Discrete Comput Geom 22:161–166 (1999)

Discrete & Computational
Geometry

Minkowski Circle Packings on the Sphere

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To the fiftieth birthday of my son Gábor

Abstract. We consider n caps on the sphere such that none of them contains in its interior the center of another. We give an upper bound for the total area of the caps, which is sharp for $n = 3$, 4, 6, and 12 and is asymptotically sharp for great values of n.

1. Introduction, Results

A *set of balls* is said to be a *Minkowski* set if none of its elements contains in its interior the center of another. Replacing in a Minkowski set of balls each ball with a concentric ball of radius half as big as the original, we obtain a packing of balls which we call a *Minkowski packing*. Minkowski sets, Minkowski packings of balls, and their variants have been investigated because of their connection with the geometry of numbers and the theory of functions [1], [2] as well as because of their intrinsic interest [3]–[9]. This paper deals with Minkowski sets and Minkowski packings of circles (spherical caps) on the sphere.

In Section 2 we prove two theorems.

Theorem 1. *The density of a Minkowski packing of $n > 2$ circles on the sphere is at most*

$$\frac{n}{2}\left(1 - \tfrac{1}{2}\sin^{-1}\omega_n\right), \tag{1}$$

where

$$\omega_n = \frac{n}{n-2}\frac{\pi}{6}.$$

Theorem 2. *The density of a Minkowski set of $n > 2$ circles on the sphere is at most*

$$n\left(1 - \tfrac{1}{4}\sin^{-2}\omega_n\right). \tag{2}$$

Both bounds are sharp for $n = 3$, 4, 6, and 12, and asymptotically sharp for great values of n.

Section 3 contains some open problems and remarks.

2. Proof of Theorems 1 and 2

Throughout this section we operate on the unit sphere. We use the same symbol for a domain and its area.

Let $c_1, \ldots, c_n$ be circles with radii $r_1, \ldots, r_n$ and centers $O_1, \ldots, O_n$, which form a Minkowski packing. We start the proof of Theorem 1 with the case of three circles. We may suppose that $r_1 > \pi/3$. Then O_2 and O_3 lie in a circle of radius $\pi - 2r_1$ which implies that $r_2 < \pi - 2r_1$ and $r_3 < \pi - 2r_1$. It follows that

$$\frac{c_1 + c_2 + c_3}{4\pi} < \tfrac{1}{2}(3 - \cos r_1 + 2\cos 2r_1),$$

which is less than the density bound 0.75 obtained from (1) for $n = 3$.

Now we show that it is enough to prove Theorem 1 for $n > 3$ circles such that $\max(r_1, \ldots, r_n) \leq 58.4^\circ$. Suppose that $r_1 = 58.4^\circ$ and that O_1 is the north pole. Then the centers $O_2, \ldots, O_n$ lie in a circle Γ of radius $R = 180^\circ - 2 \times 58.4^\circ = 63.2^\circ$ with center at the south pole. Let T_k be the maximum total area of k circles which along with c_1 constitute a Minkowski packing. It is not difficult to show that, for $k = 3$, 4, and 5, T_k consists of k congruent circles whose radius $\varrho_k = \arcsin(\sin(\pi/k)\sin R)$ is half of the side length of a regular k-gon inscribed in Γ. T_6 is attained by six circles of radius $R/2 = 31.6^\circ$. (Note that $R/2$ is just a little bit smaller than $\varrho_5 \approx 31.64$.) We list the approximate values of T_k:

k	3	4	5	6
T_k	6.8913	5.6383	4.6710	5.5898

Obviously, the biggest radius of the circles yielding T_k for $k > 6$ is less than $R/2$. Thus, for $k > 6$, T_k is certainly smaller than

$$2\pi\left(1 - \cos\frac{3R}{2}\right) \approx 6.8089$$

showing that for $k \geq 3$ we have $T_k \leq T_3$. Therefore the density of the packing is at most

$$\frac{2\pi(1 - \cos 58.4^\circ) + T_3}{4\pi} \approx 0.7864.$$

This is smaller than the bound $0.845\ldots$ obtained from (1) for $n = 4$ and hence, by the monotonity of (1), smaller than the bound obtained for any $n \geq 4$. In the case of $r_1 > 58.4^\circ$, similar considerations yield even smaller density bounds.

Let c and c' be two disjoint circles with radii r and r' and centers O and O', respectively. Let σ be the (spherical) segment OO'. Let M be the midpoint of the segment with endpoints $\sigma \cap \operatorname{bd} c$ and $\sigma \cap \operatorname{bd} c'$. Let l be the line (great circle) drawn through M

orthogonally to σ. We call l the *midline* of c and c'. We say that a point lies *nearer* to c than to c' if it lies on the same side of l as c.

Let c be a circle of the packing. Let C be the *cell* of c defined as the set of points which are nearer to c than to any other circle of the packing.

Let c' be another circle of the packing which does not contain the point diametrically opposite to O. Let Δ be the triangle bounded by the line OO', the *midline* of c and c', and a tangent to c' passing through O. Let s be the sector of c lying in Δ, and let ω be its angle. In the case when c' touches c, we write Δ^*, s^*, and ω^* for Δ, s, and ω. We show that

$$\frac{c}{C} \le \frac{s^*}{\Delta^*}, \tag{3}$$

where C is the cell of c.

First we prove that

$$\frac{s}{\Delta} \le \frac{s^*}{\Delta^*}. \tag{4}$$

Let t be the distance between O and O', and let $d = \frac{1}{2}(t + r - r')$ be the distance of O from the midline of c and c'. Let D denote the digon defined as the convex hull of O, c', and the point opposite to O. Consider the array of circles inscribed in D. In this array, d is an increasing function of t. Approaching c' toward c, s remains constant but Δ decreases so that s/Δ increases. If $\omega \le \omega^*$, we stop approaching c' at the moment when $t = 2r$. If, on the other hand, $\omega > \omega^*$, we approach c' toward c until it touches c and observe that in this situation ω/Δ is a decreasing function of ω. Therefore, replacing ω by ω^*, s/Δ increases on.

For $t = 2r$ we have

$$\frac{s}{\Delta} = \frac{\omega(1 - \cos r)}{\omega - \arcsin(\cos d \sin \omega)} = F_r(\omega), \tag{5}$$

where $d = (3r - r')/2$ and $\sin r' = \sin \omega \sin 2r$. A computer investigation of $F_r(\omega)$ and its derivative—kindly performed by J. Linhart (Salzburg)—has shown that, for $r \in (0, 58.4^\circ)$, $F_r(\omega)$ is, for $0 < \omega \le \omega^*$, strictly increasing, confirming (4). (For $r = 58.4^\circ$, the maximum of $F_r(\omega)$ is attained at $\omega \approx 73.009^\circ$, which is less than $\omega^* = \arcsin(\sin r/\sin 2r) \approx 73.125^\circ$.)

Turning to the proof of (3), let c_1 and c_2 be two circles corresponding to two consecutive sides of C. Let A be the angular region bounded by the half-lines OO_1 and OO_2. We claim that

$$\frac{A \cap c}{A \cap C} \le \frac{s^*}{\Delta^*}. \tag{6}$$

Assuming that $\overline{OO_1} < \overline{OO_2}$, we rotate c_2 about O_1 so that $\overline{OO_2}$ decreases until $\overline{OO_2} = \overline{OO_1}$. In any position of c_2 let S be the set of points which are nearer to c than either to c_1 or to c_2. Let the initial position of c_2 be such that OO_2 touches c_1. Now $A \cap c/A \cap S \le s^*/\Delta^*$ with equality only if c_1 touches c. During the rotation of c_2, $A \cap c/A \cap S$ decreases until the intersection I of the midlines of c and c_1 and c and c_2 lies outside of A. As soon as I gets into A, $A \cap c/A \cap S$ increases. In the final position, when $\overline{OO_2} = \overline{OO_1}$, c_1, and

c_2 are separated from each other by the line passing through O and the midpoint of the segment O_1O_2, so that, in view of (4),

$$\frac{A\cap c}{A\cap C}\le\frac{A\cap c}{A\cap S}\le\frac{s^*}{\Delta^*},$$

as stated. Equality holds only if c, c_1 and c_2 mutually touch each other.

In what follows, we write $\omega^*=\omega$, keeping in mind that ω satisfies

$$\sin\omega=\frac{1}{2\cos r}.$$

In the triangle Δ^*, let φ be the angle opposite to r. Then $\cos\varphi=\sin\omega\cos r=\frac{1}{2}$, so that $\varphi=\pi/3$ and $\Delta^*=\omega-\pi/6$. In view of (3), we have

$$C\ge 2\pi\left(1-\frac{\pi}{6\omega}\right)=f(\omega).$$

Consider $f(\omega)$ as a function $g(c)$ of

$$c=2\pi(1-\tfrac{1}{2}\sin^{-1}\omega). \tag{7}$$

We have

$$\frac{dg}{dc}=\frac{dg}{d\omega}:\frac{dc}{d\omega}=\frac{\pi}{3}\frac{\sin^2\omega}{\omega^2\cos\omega}.$$

Writing $h(\omega)=(\sin^2\omega)/(\omega^2\cos\omega)$, we have

$$\begin{aligned}\omega^4\cos^2\omega h'(\omega) &= \omega\sin\omega(2\omega\cos^2\omega+\omega\sin^2\omega-\sin 2\omega)\\ &= \omega\sin\omega(2\omega-\sin 2\omega-\omega\sin^2\omega)\ge\omega\sin\omega\left(\frac{8\omega^3}{6}-\omega\sin^2\omega\right)\\ &= \omega^2\sin\omega(\tfrac{4}{3}\omega^2-\sin^2\omega)\ge 0.\end{aligned}$$

This implies that $g(c)$ is convex so that we can use Jensen's inequality:

$$4\pi\ge C_1+\cdots+C_n\ge ng(\overline{c})=n2\pi\left(1-\frac{\pi}{6\omega}\right),$$

where $\overline{c}$ is the average area of the circles $c_1,\ldots,c_n$ and ω is the pertaining value of it according to (7).

The last inequality yields

$$\omega\le\frac{n}{n-2}\frac{\pi}{6}=\omega_n,$$

whence

$$\overline{c}\le 2\pi(1-\tfrac{1}{2}\sin^{-1}\omega_n), \tag{8}$$

completing the proof of Theorem 1. □

Next we turn to the proof of Theorem 2. For any circle c of radius r, let $S(c)$ denote the area of the circle whose radius is $2r$. Using the facts that $c = 2\pi(1 - \cos r)$ and $S = 2\pi(1 - \cos 2r)$, we get

$$\frac{dS}{dc} = \frac{dS}{dr} : \frac{dc}{dr} = 4\cos r,$$

which implies that $S(c)$ is concave. Thus, by Jensen's inequality, $\overline{S}$, the average area of the circles in the Minkowski set of circles associated with our Minkowski packing, satisfies $\overline{S} \leq S(\overline{c})$. In view of (8), $\overline{c} \leq 2\pi(1 - \cos r)$, where $\cos r = 1/(2\sin\omega_n)$. Therefore,

$$\overline{S} \leq 2\pi(1 - \cos 2r) = 4\pi \sin^2 r = 4\pi(1 - \tfrac{1}{4}\sin^{-2}\omega_n),$$

as asserted in Theorem 2. □

3. Concluding Remarks

In the special case when all circles are congruent, Theorem 1 has been proved independently from each other by Hadwiger and the author. This is the earliest result in the vast literature about packing the sphere with equal circles.

In the Euclidean plane, the problems of the densest Minkowski circle packing and the densest Minkowski circle arrangement are equivalent. The densest Minkowski circle packing consists of equal circles, and the densest Minkowski circle arrangement arises from the densest packing of equal circles by replacing each circle by a concentric one twice as big as the original. The situation is similar on the sphere in the case of 3, 4, 6, and 12 circles but quite different for 5 circles.

Let c be a circle with center O and radius $r \geq 45°$. Let c' be a circle centered at a distance $2r$ from O whose tangents through O make a right angle. The radius of c' is given by $\sin r' = (\sin 2r)/\sqrt{2}$. Four circles congruent to c' along with c form a Minkowski packing with density $a(r) = \frac{1}{2}(5 - \cos r - 4\cos r')$. On the other hand, the density of the associated Minkowski arrangement is $b(r) = \frac{1}{2}(5 - \cos 2r - 4\cos 2r')$. It is easy to see that the maximum of $a(r)$ and $b(r)$ yield the density of the densest Minkowski packing and the density of the densest Minkowski arrangement of five circles. However, $a(r)$ and $b(r)$ attain their maxima at different values of r, namely, at $r \sim 48.899°$ and $r \sim 48.590°$.

Are there other values of $n \neq 5$, for which the problem of the densest Minkowski packing of n circles is not equivalent with the problem of the densest Minkowski arrangement of n circles?

In the Euclidean plane we define a Minkowski arrangement of centrally symmetrical convex disks by the requirement that no disk contains in its interior the center of another disk. Several properties of such arrangement are known [3], [4]. Special attention is due to Minkowski arrangements of homothetic disks. Reducing in such an arrangement each disk by a similitude about its center in the ratio $1:2$ we obtain a Minkowski packing. It is very probable that the density of a Minkowski packing of centrally symmetric homothetic convex disks never exceeds the density of the densest lattice packing of one of the disks.

We sketch a line of attack to the problem. Let the disks $c_1, c_2, \ldots$ with centers $O_1, O_2, \ldots$ form a Minkowski packing. We assume that the disks are smooth. Let c_i and c_j be two of them, let s be the segment $O_i O_j$, and let M be the midpoint of the segment joining $s \cap \operatorname{bd} c_i$ and $s \cap \operatorname{bd} c_j$. We draw a line l through M parallel to the tangents of c_i and c_j at the points $s \cap \operatorname{bd} c_i$ and $s \cap \operatorname{bd} c_j$. We say that a point P is *nearer* to c_i than to c_j if P and c_i lie on the same side of l. Let C_i be the set of points which lie nearer to c_i than to any other disks. Now the problem is to find the arrangement of the disks for which the area C_i attains its minimum. The above conjecture would be proved if we could show that in this problem we can restrict ourselves to the case when C_i is a hexagon.

Acknowledgement

I express my thanks to E. Makai for many helpful comments.

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Received September 11, 1997, *and in revised form March* 2, 1998.