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Straight-Line Embeddings of Two Rooted Trees in the Plane

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Abstract. We prove the following theorem: Let T_1 and T_2 be two disjoint rooted trees with roots v_1 and v_2, respectively, and let P be a set of $|T_1 \cup T_2|$ points in the plane in general position containing two specified points p_1 and p_2. Then the union $T_1 \cup T_2$ can be straight-line embedded onto P such that v_1 and v_2 correspond to p_1 and p_2, respectively. Moreover, we give a $O(n^2 \log n)$ time algorithm for finding such an embedding, where n is the number of vertices contained in $T_1 \cup T_2$.

1. Introduction

We consider finite planar graphs without loops or multiple edges. Let G be a planar graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $|G|$ the order of G, that is, $|G| = |V(G)|$. Given a planar graph G, let P be a set of $|G|$ points in the plane (two-dimensional Euclidean space) in general position (i.e., no three of them are collinear). Then G is said to be *line embedded onto P* or *stright-line embedded onto P* if G can be embedded in the plane so that every vertex of G corresponds to a point of P, every edge corresponds to a straight-line segment, and no two straight-line segments intersect except their common endpoint. Namely, G is line embedded onto P if there exists a bijection $\varphi\colon V(G) \to P$ such that two points $\varphi(x)$ and $\varphi(y)$ are joined by a straight-line segment if and only if x and y are joined by an edge of G and all two distinct open straight-line segments have no point in common. We call such a bijection a *line embedding* or a *straight-line embedding* of G onto P.

In this paper we consider a line embedding having one more property. Let G be a planar graph with n specified vertices $v_1, v_2, \ldots, v_n$, and let P be a set of $|G|$ points in the plane in general position containing n specified points $p_1, p_2, \ldots, p_n$. Then we say that G is *strongly line embedded onto P* if G can be line embedded onto P so that,

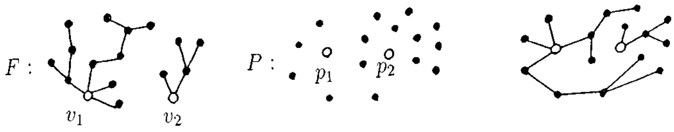

Fig. 1. A rooted forest F and its strong line embedding onto P.

for every $1 \le i \le n$, v_i corrrresponds to p_i, that is, if there exists a line embedding φ: $V(G) \to P$ such that $\varphi(v_i) = p_i$ for all $1 \le i \le n$. The line embedding mentioned above is called a *strong line embedding* of G onto P. A tree with one specified vertex v is usually called a *rooted tree* with root v. Given n disjoint rooted trees T_i with root v_i, $1 \le i \le n$, the union $T_1 \cup T_2 \cup \cdots \cup T_n$, whose vertex set is $V(T_1) \cup V(T_2) \cup \cdots \cup V(T_n)$ and whose edge set is $E(T_1) \cup E(T_2) \cup \cdots \cup E(T_n)$, is called a *rooted forest* with roots $v_1, v_2, \ldots, v_n$, which are specified vertices of it.

We begin with the following theorem, which was conjectured by Perles [5] and partially solved by Pach and Törőcsik [4]; a simpler proof can be found in [7]. Another related result can be found in [2].

Theorem A [3]. *A rooted tree T can be strongly line embedded onto every set of $|T|$ points in the plane in general position containing a specified point.*

In this paper we prove the following theorem.

Theorem 1. *A rooted forest F consisting of two rooted trees can be strongly line embedded onto every set of $|F|$ points in the plane in general position containing two specified points* (*see Fig.* 1).

Moreover, our proof of the theorem gives an $O(|F|^2 \log|F|)$ time algorithm for finding a strong line embedding. Before giving a proof, we mention that there exist rooted forests consisting of four rooted trees which cannot be strongly line embedded onto certain sets of points in the plane in general position containing four specified points. An example of such a forest and a set of points are given in Fig. 2. However, we propose the following conjecture and problem.

Conjecture B. *A rooted forest F consisting of three rooted trees can be strongly line embedded onto every set of $|F|$ points in the plane in general position containing three specified points.*

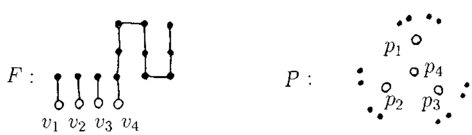

Fig. 2. A rooted forest F which cannot be strongly line embedded onto P.

Problem C. Let $F := T_1 \cup T_2 \cup \cdots \cup T_n$ be a rooted forest with roots $v_1, v_2, \ldots, v_n$ and let P be a set of $|F|$ points in the plane in general position containing n specified points $p_1, p_2, \ldots, p_n$. Find a sufficient condition for F to be strongly line embedded onto P.

2. Proof of Theorem

In order to prove our theorem, we need some notation and definitions. Let X be a set of points in the plane. We denote by $\mathrm{conv}(X)$ the convex hull of X, which is the smallest convex set containing X. A point of $\mathrm{conv}(X)$ not lying on the boundary of $\mathrm{conv}(X)$ is called an *interior point* of $\mathrm{conv}(X)$. It is obvious that if X consists of points in general position, then every point of X lying on the boundary of $\mathrm{conv}(X)$ is a vertex of $\mathrm{conv}(X)$. For two points x and y in the plane, we denote by $\overline{xy}$ the straight-line segment joining x to y.

Let G be a graph. For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G. For a subset $S \subseteq V(G)$, we denote by $G - S$ the graph obtained from G by deleting the vertices in S together with their incident edges, and if $S = \{v\}$, then we write $G - v$ for $G - \{v\}$. Furthermore, the subgraph of G induced by S is denoted by $\langle S \rangle_G$, which is equal to $G - (V(G) \backslash S)$.

Let P be a set of points in the plane in general position containing specified points. For convenience, we call a nonspecified point of P an *ordinary point*, and denote the set of ordinary points of P by $O(P)$. For a region R in the plane, we state which points of P lying on the boundary of R are contained in $P \cap R$ one by one. Thus when we introduce a new notation on a region, we do not mention its boundary.

For three noncollinear points x, y, and p in the plane, the plane is partitioned into two regions by two rays emanating from p and passing through x and y, respectively. We denote by *Rgn*(xpy) the region whose induced angle is less than π, that is, *Rgn*(xpy) denotes the internal region. Similarly, for noncollinear point x and ray r from p, and for noncollinear rays r_1 and r_2 from p, *Rgn*(xpr) and *Rgn*($r_1 p r_2$) denote the similar internal regions (see Fig. 3). Moreover, for two rays r_1 and r_2 emanating from x_1 and x_2, respectively, we define the region *Rgn*($r_1 x_1 x_2 r_2$) as Fig. 3. If we consider a region including all its boundary, then we call it a *closed region*, and if we consider a region without its boundary, then we call it an *open region*.

Before giving some lemmas, we explain a brief sketch of the proof of Theorem 1 so that readers can guess the reason why the following lemmas are given. We first deal with

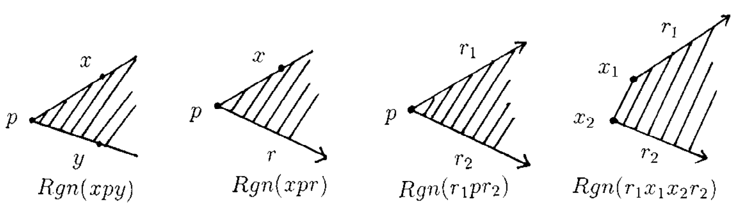

Fig. 3. Regions *Rgn*(xpy), *Rgn*(xpr), *Rgn*($r_1 p r_2$), and *Rgn*($r_1 x_1 x_2 r_2$).

the case where one of the specified points lies on the boundary of conv(P), and next consider the case where both specified points are interior points of conv(P). In order to prove the latter case, we partition the bigger rooted tree into some rooted subtrees, which have some new common roots, and also partiton the point set P into some subsets, which have some common new specified points. Then we try to embed two or more rooted subtrees onto the corresponding subset Q of P under the condition that some of the specified points of Q lie on the boundary of conv(Q).

Lemma 2. *A tree T has a vertex v such that every component of $T - v$ has order less than or equal to $|T|/2$.*

Proof. Choose a vetex v of T so that the maximum order of the components of $T - v$ is minimum among all vertices of T. Then v satisfies the condition of this lemma. □

Note that if, for a vertex x of a tree T, $T - x$ has a component C whose order is greater than $|T|/2$, then the vertex y of C adjacent to x has the property that every component of $T - y$ has order less than $|C|$. We can find the desired vertex v in the above lemma in $O(|T| \log |T|)$ time by making use of this fact.

Lemma 3. *Let T be a tree with two specified vertices v_1 and v_2, and let P be a set of $|T|$ points in the plane in general position containing two specified points p_1 and p_2. If p_1 and p_2 are consecutive vertices of* conv(P), *then T can be strongly line embedded onto P.*

Proof. We prove the lemma by induction on $|T|$. By a suitable rotation of the plane and by the symmetry of p_1 and p_2, we may assume that p_1 lies on the bottom of conv(P) and that p_2 lies to the right of p_1. Suppose first $\deg_T(v_1) = 1$. In this case we take a vertex q of conv($P \setminus \{p_1\}$) such that the straight-line segment $\overline{qp_2}$ is an edge of conv($P \setminus \{p_1\}$) and $\overline{p_1 q}$ intersects conv($P \setminus \{p_1\}$) at only q. Let u be the vertex of T adjacent to v_1. Then, by induction, the tree $T - v_1$ with two specified vertices u and v_2 is strongly line embedded onto $P \setminus \{p_1\}$ with specified points q and p_2. By adding $\overline{p_1 q}$ to this embedding, we can get the desired strong line embedding of T.

We next assume $\deg_T(v_1) \geq 2$. Let D be a component of $T - v_1$ not containing v_2. Then $1 \leq |D| \leq |T| - 2 = |P| - 2$, and so there exists a line l passing through p_1 such that the number of ordinary points of P lying on or to the left of l is equal to $|D|$. We denote the set of these ordinary points of P by Q. Then, by Theorem A, the rooted tree $\langle D \cup \{v_1\} \rangle_T$ with root v_1 is strongly line embedded onto $Q \cup \{p_1\}$ with specified point p_1. Furthermore, it follows from the inductive hypothesis that $T - V(D)$ with specified vertices v_1 and v_2 is strongly line embedded onto $P \setminus Q$ with specified points p_1 and p_2. By combining the above two embeddings, we can obtain the desired strong line embedding of T onto P. □

Lemma 4. *Let $T_1 \cup T_2$ be a rooted forest with roots v_1 and v_2, and let P be a set of $|T_1 \cup T_2|$ points in the plane in general position containing two specified points p_1 and p_2. If p_1 is a vertex of* conv(P), *then $T_1 \cup T_2$ can be strongly line embedded onto P.*

Proof. We prove the lemma by induction on $|T_1 \cup T_2|$. Suppose first $\deg_{T_1}(v_1) = 1$. We take a vertex q of $\operatorname{conv}(P \backslash \{p_1\})$ such that $q \neq p_2$ and $\overline{p_1 q}$ intersects $\operatorname{conv}(P \backslash \{p_1\})$ at only q. Let u be the vertex of T_1 adjacent to v_1. Then, by induction, the rooted forest $(T_1 - v_1) \cup T_2$ with roots u and v_2 can be strongly line embedded onto $P \backslash \{p_1\}$ with specified points q and p_2. By adding $\overline{p_1 q}$ to this embedding, we get the desired embedding of $T_1 \cup T_2$ onto P.

We next assume $\deg_{T_1}(v_1) \geq 2$. Let D be one of the smallest components of $T_1 - v_1$. Then $|D| \leq (|T_1| - 1)/2 \leq (|P| - 2)/2$, and so at least one of the two open regions detemined by the line passing through p_1 and p_2 contains at least $|D|$ ordinary points of P. Thus there exists a line l passing through p_1 such that one of the open regions determined by l, say R, contains exactly $|D|$ ordinary points of P and does not contain p_2. Then, by Theorem A, $\langle D \cup \{p_1\} \rangle_{T_1}$ with root v_1 is strongly line embedded onto $(R \cap O(P)) \cup \{p_1\}$ with specified point p_1. By the inductive hypothesis, $(T_1 - V(D)) \cup T_2$ with roots v_1 and v_2 is strongly line embedded onto $P \backslash (R \cap O(P))$ with specified points p_1 and p_2. Combining these embeddings, we can obtain the desired strong line embedding of $T_1 \cup T_2$ onto P. □

Lemma 5. *Let $F := T_1 \cup T_2 \cup T_3$ be a rooted forest with roots v_1, v_2, v_3, and let P be a set of $|F|$ points in the plane in general position containing three specified points p_1, p_2, p_3. If p_1 and p_2 are consecutive vertices of* $\operatorname{conv}(P)$, *then F can be strongly line embedded onto P.*

Proof. Without loss of generality, we may assume that p_1 lies on the bottom of $\operatorname{conv}(P)$ and that p_2 lies to the right of p_1. We consider only rays r emanating from p_1 and going upward, and so a ray means such a ray. Given a ray r, let $P(r)$ denote the set of points of P lying on or to the left of r. Then $p_1 \in P(r)$ for every ray r, and there exists a ray r_1 such that either (i) $|P(r_1)| = |T_1|$ and $P(r_1)$ does not contain p_3; or (ii) $|P(r_1)| = |T_1| + |T_3|$ and $P(r_1)$ contains p_3. If r_1 satisfies (i), then T_1 and $T_2 \cup T_3$ are strongly line embedded onto $P(r_1)$ and onto $P \backslash P(r_1)$, respectively, by Theorem A and by Lemma 4. Similarly, if r_1 satisfies (ii), then $T_1 \cup T_3$ and T_2 are strongly line embedded onto $P(r_1)$ and onto $P \backslash P(r_1)$, respectively. Therefore $T_1 \cup T_2 \cup T_3$ can be strongly line embedded onto P. □

Proof of Theorem 1. Let $F := T_1 \cup T_2$ be a rooted forest with roots v_1 and v_2, and let P be a set of $|F|$ points in the plane in general position containing two specified points p_1 and p_2. Then $O(P) = P \backslash \{p_1, p_2\}$, which is the set of ordinary points of P.

We may assume that $|T_1| \geq |T_2| \geq 2$ since if $|T_2| = 1$, then the theorem follows from the fact that a strong line embedding of T_1 onto $P \backslash \{p_2\}$ is also a strong line embedding of F onto P. Put $n_1 := |T_1| - 1$ and $n_2 := |T_2| - 1$, which are equal to the numbers of ordinary points of P added to p_1 and to p_2 to construct T_1 and T_2, respectively.

We now prove the theorem. By Lemma 4, we may assume that both p_1 and p_2 are interior points of $\operatorname{conv}(P)$. By a suitable rotation of the plane, we may assume that both p_1 and p_2 lie on the same horizontal line and that p_1 lies to the left of p_2. Moreover, since every line passing through p_1 or p_2 is an oriented line going upward, a *line passing through p_1 or p_2* means an oriented line passing through p_1 or p_2 and going upward. A line passing neither p_1 nor p_2 is an oriented line or a usual unoriented line, and we state it one by one.

Claim 1. *We may assume that for every line l passing through p_1, the number $f(l)$ of ordinary points of P lying on or to the left of l is less than n_1. In particular, we may assume that the number of ordinary points of P lying above* (*below*) *the horizontal line passing through p_1 and p_2 is less than n_1.*

Proof. Here we consider only lines passing through p_1, and so a line means such a line. Suppose that there exists a line l_1 such that $f(l_1) \geq n_1$. Since $(n_1 + n_2)/2 \leq n_1$, there exists a line l_2 with $f(l_2) \leq n_1$, which is obtained from the horizontal line passing through p_1 and p_2 by an infinitesimal clockwise or counterclockwise rotation around p_1. Since every line passes through at most one ordinary point of P, when we rotate a line from l_2 to l_1 around p_1, the value of f changes ± 1 according to whether the line hits or passes an ordinary point of P. Hence there exists a line l_3 such that $f(l_3) = n_1$. Then T_1 and T_2 are strongly line embedded onto the set of points of P lying on or to the left of l_3 and onto the set of points of P lying to the right of l_3, respectively. Therefore F can be strongly line embedded onto P. Consequently, we may assume that $f(l) < n_1$ for every l. □

Define an integer M by

$$M := \max\{f(l)\},$$

where the maximum is taken over all the lines l passing through p_1 except the horizontal line passing through p_1 and p_2, and $f(l)$ is defined as in Claim 1. Then by Claim 1 we have

$$(n_1 + n_2)/2 \leq M < n_1. \tag{1}$$

Let $D_1, D_2, \ldots, D_m$ be the components of $T_1 - v_1$ such that $|D_1| \geq |D_2| \geq \cdots \geq |D_m|$.

Claim 2. *We may assume $|D_1| > M$.*

Proof. Suppose that $|D_1| \leq M$. Here we consider only lines passing through p_1, and so a line means such a line. If there exists a line l_1 with $f(l_1) \leq |D_1|$, then there exists a line l_2 such that $f(l_2) = |D_1|$. Then, by Theorem A, the rooted tree $\langle D_1 \cup \{v_1\}\rangle_{T_1}$ with root v_1 is strongly line embedded onto the set of points of P lying on or to the left of l_2 with specified point p_1. By Lemma 4, the rooted forest $(T_1 - V(D_1)) \cup T_2$ with roots v_1 and v_2 is strongly line embedded onto the set of points of P lying on or to the right of l_2 with specified points p_1 and p_2. Hence F can be strongly line embedded onto P. Therefore we assume that, for every line l, we have $f(l) > |D_1|$.

We write r_0 for the ray emanating from p_1 and passing through p_2. Then since $f(l) > |D_i|$, for $1 \leq i \leq m-1$, we can inductively take a ray r_i emanating from p_1 and not passing through any ordinary points of P such that the open region $Rgn(r_{i-1}p_1r_i)$ contains exactly $|D_i|$ ordinary points of P. Then each rooted tree $\langle D_i \cup \{v_1\}\rangle_{T_1}$ with root v_1 is strongly line embedded onto $(Rgn(r_{i-1}p_1r_i) \cap O(P)) \cup \{p_1\}$ with specified point

p_1. Moreover, by (1), the ray r_m goes downward, and thus T_2 is strongly line embedded onto $(Rgn(r_m p_1 r_0) \cap O(P)) \cup \{p_2\}$. Therefore F can be strongly line embedded onto P. Consequently we may assume that $|D_1| > M$. □

By Lemma 2, D_1 has a vetex w_1 such that each component of $D_1 - w_1$ has order less than or equal to $|D_1|/2$. Let $A_1, A_2, \ldots, A_t$ be the components of $D_1 - w_1$ such that A_1 is the component containing the vertex adjacent to v_1 in T_1 and A_2 is the largest component among $A_2, A_3, \ldots, A_t$. Note that $A_1 = \emptyset$ if v_1 and w_1 are adjacent in T_1. Let $w_2 \in A_2$ be the vertex adjacent to w_1. Define $B_4 := A_2$ and

$$B_2 := A_1 \cup \left(\bigcup_{i=3}^{r} A_i\right) \qquad \text{if} \quad t \geq 3, \qquad \text{and otherwise} \quad B_2 := A_1,$$

where the integer r, $3 \leq r \leq t$, is chosen as large as possible subject to $|B_2| < M$. In particular, if $r < t$, then $|B_2 \cup A_{r+1}| \geq M$. Note that $|A_1| < M$ as $|A_1| \leq |D_1|/2 \leq (n_1 - 1)/2 \leq (n_1 + n_2)/2 - 1 \leq M - 1$. We define

$$B_3 := A_{r+1} \cup \cdots \cup A_t \qquad \text{if} \quad r < t, \qquad \text{and otherwise} \quad B_3 := \emptyset.$$

Moreover, put

$$B_1 := D_2 \cup \cdots \cup D_m = T_1 - (V(D_1) \cup \{v_1\}).$$

Then $V(T_1) = V(B_1) \cup V(D_1) \cup \{v_1\}$, $V(D_1) = V(B_2) \cup V(B_3) \cup V(B_4) \cup \{w_1\}$, and $w_2 \in B_4$ (see Fig. 4).

Claim 3. *We may assume that, for every line l passing through p_1, the number $f(l)$ of ordinary points of P lying on or to the left of l is greater than or equal to $|B_2| + 1$.*

Proof. Suppose that there exists a line l_1 passing through p_1 for which $f(l_1) \leq |B_2|$. Then by the definition of B_2, we have $|B_2|+1 \leq M$, and thus we can find a line l_2 through p_1 such that $f(l_2) = |B_2| + 1$ and l_2 passes through an ordinary point q of P. Let Q denote the set of ordinary points of P lying on or to the left of l_2. Then, by Lemma 3, the rooted tree $\langle B_2 \cup \{v_1, w_1\}\rangle_{T_1}$ with two specified vertices v_1 and w_1 is strongly line embedded onto $Q \cup \{p_1\}$ with two specified points p_1 and q. Similarly, by Lemma 5, the rooted forest $\langle B_1 \cup \{v_1\}\rangle_{T_1} \cup \langle B_3 \cup B_4 \cup \{w_1\}\rangle_{T_1} \cup T_2$ with roots v_1, w_1, v_2 can be strongly line embedded onto $(O(P)\backslash Q) \cup \{p_1, q, p_2\}$ with specified points p_1, q, p_2. Therefore $T_1 \cup T_2$ can be strongly line embedded onto P. □

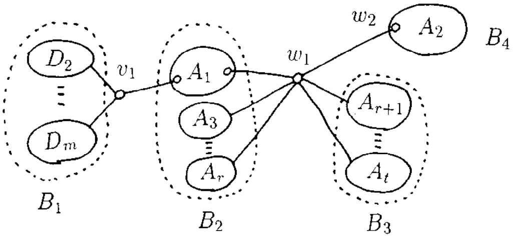

Fig. 4. The rooted tree T_1.

Claim 4. *We may assume $B_3 \neq \emptyset$.*

Proof. Suppose $B_3 = \emptyset$. Let l_0 be the horizontal unoriented line passing through both p_1 and p_2, and let r_0 be the ray emanating from p_1 and passing through p_2.

Without loss of generality, we may assume that the number of ordinary points of P lying above l_0 is greater than or equal to $(n_1 + n_2)/2$. Since $|B_4| \leq |D_1|/2 \leq (n_1 + n_2)/2 - 1$, there exists a ray r_1 emanating from p_1, going upward, and passing through an ordinary point q of P such that the closed region $Rgn(r_0 p_1 r_1)$ contains exactly $|B_4| + 1$ ordinary points of P.

By Claim 3, we can find a ray r_2 emanating from p_1 and not passing through any ordinary point of P such that the closed region $Rgn(r_1 p_1 r_2)$ contains $|B_2| + 1$ ordinary points of P. Since $|B_4| + |B_2| + 1 = |D_1| > M$, and by the definition of M, r_2 must go downward, that is, r_2 lies below l_0. Then, by Thorem A, the rooted tree $\langle B_4 \cup \{w_1\}\rangle_{T_1}$ with root w_1 is strongly line embedded onto $Rgn(r_0 p_1 r_1) \cap O(P)$ with specified point q. By Lemma 3, the rooted tree $\langle B_2 \cup \{w_1, v_1\}\rangle_{T_1}$ with two specified vertices v_1 and w_1 can be strongly line embedded onto $(Rgn(r_1 p_1 r_2) \cap O(P)) \cup \{p_1\}$ with two specified points p_1 and q. Moreover, since r_2 goes downward, the two rooted trees $\langle B_1 \cup \{v_1\}\rangle_{T_1}$ with root v_1 and T_2 with root v_2 can be strongly line embedded onto $(Rgn(r_2 p_1 r_0) \cap O(P)) \cup \{p_1, p_2\}$ by Lemma 4. Therefore $T_1 \cup T_2$ is strongly line embedded onto P. Consequently we may assume $B_3 \neq \emptyset$. $\square$

Let l_0 and r_0 denote the same line and ray as in the proof of Claim 4. Let r_0^* be the ray from p_1 which lies on l_0 and whose direction is opposite to r_0. By Claim 3, we can take a ray r_3 emanating from p_1, passing through no ordinary point of P, and going upward such that the open region $Rgn(r_0^* p_1 r_3)$ contains $|B_2|$ ordinary points of P. Since $B_3 \neq \emptyset$, we have $|B_2 \cup B_4| \geq |B_2 \cup A_{r+1}| \geq M$, which implies that the region above l_0 does not contain more than $|B_2 \cup B_4|$ ordinary points of P. Thus the region below l_0 contains at least $n_1 + n_2 - |B_2 \cup B_4|$ ordinary points of P. Since $n_1 + n_2 - |B_2 \cup B_4| \geq |B_3| + 2$, we can find a ray r_4 emanating from p_1, passing through no ordinary points of P, and going downward such that the open region $Rgn(r_0^* p_1 r_4)$ contains exactly $|B_3| + 2$ ordinary points of P.

Let l_3 be the line containing r_3 and going upward, and let r_3^* be the ray from p_1 opposite to r_3. Note that we may assume that l_3 passes through no ordinary point of P because r_3 does not pass through any ordinary points. It follows from Claim 3 that $f(l_3) \geq |B_2| + 1$, and so the open region $Rgn(r_0^* p_1 r_3^*)$ contains at least one ordinary point of P. Similarly, since $|B_2| + |B_3| + 2 > M$, the open region $Rgn(r_3^* p_1 r_4)$ contains at least one ordinary point of P. We choose two ordinary points $q_1 \in Rgn(r_0^* p_1 r_3^*)$ and $q_2 \in Rgn(r_3^* p_1 r_4)$ so that the intersection of the line segment $\overline{q_1 q_2}$ with r_3^* is closest to p_1 among all the pairs of ordinary points in $Rgn(r_0^* p_1 r_3^*)$ and those in $Rgn(r_3^* p_1 r_4)$ (see Fig. 5). In particular, the open triangle surrounded by r_0^*, r_4, and the line passing through q_1 and q_2 contains no ordinary point of P.

Let l_5 denote the unoriented line passing through q_1 and q_2. Let r_a be a ray from q_1 passing through no ordinary point of P except q_1 such that r_a does not intersect l_3 and the open region $Rgn(r_a q_1 p_1 r_3)$ contains exactly $|B_2|$ ordinary points, which is equal to the number of ordinary points in $Rgn(r_0^* p_1 r_3)$. Then since the open triangle surrounded by r_0^*, r_4, and l_5 contains no ordinary points of P, r_a lies below l_5 and above the line

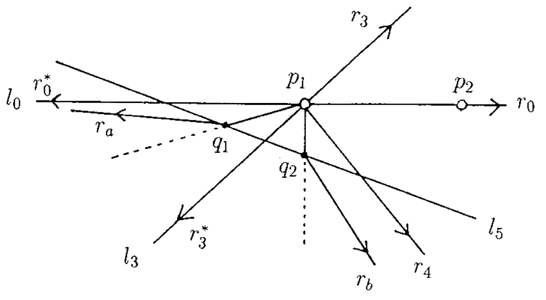

Fig. 5. $Rgn(r_a q_1 q_2 r_b)$ does not contain p_2.

passing through p_1 and q_1. Similarly, let r_b be a ray from q_2 passing through no ordinary point of P except q_2 such that r_b does not intersect l_3 and the open region $Rgn(r_a q_1 q_2 r_b)$ contains exactly $|B_3|$ ordinary points of P (see Fig. 5). Then r_b lies below l_5 and to the right of the line passing through p_1 and q_2. Moreover, note that $q_1 \notin Rgn(r_a q_1 p_1 r_3)$ and $q_1, q_2 \notin Rgn(r_a q_1 q_2 r_b)$. By the properties of r_a and r_b, the open regions $Rgn(r_a q_1 p_1 r_3)$, $Rgn(r_a q_1 q_2 r_b)$, and the new open region $Rgn(r_b q_2 p_1 r_3)$ are convex sets. We consider two cases, and show that the theorem holds in each case.

Case 1: *$Rgn(r_a q_1 q_2 r_b)$ does not contain p_2* (see Fig. 5).

Proof. By Theorem A, the rooted tree $\langle B_3 \cup \{w_1\}\rangle_{T_1}$ with root w_1 is strongly line embedded onto $(Rgn(r_a q_1 q_2 r_b) \cap O(P)) \cup \{q_1\}$ with specified point q_1. It follows from Lemma 3 that the tree $\langle B_2 \cup \{v_1, w_1\}\rangle_{T_1}$ with specified vertices v_1 and w_1 is strongly line embedded onto $(Rgn(r_a q_1 p_1 r_3) \cap O(P)) \cup \{p_1, q_1\}$ with specified points p_1 and q_1. Moreover, by Lemma 5, the rooted forest $\langle B_4\rangle_{T_1} \cup \langle B_1 \cup \{v_1\}\rangle_{T_1} \cup T_2$ with roots w_2, v_1, v_2 is strongly line embedded onto $(Rgn(r_b q_2 p_1 r_3) \cap O(P)) \cup \{q_2, p_1, p_2\}$ with specified points q_2, p_1, and p_2. By adding the line segment $\overline{q_1 q_2}$ to the above embeddings, we obtain a strong line embedding of $T_1 \cup T_2$ onto P.

Case 2: *$Rgn(r_a q_1 q_2 r_b)$ contains p_2* (see Fig. 6).

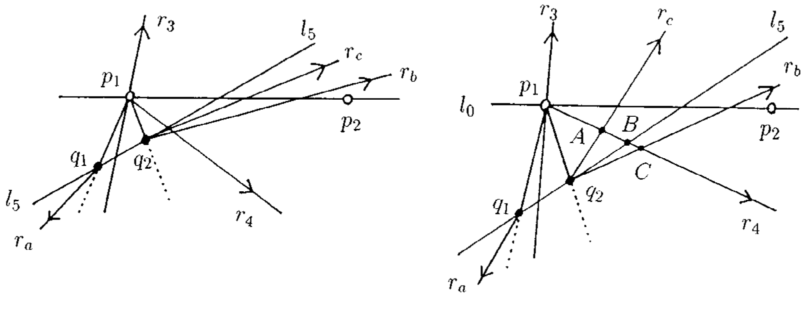

Fig. 6. $Rgn(r_a q_1 q_2 r_b)$ contains p_2.

Proof. We use Fig. 6. Since the open region $Rgn(r_b q_2 p_1 r_3)$ contains $|B_1|+|B_4|-1+n_2$ ordinary points of P, we can take a ray r_c emanating from q_2 and passing through no ordinary point of P except q_2 such that the open region $Rgn(r_c q_2 p_1 r_3)$ contains exactly $|B_1|+|B_4|-1$ ordinary points of P. In particular, the open region $Rgn(r_b q_2 r_c)$ contains exactly n_2 ordinary points of P.

If r_c lies below l_5, then the rooted forest $\langle B_1 \cup \{v_1\}\rangle_{T_1} \cup \langle B_4\rangle_{T_1}$ with roots v_1 and w_2 is strongly line embedded onto $(Rgn(r_c q_2 p_1 r_3) \cap O(P)) \cup \{p_1, q_2\}$ with two specified points p_1 and q_2. Furthermore, by Lemma 4, the rooted forest $\langle B_3 \cup \{w_1\}\rangle_{T_1} \cup T_2$ with root w_1 and v_2 is strongly line embedded onto $(Rgn(r_a q_1 q_2 r_c) \cap O(P)) \cup \{q_1, p_2\}$ with two specified points q_1 and p_2. Hence, by adding the line segment $\overline{q_1 q_2}$ to the above embeddings, we obtain a strong line embedding of $T_1 \cup T_2$ onto P. Therefore we may assume that r_c lies above l_5.

Let A, B, C be the intersection points of r_4 and r_c, of r_4 and l_5, and of r_4 and r_b, respectively (see Fig. 6). Then, by the choice of q_1 and q_2, the open triangle $\triangle(p_1 q_2 B)$ contains no ordinary point of P. Moreover, it follows from the choice of r_4 and r_b that the number of ordinary points of P contained in the open triangle $\triangle(B q_2 C)$ is equal to that of those points contained in the open region $Rgn(r_b C r_4)$. Thus the open region $Rgn(r_c A r_4)$ contains n_2 ordinary points. Then the rooted tree T_2 with root v_2 is strongly line embedded onto $(Rgn(r_4 A r_c) \cap O(P)) \cup \{p_2\}$ with specified point p_2.

By Lemma 4, the rooted forest $\langle B_1 \cup \{v_1\}\rangle_{T_1} \cup \langle B_4\rangle_{T_1}$ with roots v_1 and w_2 is strongly line embedded onto $(Rgn(r_c q_2 p_1 r_3) \cap O(P)) \cup \{p_1, q_2\}$ with two specified points p_1 and q_2. Moreover, since the open region $Rgn(r_a q_1 B r_4)$ contains $|B_3|$ ordinary points of P, the rooted tree $\langle B_3 \cup \{w_1\}\rangle_{T_1}$ with root w_1 is strongly line embedded onto $(Rgn(r_a q_1 B r_4) \cap O(P)) \cup \{q_1\}$ with specified point q_1. By adding the line segment $\overline{q_1 q_2}$ to the above embeddings, we obtain the desired strong line embedding of $T_1 \cup T_2$ onto P.

Note that if r_c intersects r_3 at a point X, then we consider the open triangle $\triangle(p_1 q_2 X)$ and the open region $Rgn(r_4 A r_c X r_3)$ instead of the open regions $Rgn(r_3 p_1 q_2 r_c)$ and $Rgn(r_4 A r_c)$, and then we obtain the desired strong line embedding of $T_1 \cup T_2$.

Consequently the proof is complete. □

We now show that the proof of Theorem 1 gives a $O(|F|^2 \log|F|)$ time algorithm for finding a strong line embedding of $F = T_1 \cup T_2$. We need the following theorem and lemma.

Theorem D [1]. *Let T be a rooted tree and let P be a set of $|T|$ points in the plane in general position containing a specified point. Then we can strongly embed T onto P in $O(|T| \log|T|)$ time.*

Theorem E (Theorem 3.7 of [6]). *The convex hull of n points in the plane can be found in $O(n \log n)$ time.*

We can find the vetex v of a tree T in Lemma 2 in $O(|T| \log|T|)$ time, and find a strong line embedding of a tree T mentioned in Lemma 3 in $O(|T|^2 \log|T|)$ time by making use of Theorems E and D. By similar arguments, we can show that a strong line embedding of a rooted forest F mentioned in Lemma 4 and one of a rooted forest F

mentioned in Lemma 5 are found in $O(|F|^2 \log|F|)$ time each. Therefore we can say that the required strong line embedding of a rooted forest F consisting of two rooted trees is also found in $O(|F|^2 \log|F|)$ time by the above results and by the proof of the theorem.

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