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Coherence and Enumeration of Tilings of 3-Zonotopes*

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Abstract. A rhombohedral tiling of a d-zonotope Z is said to be *coherent* if it may be obtained by projecting the "top faces" of some $(d+1)$-zonotope onto Z. We classify those 3-zonotopes with five or fewer distinct zones which have all rhombohedral tilings coherent, and give concise enumeration formulas for the tilings of the zonotopes in each class. This enumeration relies in equal parts on the theory of *oriented matroids* and the theory of *discriminantal arrangements* of hyperplanes.

1. Introduction

A classic problem in enumerative combinatorics is MacMahon's plane partition problem. An r by s *plane partition* is an r by s array of nonnegative integers which is weakly decreasing along rows and down columns. The problem is to determine the number $N(r, s, t)$ of r by s plane partitions with integer values bounded by t. The answer, along with a q-analogue, was originally given by MacMahon [Ma] in 1899:

$$N(r, s, t) = \frac{H(r+s+t)H(r)H(s)H(t)}{H(r+s)H(r+t)H(s+t)},$$

where $H(n) = (n-1)!\,(n-2)!\cdots 2!$ is the hyperfactorial function. A natural generalization of this problem is to attempt to enumerate all r by s by t *solid partitions* with integer values bounded by u. To date, all attempts to generalize MacMahon's result to higher-dimensional partition questions have failed.

However, there is a natural bijection between MacMahon's plane partitions and

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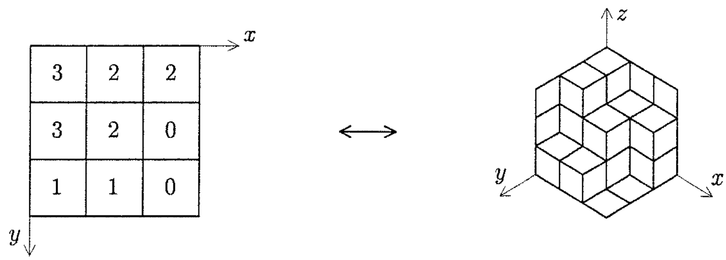

Fig. 1. A stack of cubes induces a rhombic tiling of a hexagon.

rhombic tilings of a certain class of *zonotopes*, which suggests an avenue for attacking questions which concern higher-dimensional partitions. There is an immediate bijection from r by s plane partitions with integer values bounded by t to stacks of unit cubes in an r by s by t box. Viewing such a stack of cubes from a point in general position, a rhombic tiling of a centrally symmetric hexagon with side lengths r, s, t (see Fig. 1) is seen. Such a hexagon is an example of a rank 2 zonotope.

It is this connection between plane partitions and rhombic tilings of zonotopes which partially motivates our work. A *d-zonotope* may be thought of either as the image of some projection of the n-cube into $\mathbb{R}^d$, with $n \geq d$, or as the Minkowski sum of a collection V of n vectors in $\mathbb{R}^d$ containing some basis for $\mathbb{R}^d$. A *rhombohedral tiling* of a d-zonotope $Z = Z(V)$ is a decomposition of Z into a nondisjoint union of cells, each of which is a translation of the Minkowski sum of some independent d-subset of V. Note that this definition generalizes tilings of zonotopes which arise in connection with partition problems.

A rhombohedral tiling of a d-zonotope Z is said to be *coherent* if it can be obtained by projecting the "top faces" of some $(d+1)$-zonotope onto Z. The primary goal of this paper is to explore and compare the enumeration and structure of

- the set of all tilings of Z, and
- the subset of all **coherent** tilings of Z.

Another area of study which both motivates and facilitates the study of higher-dimensional zonotopal tilings is that of *discriminantal arrangements* of hyperplanes. Discriminantal arrangements were first defined by Manin and Schechtman [MS] in 1986 as a generalization of the *braid arrangement* of type A_{n-1}. This definition was itself broadened by Bayer in 1993 [Bay]. For a particular zonotope $Z = Z(V)$, it turns out that there is a bijection between the collection of coherent rhombohedral tilings of Z and *chambers* in the corresponding discriminantal arrangement $\mathcal{D}(V)$ (see [BS]). A more comprehensive discussion of these topics and their relation to the study of zonotopes appears in the introduction of the paper by Edelman and Reiner [ER].

In this paper, we completely classify those 3-zonotopes $Z(V)$ for which $\overline{V}$, the maximal subset of pairwise distinct vectors in V, has cardinality at most five (multiplicities may occur), and which have the property that all rhombohedral tilings of Z are coherent. We further provide enumeration formulas for several infinite classes of such 3-zonotopes. The approach for this classification follows the work done by Edelman and Reiner [ER]

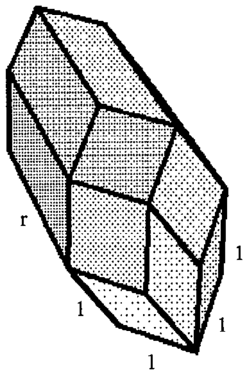

Fig. 2. A zonotope with $2(r+4)!/4!$ distinct rhombohedral tilings (Theorem 3.7).

in their classification of 2-zonotopes with this property. Given a zonotope Z, we:

(a) Enumerate all tilings of Z.
(b) Enumerate all **coherent** tilings of Z.
(c) Compare.

To count **all** tilings of Z requires combinatorial arguments together with an oriented matroid theorem of Las Vergnas. Enumeration of the coherent tilings of Z is accomplished by means of the aforementioned correspondence between coherent tilings of $Z = Z(V)$ and chambers in the discriminantal arrangement $\mathcal{D}(V)$. By a result of Zaslavsky [Za], this enumeration may be accomplished by finding the roots of the *characteristic polynomial* $\chi(\mathcal{D}(V), t)$ for $\mathcal{D}(V)$. Since it happens that $\mathcal{D}(V)$ is a *free* arrangement (see [Te]) for all zonotopes $Z(V)$ under consideration, these roots are simply the *exponents* of $\mathcal{D}(V)$, by a result of Terao [OT].

The illustrations in Figs. 2–4 summarize the main results of the paper, showing representatives from three of the four oriented matroid classes of 3-zonotopes under consideration, and giving the appropriate enumeration formula for the tiling count in each case.

2. Background

Let V be a multiset of vectors which contains a basis for $\mathbb{R}^d$, and let $\overline{V}$ be the maximal subset of pairwise distinct vectors in V. Then the d-zonotope $Z = Z(V)$ is equal to the Minkowski sum of the vectors in V. The set V is called the *generating set* of $Z = Z(V)$, and we say that V *generates* Z. Note that every face of a zonotope is again a zonotope.

Our main objects of study are *rhombohedral tilings* of zonotopes (Fig. 5). Given a zonotope $Z = Z(V)$, a *subzonotope* of Z is any zonotope $Z' = Z(V')$, where V' is a subset of V. A *tiling* T of a d-zonotope Z is the decomposition of Z into a union of d-subzonotopes, called the *tiles* of T, such that any two tiles t_1, t_2 intersect in a proper

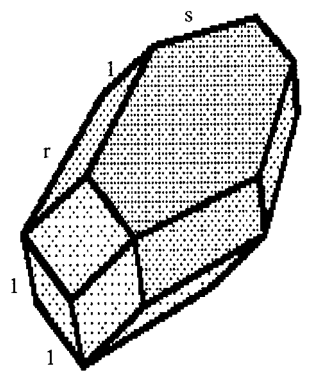

Fig. 3. A zonotope with $2(r+s+1)!(r+s+2)!/(s+2)!(r+2)!$ distinct rhombohedral tilings (Theorem 3.8).

face of each. A tiling T is a *rhombohedral tiling* if each tile t is generated by a subset of V forming a basis of $\mathbb{R}^d$.

The principal tools we use to study tilings are *arrangements of hyperplanes* (or simply *arrangements*) and *oriented matroids*. A *d-arrangement* $\mathcal{A}$ is a finite collection of codimension-one linear subspaces of $\mathbb{R}^d$ (see [OT]). We do not rule out the possibility that the hyperplanes in an arrangement $\mathcal{A}$ might appear with multiplicity, and the reader should be aware that such collections are more commonly referred to as *multiarrangements*. The hyperplanes in $\mathcal{A}$ intersect in some linear subspace S of rank $0 \le s \le d-1$. Define the *rank* of a d-arrangement $\mathcal{A}$ to be $d-s$. If $s=0$, then $\mathcal{A}$ is an *essential* arrangement. By taking normals, there is an obvious bijection between arrangements $\mathcal{A}(V)$ and vector sets V, and so consequently between arrangements $\mathcal{A}(V)$ and zonotopes $Z(V)$.

We do not introduce oriented matroids formally here, but only give an abbreviated introduction to those ideas which will be necessary in what follows. The standard refer-

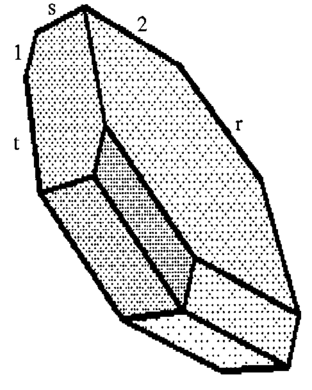

Fig. 4. A zonotope with $2(r+s+t)!\,(r+s+t+1)!/(r+1)!(s+t+1)!$ distinct rhombohedral tilings (Theorem 3.9).

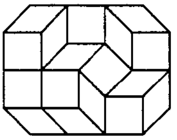

Fig. 5. A rhombohedral tiling of a 2-zonotope.

ence for oriented matroids is the book by Björner et al. [BLS$^+$]. A discussion of oriented matroids which is specific to polytopes and zonotopes appears in Chapters 6 and 7 of Ziegler's book [Zi].

For a vector collection V in $\mathbb{R}^d$ of cardinality n, let $v_1, v_2, \ldots, v_n$ be an arbitrary ordering of the vectors in V. The arrangement $\mathcal{A}(V)$ partitions $\mathbb{R}^d$ into a disjoint union of cones, where each k-cone is determined by some subarrangement of $\mathcal{A}$ with rank $d-k$. To each of these cones is associated an ordered n-tuple in $\{0, +, -\}^n$. Specifically, the n-tuple X corresponding to the cone C is defined by $X_i = \operatorname{sign}(c \cdot v_i)$, where c is any vector in C. The collection of all such n-tuples is the set of *covectors* $\mathcal{L}(V)$ which determines the oriented matroid $\mathcal{M}(V)$ associated with V.

With the notion of covectors for $\mathcal{M}(V)$ in place, we can now explicitly define a correspondence between cones in the decomposition of $\mathbb{R}^d$ induced by $\mathcal{A}(V)$ and the faces of $Z(V)$, one which exhibits the polar duality of $\mathcal{A}(V)$ and $Z(V)$. Specifically, suppose C is a cone induced by $\mathcal{A}$ with covector X. Define

$$X^- = \{i \mid X_i = -\}, \qquad X^0 = \{i \mid X_i = 0\}, \qquad X^+ = \{i \mid X_i = +\}.$$

Then the face of Z corresponding to C will be the Minkowski sum of those vectors $v_i \in V$ with $i \in X^0$, translated by $\sum_{i \in X^+} v_i - \sum_{i \in X^-} v_i$. This construction also demonstrates how to determine the covectors of $\mathcal{M}(V)$ directly from $Z(V)$. In particular, the one-dimensional rays induced by $\mathcal{A}(V)$ and the maximal dimension faces of $Z(V)$ each correspond to the *cocircuits* of $\mathcal{M}(V)$. We denote the collection of cocircuits of an oriented matroid by $\mathcal{C}^*$.

We next consider zonotopal tilings. Suppose $Z = Z(V)$ is a d-zonotope under a fixed choice of coordinates, so that one may write $v_i = (x_{1,i}, x_{2,i}, \ldots, x_{d,i})$ for each $v_i \in V \subseteq \mathbb{R}^d$. It is possible to add a $(d+1)$st coordinate $l_{v_i} = x_{d+1,i}$ to each $v_i \in V$ and add the basis vector $\mathbf{e}_{d+1}$ to V to obtain the generating set $\hat{V}$ for a $(d+1)$-zonotope $\hat{Z} = Z(\hat{V})$. Informally, the generating vectors V of $Z(V)$ are "lifted" into $\mathbb{R}^{d+1}$, and the standard basis vector $\mathbf{e}_{d+1}$ is added.

Let $\mathcal{F}$ denote the collection of *upper facets* of $\hat{Z}$, those rank d faces corresponding to cocircuits with value $+$ or 0 on $\mathbf{e}_{d+1}$, or, informally, the rank d faces which are visible from the point $k\mathbf{e}_{d+1}$ with k large. Let π_{d+1} denote the projection of $\hat{Z}$ along the basis vector $\mathbf{e}_{d+1}$. Then the collection $\{\pi_{d+1}(F) \mid F \in \mathcal{F}\}$ constitutes a tiling of Z.

Definition 2.1. If a tiling T of a d-zonotope Z can be obtained in the above manner for some choice of $\{l_v\}_{v \in V}$, then T is *coherent* (see Fig. 6). Otherwise, T is *incoherent*.

Similarly, if Z is such that T is coherent for all tilings T, then we say that Z itself

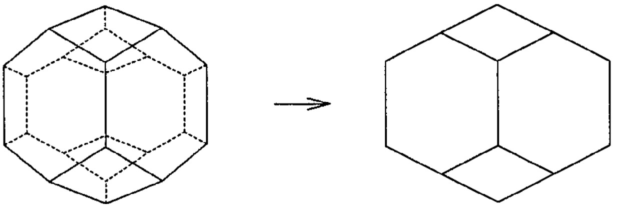

Fig. 6. A coherent tiling of a hexagon is obtained by "looking at" a 3-zonotope.

is *coherent*, otherwise *incoherent*. If T is a coherent tiling of a d-zonotope $Z = Z(V)$ obtained from $\hat{Z} = Z(\hat{V})$, then the oriented matroid $\hat{\mathcal{M}} = \mathcal{M}(\hat{V})$ is a *single-element lifting* of $\mathcal{M}(V)$ and the vector $l = (l_{v_1}, l_{v_2}, \ldots, l_{v_n})$ is called a *lifting vector*.

The principal tool for studying coherent rhombohedral tilings of a zonotope $Z(V)$ is the *discriminantal arrangement* $\mathcal{D}(V)$ (see [Bay]). The n-arrangement $\mathcal{D}(V)$ is defined by the minimally dependent sets of V as follows. The hyperplane $(a_1, a_2, \ldots, a_n)^{\perp}$ is in $\mathcal{D}(V)$ if and only if the set $V' = \{v_i \mid a_i \neq 0\}$ satisfies

$$\sum_{v_i \in V'} a_i v_i = 0$$

and V'' is independent for all proper subsets V'' of V':

Billera and Sturmfels showed [BS]

Theorem 2.2. *Let V be a vector configuration. Up to a choice of order on the parallel vectors of V, the set of coherent rhombohedral tilings of $Z(V)$ is in bijective correspondence with the set of* chambers, *or open cones of maximal dimension, in the arrangement $\mathcal{D}(V)$ (Fig. 7).*

It is clear that each chamber of $\mathcal{D}(V)$ corresponds to an equivalence class of lifting vectors. However, Z may also have an incoherent tiling T, in which case no such lifting

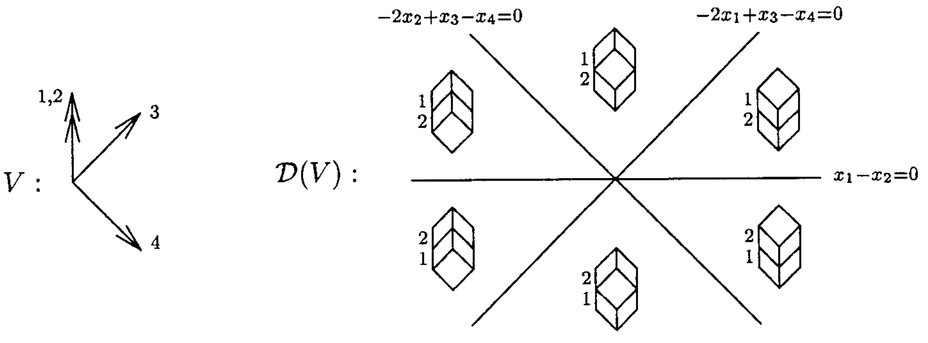

Fig. 7. For a given vector configuration V, the chambers of $\mathcal{D}(V)$ correspond to the distinct coherent rhombohedral tilings of $Z(V)$.

vector l exists (the hexagonal tiling on p. 120 is incoherent, for example). Bohne and Dress [BD, see also [RZ]] showed by passing to pseudosphere arrangements (see [BLS$^+$]) that T nevertheless corresponds to a single-element lifting $\hat{\mathcal{M}}$ of the oriented matroid $\mathcal{M} = \mathcal{M}(V)$:

Theorem 2.3 (The Bohne–Dress Theorem). *Let $Z = Z(V)$ be a zonotope. There is a bijection between the zonotopal tilings of Z and single-element liftings of $\mathcal{M}(V)$.*

By means of oriented matroid duality, we may enumerate the single-element liftings of $\mathcal{M}(V)$ by applying Las Vergnas' theorem characterizing *single-element extensions* of an oriented matroid to the dual oriented matroid $\mathcal{M}(V^*)$ [LV]. For an oriented matroid $\mathcal{M}(V)$ given by a vector collection V, the extension of V by a single element v_{n+1} will assign the symbol $\operatorname{sign}(C \cdot v_{n+1})$ to each cocircuit X of $\mathcal{M}(V)$, where C is the ray corresponding to X. The cocircuit signature $\sigma_{v_{n+1}}\colon \mathcal{C}^* \to \{+, -, 0\}$ defined in this manner is one example of a *localization*. However, many oriented matroids possess single-element extensions which are not *realizable*, and consequently have localizations which do not arise in this manner. Las Vergnas' theorem states that a cocircuit signature $\sigma\colon \mathcal{C}^* \to \{+, -, 0\}$ is a localization for the oriented matroid $\mathcal{M}$ if and only if the restriction $\sigma|_R$ is a localization for every *rank* 2 *contraction* R of $\mathcal{M}$. A rank 2 contraction of a realizable oriented matroid may be thought of as a collection of cocircuits whose corresponding rays are all contained in a rank 2 subspace arising as the intersection of elements of $\mathcal{A}(V)$.

Theorem 2.4. *Let $\mathcal{M}$ be an oriented matroid, and let*

$$\sigma\colon \mathcal{C}^* \to \{+, -, 0\}$$

be a cocircuit signature, satisfying $\sigma(-Y) = -\sigma(Y)$ for all $Y \in \mathcal{C}^*$. Then the following statements are equivalent:

(1) *σ is a localization*: *there exists a single-element extension $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ such that*

$$\{(Y, \sigma(Y)) | Y \in \mathcal{C}^*\} \subseteq \tilde{\mathcal{C}}^*.$$

(2) *σ defines a single-element extension on every contraction of $\mathcal{M}$ of rank* 2. *That is, the signature on every rank* 2 *contraction is one of the types I, II, and III shown in Fig.* 8.

(3) *The signature σ produces none of the three excluded subconfigurations* (*minors*) *of rank* 2 *on three elements, as given by Fig.* 9.

These theorems are the principal tools used in the classification. For each of the major results, we:

- Provide a description of the vector set V in a certain **normal form**.
- Classify the **circuits** of $\mathcal{M}(V)$, which are both the minimal dependences among elements of V and the cocircuits of $\mathcal{M}(V^*)$.
- Describe the resulting **rank 2 contractions** of $\mathcal{M}(V^*)$.

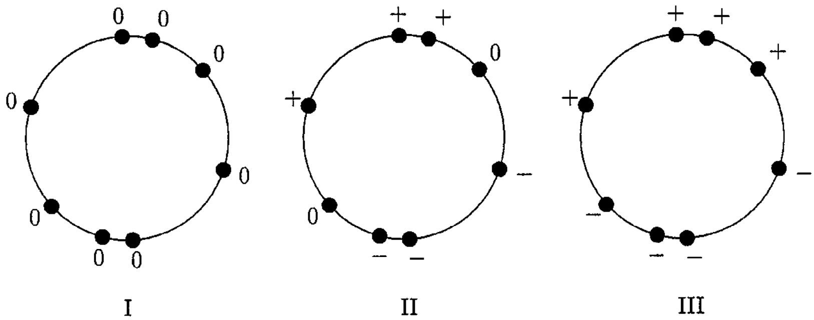

Fig. 8. The three types of allowable cocircuit signature for a rank 2 oriented matroid.

- Use the obstructions given in Theorem 2.4 to determine necessary and sufficient conditions for a cocircuit signature to be a **localization**.
- Express the conditions in terms of **bijections** to tableaux (or other combinatorial objects) for which enumeration formulas are known.
- **Compare** this total to the number of coherent tilings, using Theorem 2.2 and results from the author's thesis [Bai], available online or upon request.
- Give minimal **incoherent zonotopes** for each infinite family.

For the remainder of the paper, the term *tiling* means a rhombohedral tiling. Such tilings correspond to *uniform* localizations σ, those which map $\mathcal{C}^*$ to $\{+, -\}$.

3. Coherent 3-Zonotopes

3.1. *Introduction*

In 1996 Edelman and Reiner [ER] gave a completely combinatorial classification of coherent 2-zonotopes $Z = Z(V)$ in terms of the underlying set $\overline{V} = (v_1, v_2, \ldots, v_m)$ and the corresponding m-tuple $(r_1, r_2, \ldots, r_m)$ of vector multiplicities. The work in this section begins such a classification for 3-zonotopes. Specifically, we provide a completely combinatorial classification of coherent 3-zonotopes $Z = Z(V)$ for those vector sets V with $|\overline{V}| \leq 5$.

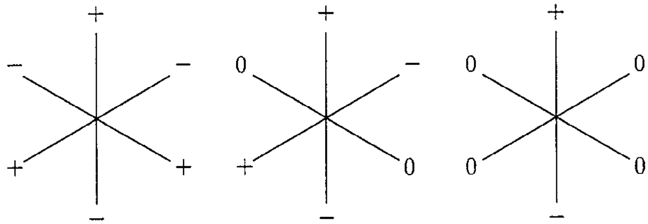

Fig. 9. The three forbidden rank 2 cocircuit signatures.

The oriented matroid equivalence classes for $Z(V)$ with $|\overline{V}| \le 5$ are as follows:

- $|\overline{V}| = 4$, which yields the *MacMahon* 3-zonotopes.
- $|\overline{V}| = 5$ and the elements of $\overline{V}$ are in general position.
- $|\overline{V}| = 5$ and $\overline{V}$ contains exactly one 3-subset of coplanar vectors.
- $|\overline{V}| = 5$ and $\overline{V}$ contains exactly two 3-subsets of coplanar vectors, with exactly one vector common to each 3-subset.

The remaining possibilities, in which four or five vectors in $\overline{V}$ are coplanar, reduce to the rank 2 case. The technique used is quite straightforward; given a d-zonotope Z:

(a) Enumerate all tilings of Z using the Bohne–Dress Theorem (Theorem 2.3) and Las Vergnas' localization theorem (Theorem 2.4).
(b) Enumerate the **coherent** tilings of Z using the result of Billera and Sturmfels (Theorem 2.2), along with techniques for counting chambers in hyperplane arrangements.
(c) Compare.

Although this is the same technique employed by Edelman and Reiner, they had the advantage that the tiling counts for step (a) were already extant in the literature. This is the first time that the computational technique in step (a) has been explained, together with sample computations. It is also, so far as we know, the first time the Las Vergnas result has been used to solve such a problem.

Enumerating the **coherent** tilings of a d-zonotope $Z = Z(V)$ is relatively straightforward. By Theorem 2.2, the set of all coherent tilings of $Z(V)$ is in bijective correspondence with the chambers of $\mathcal{D}(V)$. In all cases considered below, $\mathcal{D}(V)$ is *free* with *exponents* $b_1, b_2, \ldots, b_m$. Terao showed [Te] that these exponents are the roots of the *characteristic polynomial* $\chi(\mathcal{D}(V), t)$ of $\mathcal{D}(V)$. Zaslavsky [Za] showed that the number of chambers in an arrangement $\mathcal{A}$ is computed by $|\chi(\mathcal{A}, -1)|$. Thus we have

Theorem 3.1. *If $\mathcal{D}(V)$ is a free arrangement, then the number of coherent tilings of $Z(V)$ is counted by*

$$\prod (1 + b_i),$$

where $\{b_1, b_2, \ldots, b_m\}$ are the exponents of $\mathcal{D}(V)$.

In the cases presented below, $\mathcal{D}(V)$ lies in one of two infinite classes of free arrangements. One of these families was shown to be free by Athanasiadis [At]. The proof that the other family is free, together with an explanation linking Athanasiadis' work to the following results, may be found in [Bai].

3.2. *Coherent MacMahon Zonotopes*

A d-zonotope $Z(V)$ is a *MacMahon* zonotope if $\overline{V}$ consists of $d + 1$ distinct vectors in general position. It is clear that $\overline{V}$ is projectively equivalent to the *frame* in $\mathbb{R}^d$, namely the standard basis vectors together with the vector $(1, 1, \ldots, 1)$. Thus any MacMahon d-zonotope Z is uniquely determined by the multiplicities of its generating vectors, and it is reasonable to discuss "the" $\{r_1, r_2, \ldots, r_{d+1}\}$ MacMahon d-zonotope, where $\overline{V}$ is the frame.

Theorem 3.2. *The MacMahon $\{r_1, r_2, \ldots, r_{d+1}\}$ d-zonotope Z is coherent if and only if*

- $r_i \geq 2$ *for at most three indices* i, *and*
- $r_i \geq 3$ *for at most two indices.*

Furthermore, the MacMahon $\{r, s, 2, 1, \ldots, 1\}$ d-zonotope has exactly

$$\frac{2(r+s+1)!\,(r+s)!}{(r+1)!\,(s+1)!}$$

tilings, and the MacMahon $\{r, s, 1, \ldots, 1\}$ d-zonotope has exactly $(r+s)!$ tilings.

We begin by showing that the set of **all** tilings of the $\{r_1, r_2, \ldots, r_{d+1}\}$ MacMahon d-zonotope is in bijection with the number of ways of stacking unit $(d+1)$-hypercubes "flush into the corner" of an $r_1 \times r_2 \times \cdots \times r_{d+1}$ hyperbox.

Proposition 3.3. *The collection of tilings of the $\{r_1, r_2, \ldots, r_{d+1}\}$ MacMahon d-zonotope is in bijection with the set*

$$\mathcal{J}\left(\prod_{i=1}^{d+1}[r_i]\right) \times \prod_{i=1}^{d+1}\mathcal{S}_{r_i},$$

where $[r_i]$ denotes the poset chain of length r_i, $\mathcal{J}(P)$ denotes the set of order ideals of the poset P, and $\mathcal{S}_{r_i}$ is the symmetric group on r_i elements.

The product of symmetric groups appears because two tilings t_1, t_2 which "look" the same are considered distinct if one is obtained from the other by reordering parallel zones. We continue to enumerate tilings in this manner for the remainder of the paper.

We present a somewhat detailed proof, in order to provide a template for the more difficult results which follow.

Normal Form. Order the elements of $\overline{V}$ such that $v_1, v_2, \ldots, v_d$ are the standard basis vectors for $\mathbb{R}^d$, and $v_{d+1} = -(1, 1, \ldots, 1)$. Then V may be represented by the $d \times n$ matrix:

$$V = \begin{array}{c} \begin{matrix} \overbrace{}^{r_1} & \overbrace{}^{r_2} & & \overbrace{}^{r_d} & \overbrace{}^{r_{d+1}} \end{matrix} \\ \left(\begin{matrix} 1\ 1\ 1\ \cdots\ 1 & 0\ 0\ 0\ \cdots\ 0 & \cdots\cdots & 0\ 0\ 0\ \cdots\ 0 & -1\ -1\ -1\ \cdots\ -1 \\ 0\ 0\ 0\ \cdots\ 0 & 1\ 1\ 1\ \cdots\ 1 & \cdots\cdots & 0\ 0\ 0\ \cdots\ 0 & -1\ -1\ -1\ \cdots\ -1 \\ \vdots\ \vdots\ \vdots\ \ldots\ \vdots & \vdots\ \vdots\ \vdots\ \ldots\ \vdots & \cdots\cdots & \vdots\ \vdots\ \vdots\ \ldots\ \vdots & \vdots\ \ \vdots\ \ \vdots\ \ldots\ \vdots \\ 0\ 0\ 0\ \cdots\ 0 & 0\ 0\ 0\ \cdots\ 0 & \cdots\cdots & 0\ 0\ 0\ \cdots\ 0 & -1\ -1\ -1\ \cdots\ -1 \\ 0\ 0\ 0\ \cdots\ 0 & 0\ 0\ 0\ \cdots\ 0 & \cdots\cdots & 1\ 1\ 1\ \cdots\ 1 & -1\ -1\ -1\ \cdots\ -1 \end{matrix}\right) \end{array}.$$

Circuits. A minimal linear dependence among vectors in V is either

(a) the difference of two (identical) vectors in the same block, or
(b) the sum of $d+1$ vectors, one from each block.

Whenever vectors in V occur with multiplicity greater than one, dependences of type (a) will occur. These dependences can be easily described in terms of *braid arrangements*. The *braid arrangement* A_{n-1} is a rank $n-1$ arrangement in $\mathbb{R}^n$ defined by the hyperplanes normal to

$$\{\mathbf{e}_i - \mathbf{e}_j \mid 1 \le i < j \le n\},$$

where $\{\mathbf{e}_i\}$ is the collection of standard basis vectors in $\mathbb{R}^n$. Let h_i denote the ith partial sum $\sum_{j=1}^{i} r_j$ (in particular, $h_0 = 0$ and $h_{d+1} = n$). We define $J_i = \mathbf{0}_{h_{i-1}} \times A_{r_i-1} \times \mathbf{0}_{n-h_i}$. Then the discriminantal arrangement $\mathcal{D}(V)$ has as its set of defining vectors the columns of

$$\mathcal{D}(V) = (J_1 \mid J_2 \mid \cdots \mid J_{d+1} \mid \overline{A})$$

where the collection $\overline{A}$ is the set of all possible Cartesian products $\prod_{j=1}^{d+1} \mathbf{e}_{i_j}$, and $\{\mathbf{e}_{i_j}\}$, $1 \le i_j \le r_j$, are the standard basis vectors for $\mathbb{R}^{r_j}$, $1 \le j \le d+1$.

Rank 2 *Contractions*. A *rank* 2 *contraction* of a realizable rank l oriented matroid $\mathcal{M} = \mathcal{M}(V)$ is a rank 2 subspace of the arrangement $\mathcal{A}(V)$ which arises from the intersection of $(l-2)$ independent elements of $\mathcal{A}(V)$ (here we assume $\mathcal{A}(V)$ is essential). For any pair X, Y of cocircuits in $\mathcal{C}^*$, define $O_{X,Y}$ to be the set of indices i such that $X_i = Y_i = 0$. Define $V^*_{X,Y}$ to be the set of vectors $\{v_i \in V^* \mid i \in O_{X,Y}\}$. Then X and Y define a rank 2 contraction R of the oriented matroid $\mathcal{M}(V^*)$ if and only if $\dim(\operatorname{span}(V^*_{X,Y})) = n - d - 2$, where $n = \sum r_i$.

Thus in order to determine the rank 2 contractions, V^* must be computed. We do not compute V^* explicitly here, but only describe for the reader an easy method for obtaining V^* in order to check the description of the rank 2 contractions. Oriented matroid theory states that any $(n-d) \times n$ matrix which has full row rank with rows pairwise orthogonal to the rows of V will serve as V^*. Thus an easy way to obtain an expression for V^* is to rewrite V in the form

$$V = (I_d \mid M).$$

Then V^* may be written

$$V^* = \left(-M^T \mid I_{(n-d)}\right).$$

Finally, we may replace $-M^T$ with M^T, since doing so only *reorients* the corresponding oriented matroid, and does not alter any of the properties of concern to us. When checking rank 2 contractions, however, it is important to reorder the column vectors in V^* to ensure that they correspond to the vector ordering given in the **normal form** for V.

Weconsider only rank 2 contractions which contain three or more cocircuits, since that is the minimum number of cocircuits required for the obstructions in Fig. 9. In the case of $\mathcal{D}(V)$, the only relevant rank 2 contractions are those which are defined by a triple of vectors in some J_i, or by a triple of vectors, two of which are in $\overline{A}$, and the third in some J_i. See Fig. 10.

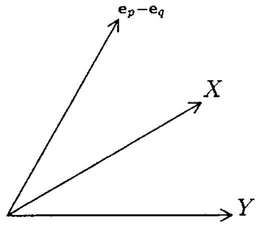

Fig. 10. A rank 2 contraction containing three cocircuits.

Here $X, Y \in \overline{A}$, $\mathbf{e}_p - \mathbf{e}_q \in J_i$ for some $1 \le i \le d+1$, and $(\mathbf{e}_p - \mathbf{e}_q) + Y = X$. For notational convenience, all future rank 2 contractions will be presented as

$$X_1 \vee X_2 \vee \cdots \vee X_k$$

to indicate that $X_1, X_2, \ldots, X_k$ all lie in a common rank 2 contraction R, and that, for each triple $(i-1, i, i+1)$, there exist positive scalars a, b such that $aX_{i-1} + bX_{i+1} = X_i$. We say that a cocircuit signature σ *respects* R if $\sigma|_R$ is a localization on R.

Let $\mathcal{R}_0$ denote the collection of rank 2 contractions arising from triples of cocircuits $X_1 \vee X_2 \vee X_3$, where $X_1, X_2, X_3 \in J_i$ for some i, and let $\mathcal{R}_1$ denote the collection of rank 2 contractions of the kind shown in Fig. 10. It is a routine matter to verify that this catalogues all rank 2 contractions.

Localization. To prove the proposition, we must show that every element in $\mathcal{J}(\prod_{i=1}^{d+1} [r_i]) \times \prod_{i=1}^{d+1} \mathcal{S}_{r_i}$ corresponds to a unique uniform localization on $\mathcal{C}^*$, and that all uniform localizations are obtained in this manner.

Recall that a uniform localization σ is simply a cocircuit signature $\sigma\colon \mathcal{C}^* \to \{+, -\}$ with special properties. Specifically, for each rank 2 contraction R of $\mathcal{M}(V^*)$, σ must assign a signature to the cocircuits in R in a realizable manner. That is, σ must be a signature of the type in Fig. 8(III) or, equivalently, σ must **avoid** the uniform signature in Fig. 9.

If σ respects all rank 2 contractions in $\mathcal{R}_0$, then σ induces an ordering on the coordinates of $\mathbb{R}^{r_i}$ for $1 \le i \le d+1$. This ordering is defined by

$$\sigma(\mathbf{e}_p - \mathbf{e}_q) = + \qquad \text{if and only if} \quad \mathbf{e}_p > \mathbf{e}_q.$$

Since σ respects all $R \in \mathcal{R}_0$, these pairwise order relations may be extended to a linear order on the coordinates $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{r_i}$ (cycles cannot occur). Conversely, it is clear that every such coordinate ordering corresponds to a cocircuit signature σ which respects all $R \in \mathcal{R}_0$. Thus we may fix an ordering on the coordinates of $\mathbb{R}^{r_i}$ for each i and multiply the localization count by $r_1!\,r_2! \cdots r_{d+1}!$. It remains to show that for each coordinate ordering ρ in $\prod \mathcal{S}_{r_i}$, there are $\left|\mathcal{J}(\prod [r_i])\right|$ distinct localizations which induce ρ.

Bijection. Without loss of generality, suppose the order relation on the coordinates of $\mathbb{R}^{r_i}$ is fixed to be $\mathbf{e}_p > \mathbf{e}_q$ if and only if $p < q$. Then each cocircuit $X_i \in J_i$ has $\sigma(X_i) = +$ for all $1 \le i \le d+1$. Consequently, when considering rank 2 contractions in $\mathcal{R}_1$, it follows that

$$\sigma(\mathbf{e}_p - \mathbf{e}_q) = + \qquad \text{implies} \quad \sigma(X) \geq \sigma(Y) \tag{$*$}$$

(in the ordering $+ > 0 > -$), where $\mathbf{e}_p - \mathbf{e}_q$, X, and Y are as in Fig. 10. To see how this yields a bijection with elements of $\mathcal{J}(\prod_{i=1}^{d+1}[r_i])$, we adopt a different notation for the elements of $\overline{A}$. Recall $\overline{A}$ is the collection of all Cartesian products of the form

$$\prod_{j=1}^{d+1} \mathbf{e}_{i_j},$$

where $\mathbf{e}_{i_j}$ is any standard basis vector in $\mathbb{R}^{r_j}$. There is an obvious bijection between elements of $\overline{A}$ and ordered $(d+1)$-tuples $(u_1, u_2, \ldots, u_{d+1})$, where $1 \leq u_i \leq r_i$ denotes the position of a unique nonzero entry among the coordinates in the interval $[h_{i-1}+1, h_i]$. Then the condition $(*)$ is equivalent to the statement:

$$\begin{aligned} &\sigma((u_1, u_2, \ldots, u_{i-1}, u_i', u_{i+1}, \ldots, u_{d+1})) \\ &\qquad \geq \sigma((u_1, u_2, \ldots u_{i-1}, u_i, u_{i+1}, \ldots, u_{d+1})) \qquad \text{if and only if} \quad u_i' \leq u_i. \end{aligned}$$

Taking the set of such relations where $u_i = u_i' + 1$ for all $1 \leq i \leq d+1$, we obtain the cover relations for the lattice $\prod_{i=1}^{d+1}[r_i]$. In particular, those cocircuits X with $\sigma(X) = +$ form an order ideal I in the lattice. Thus for a fixed ordering ρ of the coordinates, each localization σ is determined by an order ideal I of cocircuits in $\prod_{i=1}^{d+1}[r_i]$ satisfying $\sigma(X) = +$ for all $X \in I$. This completes the proof of the proposition. □

Compare. Since the count in Proposition 3.3 holds for MacMahon zonotopes of any size and rank, it is a fairly straightforward matter to determine which MacMahon zonotopes are coherent. Suppose V is an $\{r, s, 2, 1, 1, \ldots, 1\}$ MacMahon d-zonotope, where possibly $r, s \geq 2$. Since the elements of $\overline{V}$ are in general position, it makes no difference which vectors appear with multiplicity. For such a zonotope Z, Proposition 3.3 states that there are

$$|\mathcal{J}([r] \times [s] \times [2]) \times \mathcal{S}_r \times \mathcal{S}_s \times \mathcal{S}_2|$$

total tilings of Z, since the additional singleton zones do not contribute any factors to the count. By MacMahon's original formula (see p. 120), this number is

$$\frac{(r+s+1)!\,(r+s)!}{(r+1)!\,(s+1)!\,r!\,s!} \cdot 2r!\,s! = \frac{2(r+s+1)!\,(r+s)!}{(r+1)!\,(s+1)!}.$$

The discriminantal arrangement of Z is projectively equivalent to one of the arrangements studied by Athanasiadis [At], which interpolate between the cone over the braid arrangement of type A_{r+s-1} and the cone over the *Shi arrangement* of type A_{r+s-1}. Athanasiadis has shown [At, Theorem 4.1] that this class of arrangements is free with exponents

$$\{0, 1, r+1, r+2, \ldots, r+s-1, s+1, s+2, \ldots, r+s\}.$$

By Theorem 3.1, we conclude that all tilings of Z are coherent.

By Lemma 3.4, the MacMahon $\{r, s, 1, \dots, 1\}$ d-zonotope Z' is coherent as well. All tilings of Z' are counted by

$$|\mathcal{J}([r] \times [s])| \cdot r!\, s!,$$

and it is well known that $\mathcal{J}([r] \times [s])$ has cardinality $\binom{r+s}{s}$.

Incoherent Zonotopes. One nice property of **incoherent** zonotopes is that they must always contain some "minimal" incoherent zonotope.

Lemma 3.4 (Bai, Lemma 5.4). *The zonotope $Z = Z(V)$ is coherent if and only if $Z' = Z(V')$ is coherent for every $V' \subseteq V$.*

Lemma 3.4 is true even if Z and Z' are zonotopes of different dimensions. As a consequence, large infinite families of zonotopes can be dismissed as incoherent once some few relatively small obstructions are found.

Lemma 3.5. *Suppose Z is an $\{r_1, r_2, \dots, r_{d+1}\}$ MacMahon d-zonotope. Z is incoherent if $r_i \geq 3$ for three distinct values of i.*

Proof. To show that a localization σ yields an incoherent tiling requires proving that no chamber of $\mathcal{D}(V)$ corresponds to σ. If C is a chamber of $\mathcal{D}(V)$ with $l \in C$, then the signature σ_l induced by l is given by $\sigma_l(X_i) = \operatorname{sign}(l \cdot c_i)$, where c_i is the ith column of $\mathcal{D}(V)$.

If Z is the $\{r_1, r_2, \dots, r_{d+1}\}$ MacMahon d-zonotope with $r_i \geq 3$ for three distinct values of i, then by Proposition 3.3 $\Pi[r_i]$ contains a sublattice isomorphic to $[3] \times [3] \times [3]$. Therefore assume without loss of generality that $\Pi[r_i] = [3] \times [3] \times [3]$ and consider the class of localizations σ which induce the order on coordinates within each $\mathbb{R}^{r_i}$ of $\mathbf{e}_p > \mathbf{e}_q$ if and only if $p < q$. There is a tiling/localization σ corresponding to the order ideal

$$I = \langle (1, 3, 2), (2, 1, 3), (3, 2, 1), (2, 2, 2) \rangle.$$

in $\mathcal{J}([3] \times [3] \times [3])$ (this is the tiling shown in Fig. 1). That is, $\sigma(X) = +$ for all $X \in I$. If σ is coherent, then there is a chamber C in $\mathcal{D}(V)$ such that for every $l = (c_1, c_2, \dots, c_9)$ in C, we may take inner products $l \cdot X$ with $X \in I$ to conclude:

$$\left.\begin{array}{l} c_1 + c_6 + c_8 > 0 \text{ corresponding to } (1, 3, 2) \\ c_2 + c_4 + c_9 > 0 \text{ corresponding to } (2, 1, 3) \\ c_3 + c_5 + c_7 > 0 \text{ corresponding to } (3, 2, 1) \end{array}\right. \quad \Rightarrow \sum c_i > 0.$$

However, since σ may also be defined by the complementary filter

$$I^c = \langle (3, 1, 2), (2, 3, 1), (1, 2, 3) \rangle,$$

a similar set of inequalities implies $\sum c_i < 0$. Thus no chamber of the discriminantal arrangement corresponds to the localization σ, and so σ is an incoherent tiling/localization. $\square$

A similar argument, this time using the order ideal and complementary filter

$$\begin{aligned} I &= \langle (1,1,2,2), (2,2,1,1), (2,1,1,2) \rangle, \\ I^c &= \langle (1,2,1,2), (2,1,2,1), (1,2,2,1) \rangle \end{aligned}$$

gives

Lemma 3.6. *Suppose Z is an $\{r_1, r_2, \dots, r_{d+1}\}$ MacMahon d-zonotope. Z is incoherent if $r_i \geq 2$ for four distinct values of i.*

Proposition 3.3 and Lemmas 3.5 and 3.6 together prove Theorem 3.2. □

3.3. *The Case of $d+2$ Vectors in General Position in $\mathbb{R}^d$*

Let $Z = Z(V)$ be a d-zonotope such that the $d+2$ elements of $\overline{V}$ are in general position. Since the oriented matroid $\mathcal{M}(V^*)$ of the dual vector configuration V^* has rank 2, there is only one oriented matroid equivalence class of such zonotopes. Therefore, any choice of coordinates will give an identical result when enumerating the coherent tilings of Z. We assume that the underlying set $\overline{V}$ for the generating multiset V of Z is the frame together with $(1, a_1, a_2, \dots, a_{d-1})$, where $1 > a_1 > a_2 > \cdots > a_{d-1} \geq -1$, and $a_i \neq 0$ for all i. Up to projective equivalence, any collection of $d+2$ vectors in general position in $\mathbb{R}^d$ must be of this form, since coordinates can be chosen in such a way that $d+1$ of the vectors constitute the frame in $\mathbb{R}^d$, and then the final vector may be scaled to satisfy the given inequalities.

In this section we prove that all such zonotopes are coherent if at most one generating vector has multiplicity $r > 1$, and argue that this is a complete classification of coherent d-zonotopes in this class for $d = 3$.

Theorem 3.7. *Suppose $Z = Z(V)$ is a d-zonotope such that $\overline{V}$ consists of $d+2$ vectors in general position. Then Z is coherent if at most one of the generating vectors appears with multiplicity $r > 1$. If Z satisfies this condition, then the tilings of Z are enumerated by*

$$\frac{2(d+r+1)!}{(d+1)!}$$

Furthermore, if $d = 3$, then this condition is both necessary and sufficient to characterize when Z is coherent.

Proof. Again, since the elements of $\overline{V}$ are in general position, it makes no difference which vector appears with multiplicity. We choose to have $(1, a_1, a_2, \dots, a_{d-1})$ appear with multiplicity.

Normal Form.

$$V = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 & \overbrace{1 & 1 & \cdots & 1}^{r} \\ 0 & 1 & 0 & \cdots & 0 & 1 & a_1 & a_1 & \cdots & a_1 \\ 0 & 0 & 1 & \cdots & 0 & 1 & a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & a_{d-1} & a_{d-1} & \cdots & a_{d-1} \end{pmatrix}.$$

Circuits. The discriminantal arrangement $\mathcal{D}(V)$ is the $(r+d+1)\times(1+r(d+1)+\binom{r}{2})$ matrix

$$\mathcal{D}(V) = (\overline{X} \mid B_0 \mid B_1 \mid B_{a_1} \mid B_{a_2} \mid \cdots \mid B_{a_{d-1}} \mid A'_{r-1})$$

where $\overline{X}$ denotes the minimal dependence among vectors in the frame, B_x is the $(r+d+1)\times r$ matrix whose columns are all possible products $(1-x, a_1-x, a_2-x, \ldots, a_{d-1}-x, x)\times -\mathbf{e}_i$ for $1 \le i \le r$, and A'_{r-1} denotes the Cartesian product of $\mathbf{0} \in \mathbb{R}^{d+1}$ with the braid arrangement A_{r-1}.

Rank 2 *Contractions.* Let X_x denote a cocircuit in B_x for $x \in \{0, 1, a_1, \ldots, a_{d-1}\}$, and, in particular, let $X_{x,j}$ denote the unique cocircuit in B_x with nonzero entry in position $d+j+1$, so that $j \in [1, r]$. Let $\mathbf{e}_p - \mathbf{e}_q$ denote the appropriate cocircuit in A'_{r-1}. For the moment, assume $1 > a_1 > a_2 > \cdots > a_{d-1} > 0$. The following is a complete list of rank 2 contractions:

$$\begin{aligned} \mathcal{R}_0 &= \{(\mathbf{e}_p - \mathbf{e}_m) \vee (\mathbf{e}_p - \mathbf{e}_q) \vee (\mathbf{e}_m - \mathbf{e}_q) \mid p, q, m \in [d+2, d+r+1]\}, \\ \mathcal{R}_1 &= \{X_{1,j} \vee X_{a_1,j} \vee X_{a_2,j} \vee \cdots \vee X_{a_{d-1},j} \vee X_{0,j} \vee \overline{X} \mid 1 \le j \le r\}, \\ \mathcal{R}_2 &= \{(\mathbf{e}_p - \mathbf{e}_q) \vee X_{x,q} \vee X_{x,p} \mid x \in \{0, 1, a_1, \ldots, a_{d-1}\} \\ &\qquad \text{and } p, q \in [d+2, d+r+1]\}. \end{aligned}$$

In the cases where some a_i are negative, a similar collection of rank 2 contractions arises. As before, any localization σ, when restricted to the rank 2 contractions in $\mathcal{R}_0$, induces a linear order on the final r coordinates. Thus we again restrict attention to those localizations which have $\sigma(\mathbf{e}_p - \mathbf{e}_q) = +$ for all cocircuits corresponding to vectors in A'_{r-1}, and multiply the final count by $r!$.

Localization and Bijection. The linear order imposed on the coordinates by the rank 2 contractions in $\mathcal{R}_0$, together with the collection $\mathcal{R}_2$, implies $\sigma(X_{x,i}) \le \sigma(X_{x,j})$ if and only if $i < j$, within each B_x. Furthermore, assume $\sigma(\overline{X}) = +$. This requires doubling the final count.

The rank 2 contractions in $\mathcal{R}_1$ induce an order relation

$$\sigma(X_{1,j}) \le \sigma(X_{a_1,j}) \le \sigma(X_{a_2,j}) \cdots \le \sigma(X_{a_{d-1},j}) \le \sigma(X_{0,j})$$

among cocircuits in distinct blocks with the same nonzero entry in the final r coordinates. Thus all information about σ may be completely specified by a $(d+1)\times r$ tableau L

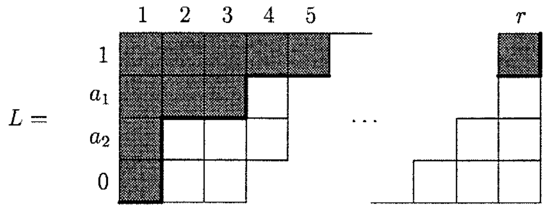

Fig. 11. The tableau of cocircuit signatures for $d = 3$, $1 > a_1 > a_2 > 0$.

with rows indexed by $1, a_1, a_2, \ldots, a_{d-1}, 0$ and columns indexed by $1, 2, \ldots, r$. Entry $L_{x,j}$ is $\sigma(X_{x,j})$. The conditions from the collections $\mathcal{R}_1$ and $\mathcal{R}_2$ imply that the entries of L must weakly increase along rows and down columns (see Fig. 11, in which shaded boxes correspond to cocircuits X with $\sigma(X) = -$). Thus the number of localizations σ is given by the number of such arrays, which is $\binom{r+d+1}{r}$. When this number is multiplied by $2r!$ to allow for the possibility that $\sigma(\overline{X}) = -$ and for other orderings of the final r coordinates, we obtain the tiling count given in the statement of the theorem.

Compare. As for coherent tilings, it happens that in this case $\mathcal{D}(V)$ is *supersolvable* with exponents $\{1, d+1, d+2, \ldots, d+r\}$ [Bai, Lemma 6.2]. Supersolvable arrangements are a proper subclass of free arrangements, first defined by Stanley [St1]. Thus $\mathcal{D}(V)$ has

$$\frac{2(r+d+1)!}{(d+1)!}$$

chambers.

Incoherent Zonotopes. All that remains is to demonstrate that $Z = Z(V)$ has an incoherent tiling when $d = 3$ and exactly two of the generating vectors for Z have multiplicity two or greater. We know of no elegant proof of this fact. However, this is a sufficiently small obstruction that sets of tilings and coherent tilings may be enumerated using symbolic manipulation packages like MAPLE and GAP [S^+] (code available from the author upon request). When $d = 3$ and exactly two vectors have multiplicity two, Z has 632 total tilings. The total number of coherent tilings is either 616, 620, or 624, depending on the choice of values for the parameters a_1 and a_2, but is always less than 632. By Lemma 3.4, this completes the proof. $\square$

3.4. *Five Vectors in $\mathbb{R}^3$ Containing a Single Three-Point Line*

Theorem 3.8. *Let $Z = Z(V)$ be a 3-zonotope such that the arrangement $\mathcal{A}(\overline{V})$ is projectively equivalent to the projectivized picture given in Fig. 12. Then Z is coherent if and only if at most two of the generating vectors have multiplicities $r, s > 1$, and these vectors with multiplicity correspond to one of the pairs* $\{(1, 3), (1, 5), (2, 3), (2, 5), (3, 4),$

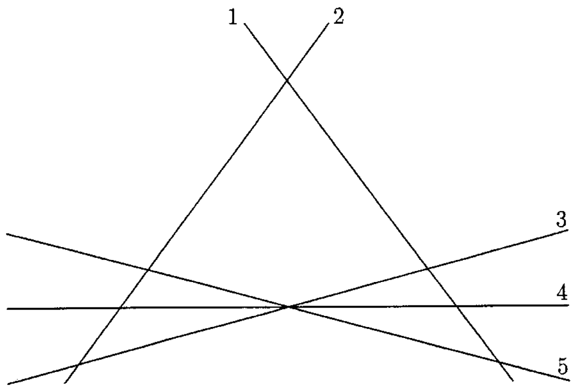

Fig. 12. Five vectors in $\mathbb{R}^3$ with a single three-point line.

$(4, 5)\}$. *If Z satisfies this condition, then the tilings of Z are enumerated by*

$$\frac{2(r+s+1)!\,(r+s+2)!}{(s+2)!\,(r+2)!}.$$

The reason for the apparent asymmetry between hyperplanes 3, 4, and 5 is that hyperplanes 3 and 5 "separate" hyperplane 4 from the intersection of hyperplanes 1 and 2. More precisely, let H_i denote the ith hyperplane for $i = 1, \ldots, 5$, and let l denote the intersection of H_1 and H_2. Then any *path* (point set homeomorphic to the unit interval) originating at l and terminating at H_4 must also contain a point in either H_3 or H_5.

Proof.

Normal Form. Any arrangement in this class may be realized by the frame together with the vector $(a, 1, 1)$, where $a \neq 0, 1$, and so the arrangement in Fig. 12 corresponds to

$$\overline{V} = ((0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 1), (a, 1, 1)).$$

For the rest of this section, we assume $a < 1$. The proof when $a > 1$ is similar. (However, be careful! When $a > 1$, H_4 and H_5 switch position.) By symmetry, it is clear that there are two cases: where vectors 2 and 5 have multiplicity, and where vectors 4 and 5 have multiplicity. We present the proof of the second case; the proof of the first case is similar.

Circuits. The vector $(1, 1, 1)$ occurs with multiplicity r and the vector $(a, 1, 1)$ occurs with multiplicity s. Thus $\mathcal{D}(V)$ is given by the $(r+s+3) \times [rs + \frac{1}{2}(r+s)(r+s+1)]$ block matrix

$$\mathcal{D}(V) = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 & 0 & 0 \\ I_r & 0 & I_r & aI_r & A_{r-1} & 0 \\ 0 & -I_s & \overset{\times}{-}I_s & \overset{\times}{-}I_s & 0 & A_{s-1} \end{pmatrix},$$

where

$$B_1 = \begin{pmatrix} -1 & \cdots & -1 \\ -1 & \cdots & -1 \\ -1 & \cdots & -1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \\ a & \cdots & a \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ a-1 & \cdots & a-1 \end{pmatrix}, \qquad B_4 = \begin{pmatrix} 1-a & \cdots & 1-a \\ 1-a & \cdots & 1-a \\ 0 & \cdots & 0 \end{pmatrix}$$

with the necessary row lengths, A_{r-1} and A_{s-1} are the matrices for the braid arrangements of rank $r-1$ and $s-1$, respectively, and the block pairs $I_r \times -I_s$ and $a I_r \times -I_s$ denote all possible Cartesian products of basis vectors $\mathbf{e}_i$ or $a\mathbf{e}_i \in \mathbb{R}^r$ with basis vectors $-\mathbf{e}_j \in \mathbb{R}^s$, respectively.

Rank 2 *Contractions*. The columns of $\mathcal{D}(V)$ may be partitioned into six blocks in the obvious way, from left to right. Let $\mathcal{C}_i^*$ denote the collection of cocircuits arising from the columns in the ith block, and let $X_{j,k}^i$ denote the cocircuit in the ith block with nonzero entries in positions $j+3$ and $k+r+3$, so that $j \in [1, r]$ and $k \in [1, s]$. Cocircuits in $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$ will be denoted by $X_j^1 = X_{j,0}^1$ and $X_k^2 = X_{0,k}^2$, respectively.

The reader can verify that the following is a complete list of rank 2 contractions:

$$\begin{aligned}
\mathcal{R}_0 &= \{(\mathbf{e}_p - \mathbf{e}_m) \vee (\mathbf{e}_p - \mathbf{e}_q) \vee (\mathbf{e}_m - \mathbf{e}_q) \mid p, q, m \in [4, r+3] \\
&\qquad \text{or } p, q, m \in [r+4, r+s+3]\}, \\
\mathcal{R}_1^i &= \{X_{j,p}^i \vee X_{j,q}^i \vee (\mathbf{e}_p - \mathbf{e}_q) \mid p, q \in [r+4, r+s+3] \text{ and } j \in [0, r]\} \\
&\quad \text{for} \quad i = 2, 3, 4, \\
\mathcal{R}_2^i &= \{X_{q,k}^i \vee X_{p,k}^i \vee (\mathbf{e}_p - \mathbf{e}_q) \mid p, q \in [4, r+3] \text{ and } k \in [0, s]\} \\
&\quad \text{for} \quad i = 1, 3, 4, \\
\mathcal{R}_3 &= \{X_j^1 \vee X_{j,k}^3 \vee X_{j,k}^4 \vee X_k^2\} \qquad \text{if} \quad 1 > a > 0.
\end{aligned}$$

A similar collection $\mathcal{R}_3$ arises for other possible values of a.

Localization and Bijections. As usual, when a localization σ is restricted to $\mathcal{R}_0$, it corresponds to a permutation in $\mathcal{S}_r \times \mathcal{S}_s$. We assume the ordering to be $\sigma(\mathbf{e}_p - \mathbf{e}_q) = +$ for $p < q$, and multiply the final count by $r!\, s!$. This ordering, together with the collections $\mathcal{R}_1^i$ and $\mathcal{R}_2^i$, implies

$$\sigma(X_{j,p}^i) \le \sigma(X_{j,q}^i) \qquad \text{for} \quad r+4 \le p < q \le r+s+3,$$

and

$$\sigma(X_{p,k}^i) \ge \sigma(X_{q,k}^i) \qquad \text{for} \quad 4 \le p < q \le r+3.$$

The signatures of cocircuits in $\mathcal{C}_3^*$ and $\mathcal{C}_4^*$ may each be entered into an $r \times s$ tableau of the kind given in Fig. 13, with certain restrictions. The collections $\mathcal{R}_1^3$, $\mathcal{R}_2^3$, $\mathcal{R}_1^4$, and $\mathcal{R}_2^4$ dictate that the signatures for each of $\mathcal{C}_3^*$ and $\mathcal{C}_4^*$ be weakly increasing along rows and

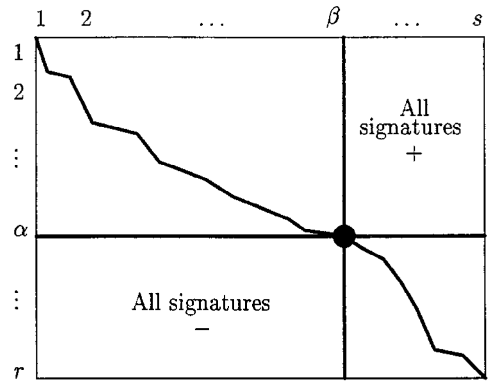

Fig. 13. The tableau of cocircuit signatures for cocircuits in $\mathcal{C}_3^*$ or $\mathcal{C}_4^*$.

weakly decreasing down columns. In Fig. 13, the path running from the upper left corner to the lower right delineates the boundary between signatures $+$ and signatures $-$.

In particular, the above inequalities hold for the cocircuits in $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$. Let $\alpha \in [0, r]$ be the greatest index such that $\sigma(X_\alpha^1) = +$ (if $\sigma(X_j^1) = -$ for all j, then $\alpha = 0$), and let $\beta \in [0, s]$ be the greatest index such that $\sigma(X_\beta^2) = -$. The signatures which have $\alpha < j \le r$ and $1 \le k \le \beta$ must all be $-$, by consideration of the collection $\mathcal{R}_3$, and similarly the signatures which have $1 \le j \le \alpha$ and $\beta < k \le s$ must be $+$.

The pair of tableaux for $\mathcal{C}_3^*$ and $\mathcal{C}_4^*$ account for all of the information in the rank 2 contractions except for the relationship between $\sigma(X_{j,k}^3)$ and $\sigma(X_{j,k}^4)$ given by the collection $\mathcal{R}_3$. When $1 \le j \le \alpha$ and $1 \le k \le \beta$, the inequality

$$+ = \sigma(X_j^1) \ge \sigma(X_{j,k}^3) \ge \sigma(X_{j,k}^4) \ge \sigma(X_k^2) = -$$

holds, so in particular $\sigma(X_{j,k}^3) \ge \sigma(X_{j,k}^4)$. Similarly when $\alpha < j \le r$ and $\beta < k \le s$, the collection $\mathcal{R}_3$ implies $\sigma(X_{j,k}^3) \le \sigma(X_{j,k}^4)$. Thus all information given by the rank 2 contractions may be encoded by superimposing the tableaux for $\mathcal{C}_3^*$ and $\mathcal{C}_4^*$ upon one another and enumerating the resulting pairs of paths in an $r \times s$ tableaux. That is, we must enumerate all $r \times s$ tableaux which contain a pair of monotonically decreasing paths from upper left to lower right, paths which may be concurrent with one another at points, but cross only once at a distinguished *root* defined by α and β (see Fig. 14). Elnitsky [El] has enumerated the collection of such paths in his thesis. There are

$$\frac{2(r+s+1)!\,(r+s+2)!}{r!\,s!\,(r+2)!\,(s+2)!}$$

such tableaux. Multiplying this count by the factor $r!\,s!$ yields the count given in the statement of the theorem.

Compare. To count the coherent tilings, the reader can verify that $\mathcal{D}(V)$ is projectively equivalent to the $(r+s) \times [rs + \frac{1}{2}(r+s)(r+s+1)]$ block matrix

$$\mathcal{D}(V) \sim (A_{r+s-1} | I_{r+s} | aI_r \times -I_s),$$

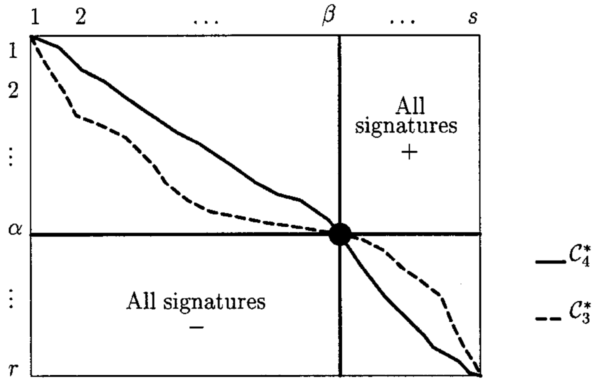

Fig. 14. A tableau which encodes all information from the rank 2 contractions.

where, as above, the notation $aI_r \times -I_s$ denotes all possible Cartesian products of elements $a\mathbf{e}_i \in \mathbb{R}^r$ with elements $-\mathbf{e}_j \in \mathbb{R}^s$ (note that all columns of $\mathcal{D}(V)$ are distinct since $a \neq 0, 1$). This arrangement is free with exponents

$$\{1, r+2, r+3, \ldots, r+s, s+2, s+3, \ldots, r+s, r+s+1\}$$

(see Corollary 7.6 of [Bai]), and so, consequently, Z has the number of coherent tilings given in the statement of the theorem.

Incoherent Zonotopes. It only remains to demonstrate that Z has an incoherent tiling in those cases for which a pair of vectors with multiplicity $r, s > 1$ is not one of the pairs listed in the statement of the theorem. Again, we know of no elegant proof of this fact. However, by the use of the programs MAPLE and GAP [S$^+$], it is possible to show that if vectors 1 and 2 occur with multiplicity two, and all other vectors are singleton, then Z has 400 total tilings, 384 of which are coherent. If the multiplicities are placed on any other forbidden pair, then Z has 304 total tilings, either 296 or 300 of which are coherent, depending on the choice of value for a. This fact, together with Lemma 3.4 finishes the proof. □

Itis interesting to note that when the multiset of multiplicities is {2, 2, 1, 1, 1} and the vectors with multiplicity two are any forbidden pair other than {1, 2}, then $\mathcal{D}(V)$ is free with exponents {1, 4, 4, 5} (unless $a = \frac{1}{2}$). This is one of the few known counterexamples to the tempting but false conjecture that if $\mathcal{D}(V)$ is free, then $Z(V)$ is coherent.

3.5. *Five Vectors in $\mathbb{R}^3$ Containing Two Three-Point Lines*

Finally, we consider the case in which the elements of $\overline{V}$ lie in two intersecting planes, P_1, P_2, with a single vector v, called the *common vector*, common to each. The remaining vectors, the *frame vectors*, may naturally be partitioned into pairs, called *partnerships*, such that the two vectors v_1, v_2 of a partnership define a rank 2 space containing the

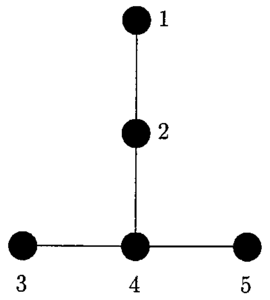

Fig. 15. Five vectors lying on two three-point lines in $\mathbb{R}^3$.

common vector. For example, in Fig. 15, 4 is the common vector, while the frame vectors 1, 2, 3, 5 form the partnerships $\{1, 2\}$ and $\{3, 5\}$. This will complete the classification of coherent 3-zonotopes $Z = Z(V)$ with $|\overline{V}| \leq 5$.

Theorem 3.9. *Let $Z = Z(V)$ be a 3-zonotope such that $\overline{V}$ is as given in Fig.* 15. *Z is coherent if and only if at most two frame vectors v_1, v_2 have multiplicity $r, s \geq 3$, some frame vector occurs with multiplicity one, and*

(1) *If v_1, v_2 form a partnership, then all other vectors, including the common vector, must have multiplicity one. In this case, the tilings of Z are enumerated by*

$$\frac{2(r+s)!\,(r+s+1)!}{(r+1)!\,(s+1)!}.$$

(2) *If v_1, v_2 do not form a partnership, then the common vector may occur with arbitrary multiplicity t, and the multiplicities of the remaining frame vectors must be at most two and one.*
   (a) *In the case that the multiplicities are $\{r, s, t, 1, 1\}$, the tilings of Z are enumerated by*

$$(r+s+t)!.$$

   (b) *In the case that the multiplicities are $\{r, s, t, 2, 1\}$, such that the vector v_3 with multiplicity two forms a partnership with the vector v_1 with multiplicity r, the tilings of Z are enumerated by*

$$\frac{2(r+s+t)!\,(r+s+t+1)!}{(r+1)!\,(s+t+1)!}.$$

Proof of Case (1).

Normal Form. One advantage to the restricted position of the vectors in this case is that there is, up to projective equivalence, only one such vector configuration in $\mathbb{R}^3$. Thus for the remainder of this section, set

$$\overline{V} = ((0,0,1), (1,1,1), (1,0,0), (1,1,0), (0,1,0)).$$

Let $(0, 0, 1)$ occur with multiplicity r and $(1, 1, 1)$ occur with multiplicity s.

Circuits. The discriminantal arrangement $\mathcal{D}(V)$ is given by the $(r+s+3) \times [2rs + \binom{r}{2} + \binom{s}{2} + 1]$ block matrix:

$$\mathcal{D}(V) = \begin{pmatrix} I_r & I_r & A_{r-1} & 0 & 0 \\ -\overset{\times}{I}_s & -\overset{\times}{I}_s & 0 & A_{s-1} & 0 \\ B_1 & B_2 & 0 & 0 & B_3 \end{pmatrix},$$

where

$$B_1 = \begin{pmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ 1 & \cdots & 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},$$

and all other entries are as described previously.

Rank 2 *Contractions*. As before, partition the cocircuits arising from $\mathcal{D}(V)$ into collections $\mathcal{C}_1^*$, $\mathcal{C}_2^*$, $\mathcal{C}_3^*$ $\mathcal{C}_4^*$ from left to right, and denote the cocircuit corresponding to the rightmost vector by $\overline{X}$. As before, let $X_{j,k}^i$ denote the cocircuit corresponding to the vector in $\mathcal{C}_i^*$ with nonzero entries in positions j and $r+k$, where $i = 1$ or 2, $j \in [1, r]$, and $k \in [1, s]$. The reader may verify that the following is a complete list of rank 2 contractions:

$$\begin{aligned} \mathcal{R}_0 &= \{(\mathbf{e}_p - \mathbf{e}_m) \vee (\mathbf{e}_p - \mathbf{e}_q) \vee (\mathbf{e}_m - \mathbf{e}_q) | p, q, m \in [1, r] \\ &\qquad \text{or } p, q, m \in [r+1, r+s]\}, \\ \mathcal{R}_1^i &= \{X_{q,k}^i \vee X_{p,k}^i \vee (\mathbf{e}_p - \mathbf{e}_q) | 1 \le p < q \le r, k \in [1, s]\} \qquad \text{for} \quad i = 1, 2, \\ \mathcal{R}_2^i &= \{X_{j,p}^i \vee X_{j,q}^i \vee (\mathbf{e}_p - \mathbf{e}_q) | j \in [1, r], 1 \le p < q \le s\} \qquad \text{for} \quad i = 1, 2, \\ \mathcal{R}_3 &= \{X_{j,k}^2 \vee X_{j,k}^1 \vee \overline{X} | j \in [1, r], k \in [1, s]\}. \end{aligned}$$

As usual, the elements of $\mathcal{R}_0$ correspond to an element in $\mathcal{S}_r \times \mathcal{S}_s$. So again we enumerate all signatures σ which fix $\sigma(\mathbf{e}_p - \mathbf{e}_q) = +$ for $1 \le p < q \le r$ and for $r+1 \le p < q \le r+s$, and also $\sigma(\overline{X}) = +$. This contributes a factor of $2r!\,s!$ to the final count.

Localization and Bijection. As in the last section, the cocircuit signatures for cocircuits in $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$ may be entered in $r \times s$ tableaux. The collections $\mathcal{R}_1^i$ require that the entries in each tableau must weakly decrease down columns, and the collections $\mathcal{R}_2^i$ require that the entries in each tableau must weakly increase along rows. Thus the cocircuit signatures for the collections $\mathcal{C}_i^*$, for $i = 1, 2$, are encoded by a tableau like the one in Fig. 16.

All that remains is to take into account the elements of $\mathcal{R}_4$. Since $\sigma(\overline{X}) = +$, it follows that $\sigma(X_{j,k}^2) \le \sigma(X_{j,k}^1)$ for all pairs $\{j, k\}$. Thus by superimposing the tableau for $\mathcal{C}_1^*$ on the tableau for $\mathcal{C}_2^*$, all information given by the rank 2 contractions may be encoded in a single $r \times s$ array containing two noncrossing paths from upper left to lower right. The collection of all possible such noncrossing paths is enumerated by

$$\frac{(r+s+1)!\,(r+s)!}{r!\,s!\,(r+1)!\,(s+1)!}$$

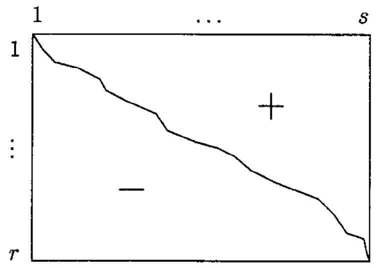

Fig. 16. The tableau of cocircuit signatures for cocircuits in $\mathcal{C}_1^*$ or $\mathcal{C}_2^*$.

(see Section 2.7 of [St2]). Multiplying by $2r!\,s!$ gives the count in the statement of the theorem.

Compare. As for the coherent tilings, $\mathcal{D}(V)$ is again projectively equivalent to an Athanasiadis-type arrangement which is free with exponents

$$\{0, 1, r+1, r+2, \dots, r+s-1, s+1, s+2, \dots, r+s\}$$

[At, Theorem 4.1]. Thus Theorem 3.1 completes the proof of case (1).

Proof of Case (2).

Normal Form. Assume the vector $(1, 0, 0)$ appears with multiplicity r, $(1, 1, 1)$ appears with multiplicity s, $(1, 1, 0)$ appears with multiplicity t, and $(0, 1, 0)$ appears with multiplicity two.

Circuits. After some row swapping, the discriminantal arrangement $\mathcal{D}(V)$ is the $(r + s + t + 3) \times (\binom{r}{2} + \binom{s}{2} + \binom{t}{2} + 2rt + 2rs + st + 1)$ block matrix

$$\mathcal{D}(V) = \begin{pmatrix} A_{r-1} & 0 & 0 & I_r & I_r & I_r & I_r & 0 & 0 \\ 0 & A_{s-1} & 0 & 0 & 0 & \overset{\times}{-}I_s & \overset{\times}{-}I_s & I_s & 0 \\ 0 & 0 & A_{t-1} & \overset{\times}{-}I_t & \overset{\times}{-}I_t & 0 & 0 & \overset{\times}{-}I_t & 0 \\ 0 & 0 & 0 & B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{pmatrix},$$

where

$$B_1 = \begin{pmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 1 & \cdots & 1 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \\ 0 & \cdots & 0 \end{pmatrix},$$

$$B_4 = \begin{pmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ 1 & \cdots & 1 \end{pmatrix}, \qquad B_5 = \begin{pmatrix} -1 & \cdots & -1 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}, \qquad B_6 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

and all other entries are as described above.

Rank 2 *Contractions*. As above, the cocircuits arising from the column vectors of $\mathcal{D}(V)$ may be partitioned in a natural way into eight classes $\mathcal{C}_1^*, \ldots, \mathcal{C}_8^*$, with the final, single cocircuit denoted by $\overline{X}$. Let $X_{j,k,l}^i$ denote the cocircuit vector in $\mathcal{C}_i^*$ with nonzero entries in positions j, $r+k$, and $r+s+l$, with $j \in [1,r]$, $k \in [1,s]$, and $l \in [1,t]$. If some cocircuit has a zero entry in all positions in the interval $[1,r]$, for example, then set $j=0$. So all elements of $\mathcal{C}_7^*$ are written in the form $X_{0,k,l}^7$, and similarly for other $\mathcal{C}_i^*$. The reader may verify that the following is a complete list of rank 2 contractions:

$$\begin{aligned}
\mathcal{R}_0 &= \{(\mathbf{e}_p - \mathbf{e}_m) \vee (\mathbf{e}_p - \mathbf{e}_q) \vee (\mathbf{e}_m - \mathbf{e}_q) | p, q, m \in [1,r] \text{ or } [r+1, r+s] \\
&\qquad \text{or } [r+s+1, r+s+t]\},\\
\mathcal{R}_1^i &= \{X_{q,k,l}^i \vee X_{p,k,l}^i \vee (\mathbf{e}_p - \mathbf{e}_q) \mid p, q \in [1,r]\} \qquad \text{for} \quad i = 4,5,6,7,\\
\mathcal{R}_2^i &= \{X_{j,k,p}^i \vee X_{j,k,q}^i \vee (\mathbf{e}_p - \mathbf{e}_q) \mid p, q \in [r+s+1, r+s+t]\} \qquad \text{for} \quad i = 4,5,8,\\
\mathcal{R}_3^i &= \{X_{j,p,0}^i \vee X_{j,q,0}^i \vee (\mathbf{e}_p - \mathbf{e}_q) \mid p, q \in [r+1, r+s]\} \qquad \text{for} \quad i = 6,7,\\
\mathcal{R}_4 &= \{X_{0,q,l}^8 \vee X_{0,p,l}^8 \vee (\mathbf{e}_p - \mathbf{e}_q) \mid p, q \in [r+1, r+s]\},\\
\mathcal{R}_5^i &= \{X_{j,k,l}^{i+1} \vee X_{j,k,l}^i \vee \overline{X}\} \qquad \text{for} \quad i = 4,6,\\
\mathcal{R}_6^i &= \{X_{j,k,0}^i \vee X_{j,0,l}^{i-2} \vee X_{0,k,l}^8\} \qquad \text{for} \quad i = 6,7.
\end{aligned}$$

Again, the elements of $\mathcal{R}_0$ define a permutation in $\mathcal{S}_r \times \mathcal{S}_s \times \mathcal{S}_t$. So we enumerate those localizations σ which fix $\sigma(\mathbf{e}_p - \mathbf{e}_q) = +$ for $p < q$ and $\sigma(\overline{X}) = +$, and multiply this count by $2r!\,s!\,t!$.

Localization and Bijections. Once again, the cocircuit signatures for the cocircuits in the remaining classes $\mathcal{C}_4^*, \ldots, \mathcal{C}_8^*$ may be entered in tableaux with entries weakly increasing along rows and weakly decreasing down columns. Furthermore, as in earlier cases, certain similar cocircuit classes can be paired off, with their tableaux superimposed upon one another. The reader can verify that the collection of tableaux in Fig. 17 encodes the information from all rank 2 contractions except $\mathcal{R}_6^6$ and $\mathcal{R}_6^7$.

If $L_{r,t}$, $L_{r,s}$, and $L_{s,t}$ encoded the information from **all** rank 2 contractions, then the final count would be obtained by simply enumerating all possible tableaux of each type and taking the product. However, the information from the rank 2 contractions $\mathcal{R}_6^6$ and $\mathcal{R}_6^7$ still has to be taken into account. It turns out that these rank 2 contractions may be used to define a bijection between the collection of localizations and a somewhat simpler collection of tableaux.

Each of the tableaux $L_{r,t}$ and $L_{r,s}$ may be thought of as a collection of columns, ordered from left to right. Specifically, each column h of each tableau may be indexed with an ordered pair (j_1, j_2), where j_1 is the greatest row index of a cell in h lying above the dotted path, and j_2 is the greatest row index of a cell lying above the solid path. Then a collection of columns of the same size may be partially ordered by the product partial order on pairs (j_1, j_2), namely, $(j_1, j_2) \le (j_1', j_2')$ if and only if $j_1 \le j_1'$ and $j_2 \le j_2'$. It is clear that a tableau L contains two noncrossing, monotonically decreasing paths if and only if the columns of L define some linear extension of this partial order. Let the columns of $L_{r,t}$ be indexed by $(\alpha_1, \alpha_2, \ldots, \alpha_t)$, and the columns of $L_{r,s}$ by $(\beta_1, \beta_2, \ldots, \beta_s)$.

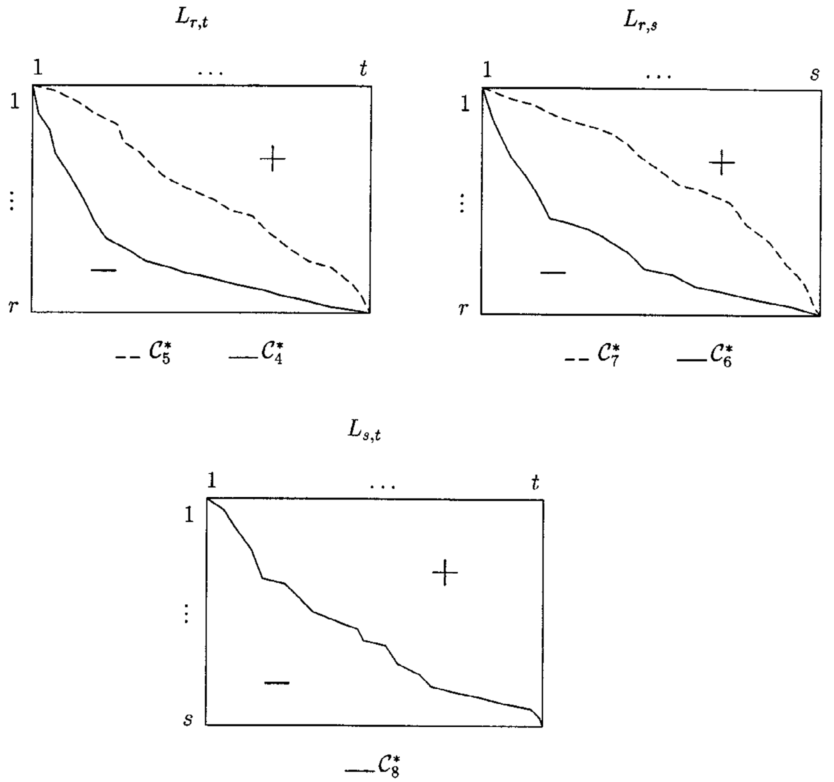

Fig. 17. The tableaux of cocircuit signatures for Z.

The single path P in $L_{s,t}$ defines an *interweaving* of the columns $\alpha_a \in L_{r,t}$ and $\beta_b \in L_{r,s}$ in the following manner. For each unit segment z of P, give z the label α_l if z is a horizontal segment adjacent to cells with column index l, and give z the label β_k if s is a vertical segment adjacent to cells with row index k. By following P from the upper left corner of $L_{s,t}$ to the lower right and reading off the labels, an interweaving ω of the α_a with the β_b is obtained which preserves the original linear order for each collection of columns.

Lemma 3.10. *Let σ be a cocircuit signature on the cocircuits of $Z(V^*)$ respecting the restrictions imposed by $\mathcal{R}_0, \mathcal{R}_1^i, \mathcal{R}_2^i, \mathcal{R}_3^i, \mathcal{R}_4$, and $\mathcal{R}_5^i$, and furthermore satisfying $\sigma(\mathbf{e}_p - \mathbf{e}_q) = +$ for all possible p, q and $\sigma(\overline{X}) = +$.*

Let $L_{r,t}$, $L_{r,s}$, and $L_{s,t}$ be as described above. The cocircuit signature σ respects the rank 2 *contractions of $\mathcal{R}_6^i$* (*and thus is a localization*) *if and only if the interweaving ω orders the columns of $L_{r,t}$ and $L_{r,s}$ in a manner consistent with the partial order on columns.*

Proof. Suppose the path P in $L_{s,t}$ is such that the adjacent labels α_l and β_k are encountered in order in a walk from the upper left corner to the lower right corner (they form a "northeast corner" in P). In particular, this implies that $\sigma(X^8_{0,k,l}) = -$. The interweaving ω implies $\alpha_l \leq \beta_k$.

Suppose instead that either $\alpha_l > \beta_k$ or the two are incomparable under the partial order on columns. This will happen if and only if the statement

$$\text{there exists an index } j \text{ such that } \sigma(X^{i-2}_{j,0,l}) > \sigma(X^i_{j,k,0})$$

holds for at least one of $i = 6$ or $i = 7$. However, if σ respects $\mathcal{R}^i_6$, then this statement implies that $\sigma(X^8_{0,k,l}) = +$, which is a contradiction. This demonstrates the necessity of the condition in the lemma.

To demonstrate sufficiency, suppose there exist indices j, k, l such that

$$\sigma(X^i_{j,k,0}) = -, \qquad \sigma(X^{i-2}_{j,0,l}) = +, \qquad \sigma(X^8_{0,k,l}) = - \tag{$*$}$$

for $i = 6$ or $i = 7$. If the cell (k, l) of $L_{s,t}$ is bordered by P above and on the right, then ω implies $\alpha_l \leq \beta_k$. If the columns α_l and β_k satisfy this relation, then necessarily $\sigma(X^{i-2}_{j,0,l}) \leq \sigma(X^i_{j,k,0})$ for all j and $i = 6, 7$. However, this already contradicts the assumption $(*)$.

If the cell (k, l) is not bordered by P in the manner described above, then it is still possible to move from the cell (k, l) to a cell (k', l') which is bordered by P and satisfies a condition like the one given in $(*)$. Since σ respects all rank 2 contractions except $\mathcal{R}^i_6$ for $i = 6, 7$, moving from (k, l) in the direction of decreasing k and increasing l preserves the signatures in $(*)$. Then the condition $(*)$ for the cell (k', l') yields a contradiction also. This demonstrates that if σ is not a localization, then the ordering ω will not agree with the natural ordering on the columns of $L_{r,t}$ and $L_{r,s}$. □

As a result of Lemma 3.10, we now see that for a given localization σ, **all** necessary information from the rank 2 contractions may be encoded by taking a collection of tableaux as given in Fig. 17 and interweaving the columns of $L_{r,t}$ in $L_{r,s}$ to obtain a single $r \times (s + t)$ tableau L such that the columns of L define a linear extension of the partial order on columns. In particular, L must contain two noncrossing, monotonically decreasing paths. An example of this interweaving is given in Fig. 18.

Thus the total number of localizations is counted by multiplying the number of $r \times (s + t)$ tableaux L containing two noncrossing paths by the number of ways of partitioning the columns of such a tableau L into sets of sizes s and t. Again using the result of Stanley [St2, Section 2.7], this number is

$$\frac{(r + s + t)!\,(r + s + t + 1)!}{r!\,(r + 1)!\,(s + t)!\,(s + t + 1)!} \cdot \frac{(s + t)!}{s!\,t!}.$$

Multiplying this last number by $2r!\,s!\,t!$ gives the result in the statement of the theorem.

Compare. To enumeratethe coherent tilings of Z, the $\mathcal{D}(V)$ is an Athanasiadis-type arrangement which is free with exponents

$$\{0, 1, r + 1, r + 2, \ldots, r + s + t - 1, s + t + 1, s + t + 2, \ldots, r + s + t\}$$

[At, Theorem 4.1], and so Z has the desired number of coherent tilings as well.

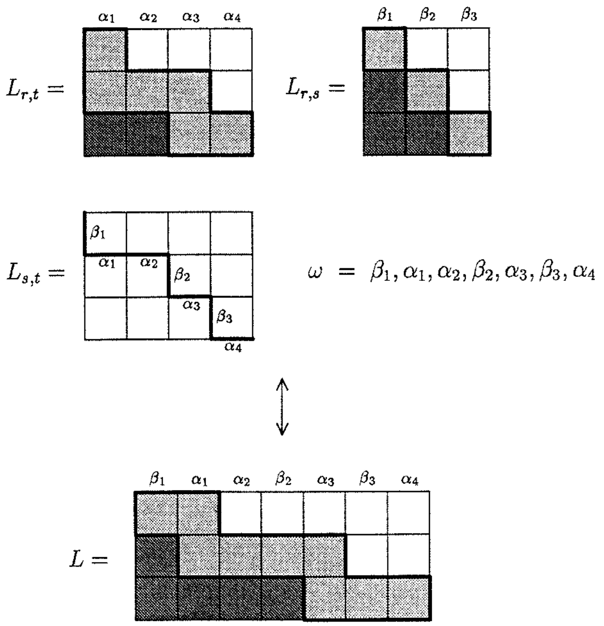

Fig. 18. The rank 2 contractions $\mathcal{R}_6^i$ define an interweaving of the columns of $L_{r,t}$ and $L_{r,s}$.

Next consider the case where two frame vectors have multiplicity one, and the other multiplicities are r, s, t as in part (2a) of the theorem. By Lemma 3.4, we already know that Z is coherent. Here $\mathcal{D}(V)$ may be written in block form as

$$\mathcal{D}(V) = \begin{pmatrix} A_{r-1} & 0 & 0 & I_r & I_r & 0 \\ 0 & A_{s-1} & 0 & \overset{\times}{-} I_s & 0 & I_s \\ 0 & 0 & A_{t-1} & 0 & \overset{\times}{-} I_t & \overset{\times}{-} I_t \\ 0 & 0 & 0 & B_1 & B_2 & B_3 \end{pmatrix},$$

where

$$B_1 = \begin{pmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 1 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} -1 & \cdots & -1 \\ 0 & \cdots & 0 \end{pmatrix}.$$

The reader can verify that this arrangement is projectively equivalent to the braid arrangement $A_{r+s+t-1}$, which is known to be free with exponents $\{1, 2, \ldots, r+s+t-1\}$.

Incoherent Zonotopes. Finally, we must show that Z is incoherent in the case that the multiplicities on its vectors do not satisfy the hypotheses of the theorem. Again, we resort to brute-force computation in GAP [S$^+$] to show that this is the case.

First, if the vector multiplicities are {3, 3, 2, 1, 1} where the vectors with multiplicity three form a partnership, then Z has 211,680 tilings, 210,816 of which are coherent. This gives the necessity of the condition in part (1) of the theorem. Furthermore, if all frame vectors have multiplicity two and the common vector has multiplicity one, then Z

has 25,408 tilings, 23,136 of which are coherent. This demonstrates the necessity of the condition that one frame vector has multiplicity one. Together, these conditions show the necessity of the statement that at most two frame vectors may have multiplicities of three or greater. This completes the proof of the theorem. □

Acknowledgment

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