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Obstructions to Shellability*

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Abstract. We consider a simplicial complex generalization of a result of Billera and Myers that every nonshellable poset contains the smallest nonshellable poset as an induced subposet. We prove that every nonshellable two-dimensional simplicial complex contains a nonshellable induced subcomplex with less than eight vertices. We also establish CL-shellability of interval orders and as a consequence obtain a formula for the Betti numbers of any interval order.

A recent result of Billera and Myers [BM] implies that every nonshellable poset contains as an induced subposet the four-element poset Q consisting of two disjoint two-element chains. (Throughout this paper shellability refers to the general notion of nonpure shellability introduced in [BW2].) Note that Q is the nonshellable poset with the fewest number of elements. Of course, a shellable poset can also contain Q; e.g., the lattice of subsets of a three-element set. So the condition of not containing Q as an induced subposet is only sufficient for shellability; it does not characterize shellability. It is, however, a well-known characterization of a class of posets called interval orders and the question of whether all interval orders are shellable is what Billera and Myers were considering in the first place.

In this note we suggest a way to generalize the poset result to general simplicial complexes. We also give a simple proof of the poset result and prove the stronger result that any poset that does not contain Q as an induced subposet is CL-shellable. This yields a recursive formula for the Betti numbers of the poset.

We assume familiarity with the general theory of shellability [BW2], [BW3]. Recall

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that a simplicial complex is said to be *shellable* if its facets can be arranged in linear order $F_1, F_2, \ldots, F_t$ in such a way that the subcomplex

$$(\bar{F}_1 \cup \bar{F}_2 \cup \cdots \cup \bar{F}_{i-1}) \cap \bar{F}_i$$

is pure and $(\dim F_i - 1)$-dimensional for all $i = 2, \ldots, t$. (For any face F of a simplicial complex, $\bar{F}$ denotes the subcomplex consisting of F and all its subsets.) Such an ordering is called a *shelling*. The notation and terminology used throughout this paper is defined in [BW2] and [BW3].

The most simple-minded conjecture one could make is that every nonshellable simplicial complex contains the induced subcomplex consisting of edges $\{a, b\}$ and $\{c, d\}$, where a, b, c, d are distinct vertices. A simple counterexample is given by the five-vertex simplicial complex consisting of facets $\{a, b, c\}, \{c, d, e\}, \{a, d\}$. Indeed the situation for simplicial complexes turns out to be much more complicated than it is for posets.

The most natural thing to do next is to look for other "obstructions" to simplicial complex shellability. Is there a finite list? Below we see that the answer is no. Define an *obstruction* to shellability to be a nonshellable simplicial complex all of whose proper induced subcomplexes are shellable. The four- and five-vertex simplicial complexes given above are examples of one- and two-dimensional obstructions, respectively. The following observation was made by Stanley [S1].

Proposition 1. *For every positive integer d there is an obstruction to shellability of dimension d.*

Proof. Let K be the $(d-1)$-skeleton of the simplex on vertex set $\{1, 2, \ldots, d+3\}$ together with two d-dimensional faces $\{1, 2, \ldots, d+1\}$ and $\{3, 4, \ldots, d+3\}$. We claim that K is a d-dimensional obstruction. If K were shellable, then by the Rearrangement Lemma of [BW2] there would be a shelling order in which the maximal dimensional facets come first; namely, $\{1, 2, \ldots, d+1\}$ and $\{3, 4, \ldots, d+3\}$ come first. However, this is impossible because these two facets intersect in a face of dimension $d-2$. Hence K is not shellable.

Every proper induced subcomplex of K is either a simplex or consists of a single d-face in a $(d+1)$-simplex together with the $(d-1)$-skeleton of the $(d+1)$-simplex. Certainly the simplex is shellable. Let J be the $(d-1)$-skeleton of the simplex on vertex set $\{1, 2, \ldots, d+2\}$ together with the face $\{1, 2, \ldots, d+1\}$. It is easy to see that the lexicographical order on the facets of J is a shelling of J. (The lexicographical order on subsets of $\{1, 2, \ldots, d+2\}$ is defined by $\{a_1 < \cdots < a_k\} \le \{b_1 < \cdots < b_j\}$ if the word $a_1 \cdots a_k$ is less than the word $b_1 \cdots b_j$ in lexicographical order.) Therefore all proper induced subcomplexes of K are shellable. □

We now consider the following problem.

Problem. Determine whether or not there is a finite number of d-dimensional obstructions to shellability for each d. If so, find bounds on the number of vertices that a d-dimensional obstruction can have.

In this paper we solve this problem only for dimensions $d = 1, 2$ and we leave open the problem for general d.

Proposition 2. *The only one-dimensional obstruction to shellability is the complex J generated by facets $\{a, b\}$, $\{c, d\}$ where a, b, c, d are distinct.*

Proof. A one-dimensional simplicial complex is shellable if and only if it has at most one connected component with more than one vertex. If K is a nonshellable one-dimensional simplicial complex, then let $\{a, b\}$ be an edge in one component of K and let $\{c, d\}$ be an edge in another component. The subcomplex induced by a, b, c, d is J. Hence K is an obstruction if and only if $K = J$. □

Already in dimension 2 the situation is much more complicated. We use the following notation: For any subset U of V and simplicial complex K on vertex set V, let $K(U)$ be the subcomplex of K induced by U. Also let the *pure part* of K, denoted $\mathrm{pure}(K)$, be the subcomplex of K generated by the facets of maximum dimension. For $v \in V$, the *link* of v in K is denoted $\mathrm{lk}_K(v)$ and is defined to be $\{F \in K \mid F \cup \{v\} \in K \text{ and } v \notin F\}$. For any $v \in V$ and subcomplex J of $\mathrm{lk}_K(v)$, the *join* of v and J is denoted $v * J$ and is defined to be $\{F \in K \mid v \in F \text{ and } F \backslash \{v\} \in J\}$. The ith reduced simplicial homology of K over the ring of integers is denoted by $\tilde{H}_i(K)$.

Theorem 3. *The number of vertices in a two-dimensional obstruction is either five, six or seven.*

Proof. It is easy to see that any two-dimensional simplicial complexes with less than five vertices is shellable and hence cannot be an obstruction.

Let K be a two-dimensional simplicial complex on vertex set V where $|V| > 7$. Assume all induced proper subcomplexes are shellable. We shall show that K is shellable by showing that $\mathrm{pure}(K)$ is shellable and the 1-skeleton of K is connected (except for isolated points). That the 1-skeleton is connected follows immediately from the fact that no induced subcomplex consists only of a pair of disjoint edges.

To prove that $\mathrm{pure}(K)$ is shellable, choose any vertex v of $\mathrm{pure}(K)$. Let

$$K_1 = \mathrm{pure}(K(V \backslash \{v\})).$$

Since the pure part of a shellable complex is shellable (by the Rearrangement Lemma of [BW2]), K_1 is shellable. Let

$$K_2 = v * \mathrm{pure}(\mathrm{lk}_K(v)).$$

We claim that $\mathrm{pure}(\mathrm{lk}_K(v))$ is a connected one-dimensional complex. If not there would be distinct vertices $a, b, c, d \in V \backslash \{v\}$ such that edges $\{a, b\}$ and $\{c, d\}$ are in different components of $\mathrm{pure}(\mathrm{lk}_K(v))$. Since $|V| > 5$, the induced subcomplex $K(\{v, a, b, c, d\})$ would be shellable which would imply that $\mathrm{lk}_{K(\{v,a,b,c,d\})}(v)$ is shellable since the link of any vertex in a shellable complex is shellable [BW3]. However, this is impossible since $\mathrm{lk}_{K(\{v,a,b,c,d\})}(v)$ has only two facets $\{a, b\}$ and $\{c, d\}$. It follows from this claim that K_2 is shellable and two-dimensional.

Next we dispose of the special cases that K_1 is zero or one-dimensional. Clearly, K_1 cannot be zero-dimensional since v belongs to a 2-face. If K_1 is one-dimensional, then $\mathrm{pure}(K) = K_2$ which is shellable.

Now we can assume that K_1 and K_2 are both shellable and two-dimensional. It follows that

$$\mathrm{pure}(K) = K_1 \cup K_2.$$

Let

$$A = K_1 \cap K_2.$$

We shall show that A is connected and one-dimensional. Suppose not. Then either (1) $A = \{\emptyset\}$, (2) A contains an isolated point, or (3) there are edges in different components of A. For the first case, choose distinct vertices a, b, c, d, e such that $\{a, b\} \in \mathrm{pure}(\mathrm{lk}_K(v))$ and $\{c, d, e\} \in K_1$. Since $|V| > 6$, $K(\{v, a, b, c, d, e\})$ is shellable. It follows that $\{v, a, b\}$ and $\{c, d, e\}$ cannot be the only 2-faces of $K(\{v, a, b, c, d, e\})$. Hence there is a third 2-face F. Note $v \notin F$ because otherwise one of the other vertices of F would be in A. So $F \in K_1$. Since either a or b is a vertex of F as well as of $\mathrm{pure}(\mathrm{lk}_K(v))$, a or b is a vertex of $K_1 \cap K_2$. Hence A cannot be $\{\emptyset\}$.

For the second case, let a be the isolated point. Then K_1 and K_2 contain 2-faces $\{a, c, d\}$ and $\{v, a, b\}$, respectively, which intersect only at a. Since $|V| > 5$, $K(\{v, a, b, c, d\})$ is shellable. This means that there is a third 2-face in the induced subcomplex that intersects each of the 2-faces along edges that contain a. If the third 2-face contains v, then it is either $\{v, a, c\}$ or $\{v, a, d\}$. This implies that either $\{a, c\}$ or $\{a, d\}$ is in A, which contradicts the fact that $\{a\}$ is a facet of A. Hence the third 2-face must be $\{a, b, c\}$ or $\{a, b, d\}$. It follows that $\{a, b\}$ is a facet of A, which is still a contradiction.

For the third case, suppose that $\{a, b\}$ and $\{c, d\}$ are edges in different components of A. Let $x, y \in V \backslash \{v\}$ be such that $\{a, b, x\}$ and $\{c, d, y\}$ are facets of K_1. Since $|V| > 7$, $J = K(\{v, a, b, c, d, x, y\})$ is shellable. Let

$$B = (v * \mathrm{pure}(\mathrm{lk}_J(v))) \cap \mathrm{pure}(K(\{a, b, c, d, x, y\})).$$

Since B is a subcomplex of A, $\{a, b\}$ and $\{c, d\}$ are in different components of B. It follows that $\tilde{H}_0(B) \neq 0$. Since $v * \mathrm{pure}(\mathrm{lk}_J(v))$ is contractible, we also have $\tilde{H}_i(v * \mathrm{pure}(\mathrm{lk}_J(v))) = 0$ for all i. By the Rearrangement Lemma of [BW2], $\mathrm{pure}(K(\{a, b, c, d, x, y\}))$ is shellable since the induced subcomplex $K(\{a, b, c, d, x, y\})$ is. Hence $\tilde{H}_i(\mathrm{pure}(K(\{a, b, c, d, x, y\}))) = 0$ for $i \leq 1$. Since

$$\mathrm{pure}(J) = (v * \mathrm{pure}(\mathrm{lk}_J(v))) \cup \mathrm{pure}(K(\{a, b, c, d, x, y\})),$$

by Mayer–Vietoris we have that $\tilde{H}_1(\mathrm{pure}(J)) = \tilde{H}_0(B) \neq 0$. This contradicts the fact that J is shellable and two-dimensional. Hence we may conclude that A is connected and one-dimensional.

Since A and $\mathrm{pure}(\mathrm{lk}_K(v))$ are connected we can get a shelling of $\mathrm{pure}(\mathrm{lk}_K(v))$ by first listing the edges of A and then listing the remaining edges of $\mathrm{pure}(\mathrm{lk}_K(v))$ so that each edge is connected to the previous ones. We claim that we can obtain a shelling of $\mathrm{pure}(K)$ by first listing the facets of K_1 in the order given by any shelling of K_1 and then listing the facets of $K_2 = v * \mathrm{pure}(\mathrm{lk}_K(v))$ in the order indicated by a shelling of $\mathrm{pure}(\mathrm{lk}_K(v))$ in which the edges of A come first. Let $F_1, F_2, \ldots, F_t$ be the resulting ordered list of facets of $\mathrm{pure}(K)$.

If $F_i \in K_1$, then it is clear that $(\bar{F}_1 \cup \bar{F}_2 \cup \cdots \cup \bar{F}_{i-1}) \cap \bar{F}_i$ is pure one-dimensional. If $F_i = \{v, a, b\}$ where $\{a, b\} \in A$, then it is easy to see that $\{a, b\}$ is a facet of $(\bar{F}_1 \cup \bar{F}_2 \cup \cdots \cup \bar{F}_{i-1}) \cap \bar{F}_i$ and that, for all but the first F_i in $v * A$, $\{v, a\}$ or $\{v, b\}$ is also a facet of $(\bar{F}_1 \cup \bar{F}_2 \cup \cdots \cup \bar{F}_{i-1}) \cap \bar{F}_i$. Hence $(\bar{F}_1 \cup \bar{F}_2 \cup \cdots \cup \bar{F}_{i-1}) \cap \bar{F}_i$ is pure one-dimensional for all $F_i \in v * A$.

Now suppose $F_i = \{v, a, b\}$ where $\{a, b\} \notin A$. Clearly, either $\{v, a\}$ or $\{v, b\}$ is a facet of $(\bar{F}_1 \cup \bar{F}_2 \cup \cdots \cup \bar{F}_{i-1}) \cap \bar{F}_i$; assume without loss of generality that $\{v, a\}$ is the facet. We claim that $\{b\}$ is not a facet of $(\bar{F}_1 \cup \bar{F}_2 \cup \cdots \cup \bar{F}_{i-1}) \cap \bar{F}_i$. Suppose it is. Then b is in some facet F_j where $j < i$ and $v, a \notin F_j$. Let $F_j = \{b, c, d\}$. Consider the induced subcomplex $L = K(\{v, a, b, c, d\})$. Since L is shellable and contains facets $\{v, a, b\}$ and $\{b, c, d\}$, one of the following sets must be a facet of L and therefore also of $\mathrm{pure}(K)$: $\{a, b, c\}$, $\{a, b, d\}$, $\{v, b, c\}$, $\{v, b, d\}$. The first two are impossible since $\{a, b\} \notin A$. If $\{v, b, c\}$ were a facet then it would have to precede $\{v, a, b\}$ in the shelling because $\{b, c\}$ would be in A. It would then follow that $\{v, b\} \in (\bar{F}_1 \cup \bar{F}_2 \cup \cdots \cup \bar{F}_{i-1}) \cap \bar{F}_i$ which contradicts the assumption that $\{b\}$ is a facet. Similarly $\{v, b, d\}$ cannot be a facet of L. Hence $\{b\}$ is not a facet of $(\bar{F}_1 \cup \bar{F}_2 \cup \cdots \cup \bar{F}_{i-1}) \cap \bar{F}_i$ which implies that $(\bar{F}_1 \cup \bar{F}_2 \cup \cdots \cup \bar{F}_{i-1}) \cap \bar{F}_i$ is pure one-dimensional. Therefore $F_1, F_2, \ldots, F_n$ is indeed a shelling of $\mathrm{pure}(K)$. □

Corollary 4. *Every nonshellable two-dimensional simplicial complex has a nonshellable induced subcomplex with n vertices where* $4 \leq n \leq 7$.

Lemma 5. *For each* $n = 5, 6, 7$*, there is a two-dimensional obstruction with n vertices.*

Proof. Let M_n be the simplicial complex on vertex set $\{1, \ldots, n\}$ with facets $\{1, 2, 3\}$, $\{2, 3, 4\}$, ..., $\{n-2, n-1, n\}$, $\{n-1, n, 1\}$, $\{n, 1, 2\}$. For $n \geq 5$, M_n triangulates a cylinder when n is even and a Möbius strip when n is odd. Hence M_n is not shellable. We leave it to the reader to check that every induced proper subcomplex of M_n is shellable when $n \leq 7$. □

The obstructions given in the proof of Theorem 5 are two-dimensional pseudomanifolds *with* boundary. We show next that a general n-dimensional pseudomanifold *without* boundary cannot be an obstruction.

Lemma 6. *Let K be a simplicial complex on vertex set V. Suppose that $v \in V$ is such that $K(V \backslash \{v\})$ and $\mathrm{lk}_K(v)$ are shellable and no facet of $\mathrm{lk}_K(v)$ is a facet of $K(V \backslash \{v\})$. Then K is shellable.*

Proof. First list the facets of $K(V \backslash \{v\})$ in any shelling order and then list the facets of $v * \mathrm{lk}_K(v)$ in the order indicated by the shelling of $\mathrm{lk}_K(v)$. It is easy to see that this is a shelling of K. □

Theorem 7. *Let K be a simplicial complex for which every nonfacet face is contained in at least two facets. Then K is not an obstruction. Consequently, there are no obstructions that are pseudomanifolds (without boundary) or triangulations of manifolds (without boundary).*

Proof. The proof is by induction on $\dim K$. When $\dim K = 1$ the result follows immediately from Proposition 2. Suppose $\dim K > 1$ and every proper induced subcomplex of K is shellable. We will show that K must also be shellable. Let V be the vertex set of K and choose any $v \in V$. Then $K(V \backslash \{v\})$ is shellable. It follows from the fact that every nonfacet face is in at least two facets of K that no facet of $\mathrm{lk}_K(v)$ is a facet of $K(V \backslash \{v\})$.

To apply Lemma 6 we need only show that $\mathrm{lk}_K(v)$ is shellable. Note that the property that every nonfacet face is contained in at least two facets, is inherited by $\mathrm{lk}_K(v)$. Hence if $\mathrm{lk}_K(v)$ is not shellable, then by induction it contains an obstruction $\mathrm{lk}_K(v)(U)$, where $U \subsetneq V \backslash \{v\}$. We have that $K(U \cup \{v\})$ is shellable since it is a proper induced subcomplex of K. Since $\mathrm{lk}_K(v)(U) = \mathrm{lk}_{K(U \cup \{v\})}(v)$ and any link in a shellable complex is shellable, we have that $\mathrm{lk}_K(v)(U)$ is also shellable, contradicting the fact that $\mathrm{lk}_K(v)(U)$ is an obstruction. Therefore $\mathrm{lk}_K(v)$ is shellable and, by Lemma 6, K is shellable. So K is not an obstruction. □

A “pure” version of Lemma 6 is used implicitly in Provan and Billera’s proof of the shellability of matroid complexes [PB]. A *matroid complex* is a simplicial complex for which all induced subcomplexes are pure. Lemma 6 can, in fact, be used to prove the following stronger result.

Proposition 8. *If every proper induced subcomplex of a simplicial complex K is pure then K is shellable.*

Proof. The proof is by induction on the size of the vertex set V. Suppose that K is not a simplex. Let F be any d-dimensional facet of K where $d = \dim K$. Choose $v \in V \backslash F$. Clearly, $K(V \backslash \{v\})$ is pure d-dimensional since it contains F. It follows that no facet of $\mathrm{lk}_K(v)$ is a facet of $K(V \backslash \{v\})$.

By induction, $K(V \backslash \{v\})$ is shellable. To apply Lemma 6, we need only show that $\mathrm{lk}_K(v)$ is also shellable. For any $U \subsetneq V \backslash \{v\}$, $K(U \cup \{v\})$ is pure. Since $\mathrm{lk}_K(v)(U) = \mathrm{lk}_{K(U \cup \{v\})}(v)$ and any link in a pure complex is pure, we have that $\mathrm{lk}_K(v)(U)$ is pure. Hence every proper induced subcomplex of $\mathrm{lk}_K(v)$ is pure. It follows by induction that $\mathrm{lk}_K(v)$ is shellable. □

Remark. Provan and Billera [PB] prove that matroid complexes are shellable by showing that they are vertex decomposable. The proof of Proposition 8 given here is a slight modification of the Provan–Billera proof and also yields the conclusion that K is vertex decomposable, but in the nonpure sense described in [BW3].

Define an *obstruction to purity* to be a nonpure simplicial complex for which all proper induced subcomplexes are pure. Proposition 8 extends the Provan–Billera result from matroid complexes to obstructions to purity. It turns out that there are really very few obstructions to purity in each dimension.

Proposition 9. *For each $d \geq 1$, every d-dimensional obstruction to purity has exactly $d+2$ vertices. Moreover there exists a d-dimensional obstruction to purity for each d.*

Proof. Let K be a d-dimensional simplicial complex with vertex set V. Suppose $|V| > d+2$ and all proper induced subcomplexes of K are pure. We will show that K is also pure.

Let $v \in V \backslash F$, where F is any d-dimensional face of K. It follows that $K(V \backslash \{v\})$ is pure d-dimensional. To show that K is pure we need only show that $\mathrm{lk}_K(v)$ is pure $(d-1)$-dimensional. Let G be a face of $\mathrm{lk}_K(v)$. Then since $K(V \backslash \{v\})$ is pure d-dimensional, G is contained in some d-dimensional facet H of $K(V \backslash \{v\})$. Since $H \cup \{v\} \neq V$, it follows that $K(H \cup \{v\})$ is pure d-dimensional. Since $G \cup \{v\} \in K(H \cup \{v\})$, it follows that $G \cup \{v\}$ is contained in a facet of dimension d in $K(H \cup \{v\})$. This implies that G is contained in a $(d-1)$-dimensional face of $\mathrm{lk}_K(v)$, which means that $\mathrm{lk}_K(v)$ is pure and $(d-1)$-dimensional.

An example of a d-dimensional obstruction to purity is given by the $(d-1)$-skeleton of the simplex on vertex set $\{1, 2, \ldots, d+2\}$ together with the d-dimensional face $\{1, 2, \ldots, d+1\}$. □

Lemma 6 also yields a simple proof of the shellability of interval orders which we give below. Recall that a *bounded poset* is a poset that has a minimum element $\hat{0}$ and a maximum element $\hat{1}$. If P is a bounded poset, then $\bar{P}$ denotes the induced subposet $P \backslash \{\hat{0}, \hat{1}\}$. The *length* of bounded poset P is the length of the longest chain from $\hat{0}$ to $\hat{1}$. For any $a \leq b$ in P, the open interval $\{x \in P \mid a < x < b\}$ is denoted by (a, b) and the closed interval $\{x \in P \mid a \leq x \leq b\}$ is denoted by $[a, b]$. The *order complex* of P is the simplicial complex of chains of P and is denoted by $\Delta(P)$.

Proposition 10 [BM]. *Every interval order is shellable.*

Proof. Let P be an interval order. By the well-known characterization of interval orders, P does not contain Q (the poset with two disjoint two-element chains) as an induced subposet. We may assume without loss of generality that P is bounded and that P has more than one atom.

The fact that P does not contain Q enables us to choose an atom a such that each of the covers of a is greater than some other atom. Since $\mathrm{lk}_{\Delta(\bar{P})}(a) = \Delta((a, \hat{1}))$, this implies that no facet of $\mathrm{lk}_{\Delta(\bar{P})}(a)$ is a facet of $\Delta(\bar{P} \backslash \{a\})$. Also the interval $(a, \hat{1})$ and the induced subposet $\bar{P} \backslash \{a\}$ both inherit the property of not containing Q as an induced subposet. Hence by induction they are shellable. We conclude that $\Delta(\bar{P})$ and hence P are shellable by Lemma 6. □

Remark. Björner [B] has independently used the same idea to prove more generally that all interval orders are vertex decomposable.

Another proof that interval orders are shellable can be obtained using the technique of lexicographical shellability [BW2].

Theorem 11. *Every bounded interval order is CL-shellable.*

Proof. Let P be a bounded interval order. Partially order the atoms of P by letting $a \prec b$ if a has a cover that is not greater than b. Antisymmetry and transitivity follow readily from the forbidden induced subposet characterization of interval order. It is straightforward to verify that any linear extension of $\preceq$ is a recursive atom ordering of P by induction on $|P|$. Therefore P is CL-shellable. □

For any bounded poset P of length at least 2, let $\beta_i(P)$ be the ith reduced Betti number of $\Delta(\bar{P})$. If the length of P is 1 then let $\beta_i(P) = 0$ for all i except for $i = -1$ in which case $\beta_i(P) = 1$.

We refer to an atom of a bounded interval order P as a *minimal atom* if it is minimal in the partial order on atoms given in the proof of Theorem 11. Such atoms can be characterized as those atoms that are smaller than every element of P that is neither an atom nor $\hat{0}$.

Corollary 12. *Let P be a bounded interval order of length ≥ 2, let A be its set of atoms, and let a_0 be a minimal atom. Then, for $i \geq 0$,*

$$\beta_i(P) = \sum_{a \in A \setminus \{a_0\}} \beta_{i-1}([a, \hat{1}]).$$

Proof. By Theorem 5.9 of [BW2], $\beta_i(P)$ is the number of falling maximal chains of length $i+2$ with respect to the CL-labeling induced by the recursive atom ordering given in the proof of Theorem 11. So we need to describe these falling chains. Each falling chain from $\hat{0}$ to $\hat{1}$ of length $i+2$ is of the form $\{\hat{0}\} \cup c$ where c is a falling chain of length $i+1$ from a to $\hat{1}$ for some atom a. We need to determine which atoms a and falling chains c from a to $\hat{1}$ are such that $\{\hat{0}\} \cup c$ is falling. The proof of Theorem 3.2 of [BW1] produces a CL-labeling from a recursive atom ordering (although it is done in the pure case in [BW1], it easily carries over to general case, see [BW2]). A maximal chain has a descent on the subchain $\hat{0} \to a \to b$ if and only if b is greater than some atom that precedes a in the recursive atom ordering. This happens for every maximal chain through $a \neq a_0$ and for no maximal chain through a_0. Hence the maximal chains of the form $\{a\} \cup c$, where $a \neq a_0$ and c is a falling chain of $[a, \hat{1}]$, are the falling chains of P. □

The problem of studying obstructions could conceivably be made easier by considering special classes of simplicial complexes that are closed under taking induced subcomplexes. A natural class, suggested by Björner [B], which generalizes that of order complexes is the class of flag complexes. A *flag complex* is a simplicial complex for which every minimal nonface has exactly two elements. See [S2] for further information on flag complexes. One might ask whether the pair of disjoint edges is the only obstruction for flag complexes. It turns out that this is not the case. The obstruction M_7

given in the proof of Theorem 5 is a flag complex. However, obstructions M_5 and M_6 are not flag complexes. Also the obstructions given in the proof of Proposition 1 are not flag complexes. We leave open the question of whether there is a finite number of obstructions that are flag complexes.

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