepsilon(z_1 - w_1) + w_1 - w_2 + F_2)$$

is not empty for all $\varepsilon \in ]0, \varphi[$. By Lemma 3, every $F(\varepsilon)$, $\varepsilon \in ]0, \varphi[$, is an $(n-1)$-simplex homothetic to $F_1$.

Finally, consider the family of convex $n$-polytopes

$$\{M(\varepsilon) := K_1 \cap (\varepsilon(z_1 - w_1) + K_2'),\ \varepsilon \in ]0, \varphi[\}.$$

Clearly, $F(\varepsilon)$ is the facet of $M(\varepsilon)$ lying in $H$, and the size of $F(\varepsilon)$ continuously increases to the size of $F_1$ when $\varepsilon \to 0$. On the other hand, the width of $M(\varepsilon)$ in the direction $l$ equals the distance between $H$ and $H'$, i.e., it is independent of $\varepsilon$.

Summing up, we deduce that the family $\{M(\varepsilon), \varepsilon \in ]0, \varphi[\}$ contains uncountably many pairwise nonhomothetic convex $n$-polytopes, contradicting (3). Hence $K_1$ is a simplex. □

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