Discrete Comput Geom 21:243–255 (1999)

Discrete & Computational
Geometry

Resolvable Representation of Polyhedra*

K. Sugihara

Department of Mathematical Engineering and Information Physics,
University of Tokyo, Hongo, Bunkyo-ku,
Tokyo 113-8656, Japan

Abstract. The paper proposes a new method for the boundary representation of three-dimensional (not necessarily convex) polyhedra, called a *resolvable representation*, in which small numerical errors do not violate the symbolic part of the representation. In this representation, numerical data are represented partly by the coordinates of vertices and partly by the coefficients of face equations in such a way that the polyhedron can be reconstructed from the representation in a step-by-step manner. It is proved that any polyhedron homeomorphic to a sphere has a resolvable representation, and an algorithm for finding such a representation is constructed.

1. Introduction

Establishing numerically robust algorithms is one of the most important problems in practical geometric design systems [3], [4], and indeed many methods have been proposed for this purpose [1], [2], [5], [9], [12], [13]. However, those algorithms are valid only when the inputs are given correctly. Hence, it is another important problem to guarantee the consistency of the inputs to geometric algorithms. This problem is not trivial if the inputs are polyhedra. This paper concentrates on the way of representing three-dimensional polyhedral objects which gives no inconsistency if numerical errors are small in comparison with the minimum separation of geometric elements.

The boundary representation of a polyhedron consists of symbolic data and numerical data [4], [8]. The symbolic data describe the combinatorial topological structure of the polyhedron; they include the incidence relations among vertices, edges, and faces, and the nesting relations between faces and holes. The numerical data, on the other hand,

* This work was partly supported by the Grant in Aid for Scientific Research of the Ministry of Education, Science, Sports, and Culture of Japan.

describe the location of geometric elements; typical data include the coordinates of vertices and/or the coefficients of the equations of the planes containing faces.

Numerical errors sometimes give inconsistency in the representation. For example, if the locations of all the vertices are specified by their coordinates, the errors may violate the coplanarity of four vertices that are symbolically on a common face. This kind of difficulty mainly stems from redundancy of the numerical data; each part of redundant data is contaminated by errors independently and thus becomes inconsistent with other parts. Hence, to choose a subset of numerical data that is nonredundant and that is still enough to specify the polyhedron unambiguously is one of the most important problems for establishing numerically robust representations of polyhedra.

In this paper we define one way of nonredundant representation, called "resolvable representations," for polyhedral objects, and show that a certain class of polyhedra always have such representations. We also present an algorithm for finding a resolvable representation.

2. Step-by-Step Reconstructibility

Throughout this paper a *polyhedron* means a closed set of points in the three-dimensional Euclidean space bounded by a finite number of planar faces, and its boundary consists of a finite number of closed 2-manifolds.

First, we consider a simple example of a polyhedron, a pyramid with a quadrilateral base depicted in Fig. 1, and describe the basic idea of our new representation. This polyhedron has five vertices $v_1, v_2, \ldots, v_5$ and five faces $f_1, f_2, \ldots, f_5$, among which v_1 is on four faces and f_5 has four vertices.

Suppose that, for each vertex v_i $(i = 1, 2, \ldots, 5)$, the three-dimensional coordinates (x_i, y_i, z_i) are given, and, for each face f_j $(j = 1, 2, \ldots, 5)$, the coefficients a_j, b_j, c_j, d_j of the equation

$$a_j x + b_j y + c_j z + d_j = 0$$

of the plane containing f_j are given.

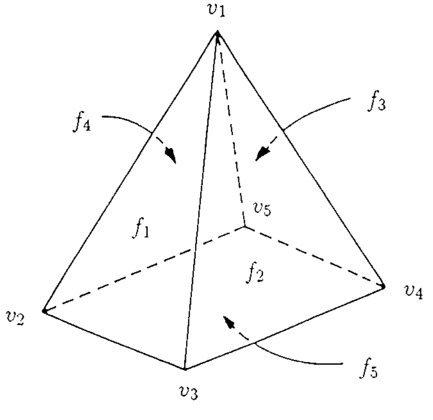

Fig. 1. Pyramid with a quadrilateral base.

Since the data are of finite precision, they may have rounding errors and consequently they may not be consistent with the symbolic data; the four points (x_i, y_i, z_i) $(i = 2, 3, 4, 5)$ may not be coplanar, or the four planes specified by (a_j, b_j, c_j, d_j) $(j = 1, 2, 3, 4)$ may not have a common point of intersection. So we consider these numerical data "tentative," and seek a mechanism for defining the polyhedron without inconsistency.

Let S be the sequence of the vertices and the faces defined by

$$S = (v_1,\ f_5, v_2, v_3, v_4, v_5,\ f_1,\ f_2,\ f_3,\ f_4).$$

The vertices and the planes can be placed in the space one by one in the order specified by S in the following manner. First, v_1 is placed at the point defined by (x_1, y_1, z_1). Next, f_5 is placed as the plane specified by $a_5x + b_5y + c_5z + d_5 = 0$. Then vertex v_2, the third element in S, cannot in general be placed at (x_2, y_2, z_2), because v_2 should be on f_5 and f_5 has already been placed. So we place v_2 at the foot of the perpendicular dropped from (x_2, y_2, z_2) to the plane f_5. Similarly v_3, v_4, and v_5 are placed on f_5 nearest to (x_i, y_i, z_i) $(i = 3, 4, 5)$. Next, plane f_1, the seventh element in S, is placed in such a way that it contains v_1, v_2, and v_3. Planes f_2, f_3, and f_4 are placed similarly.

Next, for the same pyramid, we consider another sequence

$$S' = (v_1, v_2, v_3, v_4, v_5,\ f_1,\ f_2,\ f_3,\ f_4,\ f_5).$$

The step-by-step reconstruction based on this sequence will fail. Indeed, we first place all five vertices, and then try to place the planes, but plane f_5 cannot be placed because the four vertices, v_2, v_3, v_4, v_5, are not guaranteed to be coplanar. We need to backtrack in order to adjust the locations of the vertices.

From the above observation, we see that some sequences allow us to reconstruct the polyhedron in the step-by-step manner, but some do not. The goal of the present paper is to characterize the former class of sequences and to construct an algorithm for finding them.

3. Resolvable Sequence

Let P be a polyhedron in the three-dimensional space, and let V and F be the set of vertices and that of faces of P. Let R be the set of all pairs $(v,\ f)$ of vertex v $(\in V)$ and face f $(\in F)$ such that v is on f. If $(v,\ f) \in R$, we say that v and f are *incident* to each other. Triple $I = (V, F, R)$ is called the *incidence structure* of P. The incidence structure can be visually represented by a bipartite graph having the left node set V, the right node set F, and the arc set R.

Let $n = |V \cup F|$, where $|X|$ denotes the number of elements of set X. Let

$$S = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

be a sequence obtained by giving a total order to the elements of $V \cup F$; thus $\alpha_i \in V \cup F$ for $i = 1, \ldots, n$, and $\alpha_i \neq \alpha_j$ for $i \neq j$. S is said to be *resolvable* if the following three conditions are satisfied:

(C1) For all i $(1 \leq i \leq n)$, α_i is incident to at most three preceding elements.

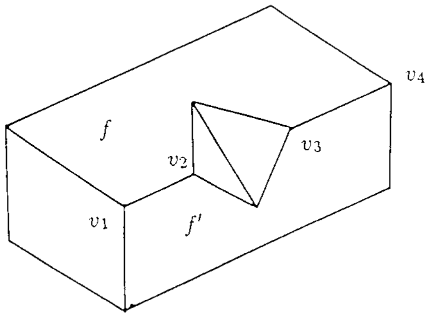

Fig. 2. Polyhedron whose incidence structure contains $K_{3,2}$.

(C2) If two faces f and f' are incident to three or more common vertices, both f and f' appear earlier than the third of the common vertices.
(C3) If two vertices v and v' are incident to three or more faces, both v and v' appear earlier than the third of the common faces.

Consider the pyramid in Fig. 1 again. In this polyhedron no two faces are incident to three or more common vertices and no two vertices are incident to three or more common faces, and hence (C2) and (C3) are trivially satisfied. The sequence $S = (v_1, f_5, v_2, v_3, v_4, v_5, f_1, f_2, f_3, f_4)$ satisfies (C1) while $S' = (v_1, v_2, v_3, v_4, v_5, f_1, f_2, f_3, f_4, f_5)$ does not, and hence S is resolvable whereas S' is not.

Next, consider the polyhedron shown in Fig. 2, where two faces f and f' are incident to four common vertices v_1, v_2, v_3, and v_4. The sequence

$$S = (\cdots v_1 \cdots v_2 \cdots f \cdots f' \cdots v_3 \cdots v_4 \cdots)$$

satisfies (C2), whereas the sequence

$$S' = (\cdots v_1 \cdots v_2 \cdots f \cdots v_3 \cdots f' \cdots v_4 \cdots)$$

does not, because in S' the face f' appears later than the third of the four vertices.

For the polyhedron shown in Fig. 3, the two vertices v and v' are incident to three common faces f_1, f_2, and f_3. The sequence

$$S = (\cdots f_1 \cdots v \cdots v' \cdots f_2 \cdots f_3 \cdots)$$

satisfies (C3), whereas the sequence

$$S' = (\cdots f_1 \cdots v \cdots f_2 \cdots f_3 \cdots v' \cdots)$$

does not, because in S' the vertex v' appears later than the three faces.

Suppose that $S = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a resolvable sequence of a polyhedron P. Then we can locate the vertices and the planes in the space step by step in this order. This can be understood in the following way.

First, assume that α_i is a face. Condition (C1) guarantees that α_i is incident to at most three preceding vertices. If α_i is incident to no preceding vertex, α_i is placed as the plane

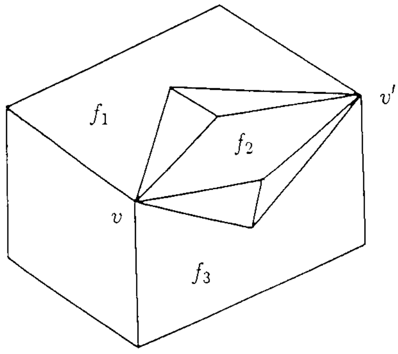

Fig. 3. Polyhedron whose incidence structure contains $K_{2,3}$.

$a_i x + b_i y + c_i z + d_i = 0$. If α_i is incident to one preceding vertex, say v_j, then α_i is placed as the plane $a_i x + b_i y + c_i z + d = 0$ where d is chosen in such a way that the plane contains v_j. If α_i is incident to two preceding vertices, say v_j and v_k, then α_i is placed as the plane that contains v_j and v_k and whose normal is nearest to the normal of the plane $a_i x + b_i y + c_i z + d_i = 0$. If α_i is incident to three preceding vertices, say v_j, v_k, and v_l, then α_i is placed as the plane containing v_j, v_k, and v_l; this procedure fails if there is a preceding face f_m which is also incident to all three vertices v_j, v_k, and v_l (an example of this case arises if $\alpha_i = f$, $v_j = v_1$, $v_k = v_2$, $v_l = v_3$, and $\alpha_m = f'$ in Fig. 2), but this does not happen because of condition (C2).

Next, assume that α_i is a vertex. Condition (C1) guarantees that α_i is incident to at most three preceding faces, and the situation is dual to the case where α_i is a face. If α_i is incident to none, one, two, or three preceding faces, then α_i is placed at the point on the associated plane(s) nearest to (x_i, y_i, z_i). The procedure fails if α_i is incident to three preceding faces and there is another vertex α_j that precedes α_i and that is also incident to all three faces (an example of this case arises if $\alpha_i = v$ and $\alpha_j = v'$ in Fig. 3); however, this situation does not happen because of (C3).

Thus, the resolvable sequence indicates the way to specify the locations of the vertices and the faces in a step-by-step manner. We say a polyhedron P is *resolvable* if it has a resolvable sequence S. The next question we ask is: what class of polyhedra have resolvable sequences and how can they be found?

4. Resolvable Polyhedra

A polyhedron is not necessarily resolvable. An example of an unresolvable polyhedron is shown in Fig. 4. This polyhedron is composed of three triangular prisms connected together, forming an object homeomorphic to a torus. Every vertex is incident to four faces and every face is incident to four vertices. Hence, for any order $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of elements in $V \cup F$, the last element α_n is incident to four preceding elements, which violates the condition for a resolvable sequence. Difficulty in numerical specification of this object was also pointed out from another point of view [11].

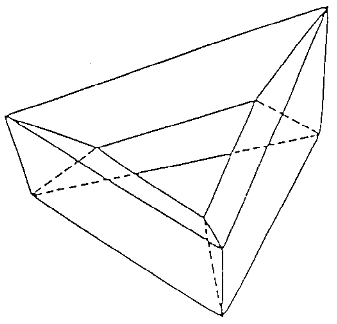

Fig. 4. Unresolvable polyhedron.

Let V and E be the set of vertices and that of edges of polyhedron P. The pair (V, E) can be considered a graph. We call this graph the *skeleton* of P. Graph G is said to be *planar* if G can be embedded in the plane without any intersection of edges except at their end vertices. For positive integer i, G is said to be *i-connected* if deletion of any $(i-1)$ vertices does not make the resulting graph disconnected. "1-connected" is simply called "connected."

A polyhedron is said to be *simply connected* if it is homeomorphic to a ball. This definition is equivalent to saying that a polyhedron is simply connected if its boundary is homeomorphic to a sphere.

The next theorem is well known.

Theorem 1 [10]. *A graph G is the skeleton of a convex polyhedron if and only if G has at least four vertices, and G is planar and* 3*-connected.*

In the textbook [7], Lyusternik proved the next theorem and used it as a lemma to prove Theorem 1 (though he did not use the term "resolvable").

Theorem 2 [7]. *Any polyhedron whose skeleton is planar and* 3*-connected is resolvable.*

Theorem 2 together with Theorem 1 implies that every convex polyhedron is resolvable. The proof of Theorem 2 is rather complicated. Here, we prove the following stronger theorem; the proof also gives a much simpler proof to Theorem 2.

Theorem 3. *Any simply connected polyhedron is resolvable.*

In Section 5 we give a new and simpler proof to Theorem 2, and in Section 6 we prove Theorem 3.

5. Proof of Theorem 2

Let P be a polyhedron having the vertex set V and the face set F, and let $I = (V, F, R)$ be the incidence structure of P. For any subsets $V' \subseteq V$ and $F' \subseteq F$, we define $R(V', F')$ as the set of incidence pairs related to the vertices and the faces in $V' \cup F'$. That is,

$$R(V', F') = R \cap (V' \times F').$$

The triple $(V', F', R(V', F'))$ is called the *substructure* of the incidence structure (V, F, R) induced by V' and F'. Recall that (V, F, R) can be considered a bipartite graph; hence the substructure $(V', F', R(V', F'))$ can be considered a subgraph of the bipartite graph, and so it is also bipartite.

Lemma 1. *Let P be a simply connected polyhedron, and let (V, F, R) be the incidence structure of P. For any subsets $V' \subseteq V$ and $F' \subseteq F$ the next inequality holds*:

$$|R(V', F')| \leq 2|V' \cup F'| - 4. \tag{1}$$

Proof. Let P be the polyhedron stated in the lemma, and let $G = (V, E)$ be its skeleton. Since the boundary of P is homeomorphic to a sphere, G has a natural embedding in the sphere that partitions the sphere into connected regions that have the one-to-one correspondence to the faces of P. From this embedding we generate another embedded graph $H = (N, A)$ whose node set is $N = V \cup F$ and whose arc set is $A = R$ (note that we refer to *nodes* and *arcs*, instead of vertices and edges, when we talk about the new graph $H = (N, A)$). The graph H is nothing but an embedded version of the bipartite graph $I = (V, F, R)$. H is planar, and gives another partition of the sphere into connected regions. Each region is bounded by exactly four arcs (note that each region has four nodes corresponding to the two endpoints of an edge and the two side faces).

Next, we delete from the graph $H = (N, A)$ the nodes (i.e., vertices and faces) in $V \cup F - V' \cup F'$ and the arcs incident to these nodes, and let the resulting graph be $H' = (N', A')$. H' coincides with the bipartite graph $(V', F', R(V', F'))$ stated in the lemma, and hence $N' = V' \cup F'$ and $A' = R(V', F')$. Let c be the number of connected components of H'. H' is also embedded on the sphere, so that H' partitions the sphere into connected regions; let W' be the set of these connected regions.

From Euler's formula, we get

$$|N'| - |A'| + |W'| = 1 + c. \tag{2}$$

Since each region in W is bounded by four or more edges, we have

$$2|A'| \geq 4|W'|. \tag{3}$$

From (2), (3), and $c \geq 1$, we obtain $2|N'| \geq |A'| + 4$; thus we get the lemma. □

Proof of Theorem 2. Suppose that P is a polyhedron such that the skeleton G of P is planar and 3-connected. Then, since G is 3-connected, no two faces are incident to three common vertices, and no two vertices are incident to three common faces. Hence conditions (C2) and (C3) are trivially satisfied.

Let (V, F, R) be the incidence structure of P. From Lemma 1, we get

$$\frac{2|R(V, F)|}{|V \cup F|} \leq 4 - \frac{8}{|V \cup F|} < 4.$$

The left-hand side of this inequality represents the average number of incidence relations per element, and the inequality implies that this average number is less than 4. Hence, there is an element in $V \cup F$ that is incident to three or fewer elements. Let this element be α_n, where $n = |V \cup F|$, and consider α_n the last element in the sequence S which we want to construct.

Next, we delete from the bipartite graph (V, F, R) the node α_n and the arcs incident to α_n; let the resulting incidence structure be $(V', F', R(V', F'))$. From Lemma 1, we get

$$\frac{2|R(V', F')|}{|V' \cup F'|} < 4,$$

which implies that there is an element in $V' \cup F'$ that is incident to three or fewer elements in $V' \cup F'$. We denote this element α_{n-1}, and add it to S as the second element from the end. Repeating this procedure, we obtain a resolvable sequence. □

6. Proof of Theorem 3

As before, we regard the incidence structure $I = (V, F, R)$ as the bipartite graph whose "left" node set is V and "right" node set is F. Let $K_{i,j}$ be the complete bipartite graph with i left nodes and j right nodes. For two sequences S and S', let $S \circ S'$ denote the concatenation of S and S'.

Lemma 2. *Let $G = (V, E)$ be a planar 3-connected skeleton and let $I = (V, F, R)$ be the bipartite graph associated with G. Let $Q \subseteq V \cup F$ be a subset such that the nodes in Q form a subgraph of I isomorphic to $K_{1,1}$, $K_{1,2}$, $K_{2,1}$, or $K_{2,2}$. Let S' be an arbitrary sequence of all the elements of Q. There exists a sequence S'' of all the elements of $V \cup F - Q$ such that $S = S' \circ S''$ is a resolvable sequence of I.*

Proof. We prove the case where Q forms a graph isomorphic to $K_{2,2}$; the other cases are easier to prove. We name the two vertices in Q as v_1 and v_2, and the two faces in Q as f_1 and f_2. We can construct $S'' = (\alpha_5, \alpha_6, \ldots, \alpha_n)$ (where $\alpha_i \in V \cup F - Q$, $i = 5, 6, \ldots, n$, $n = |V \cup F|$) from the tail backward in the following way. Assume that we have chosen the last $n - k$ $(k \geq 5)$ elements $\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n$ of S'' such that $\{\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n\} \cap Q = \emptyset$. Let $V' = V - \{\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n\}$, $F' = F - \{\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n\}$, and $I' = (V', F', R(V', F'))$.

Case 1: *Suppose that I' is connected*. For $\alpha \in V' \cup F'$, let $\mu(\alpha)$ be the number of elements in $V' \cup F'$ that are incident to α, and let $T = \{\alpha \mid \alpha \in V' \cup F',\ \mu(\alpha) \leq 3\}$. We get

$$\begin{aligned} 2|R(V', F')| &= \sum_{\alpha \in V' \cup F'} \mu(\alpha) \\ &\geq \sum_{\alpha \in T} \mu(\alpha) + 4|V' \cup F' - T|. \end{aligned} \tag{4}$$

From inequalities (1) and (4), we have

$$\sum_{\alpha \in T} (4 - \mu(\alpha)) \geq 8. \tag{5}$$

On the other hand, we have

$$\sum_{\alpha \in Q} (4 - \mu(\alpha)) \leq 7, \tag{6}$$

because the nodes in Q form the complete bipartite graph $K_{2,2}$ and there is at least one arc connecting Q and $V' \cup F' - Q$ (recall that I' is connected). From (5) and (6), we conclude that $T - Q \neq \emptyset$, that is, there is at least one node in $V' \cup F'$ that is incident to at most three elements in $V' \cup F'$ and that is different from v_1, v_2, f_1, or f_2. We name this node α_k.

Case 2: *Suppose that I' is not connected*. We rename a connected component that does not include Q as I'. For this component we get inequality (5), which tells that there is a node, say α_k, that is incident to at most three nodes in $V' \cup F'$.

In both cases, we add α_k at the head of the sequence $(\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n)$, thus obtaining a one-longer sequence $(\alpha_k, \alpha_{k+1}, \ldots, \alpha_n)$. Repeating this step, we finally get the desired sequence $S = S' \circ (\alpha_5, \alpha_6, \ldots, \alpha_n)$. □

Let P be a simply connected polyhedron, and let $G = (V, E)$ and $I = (V, F, R)$ be the skeleton and the incidence structure, respectively, of P. Since P is simply connected, G can be considered a graph embedded on the sphere. This embedded graph partitions the sphere into connected regions and their boundaries. The connected regions are in one-to-one correspondence with the faces of P. In order to represent this embedded structure explicitly, we sometimes denote it by $\bar{G} = (V, E, F)$ instead of $G = (V, E)$, where F is the set of the faces of P as before, and each face $f \in F$ is represented by one or more cycles corresponding to the boundary of the face. We call $\bar{G} = (V, E, F)$ an *embedded graph*.

We decompose G into 3-connected components. For this purpose, in the first stage we decompose G into connected components, next decompose them into 2-connected components, and finally decompose them into 3-connected components. The three decomposition stages are similar, and hence we show the decomposition of a 2-connected component into 3-connected components.

Let G be a 2-connected component obtained in the second stage, and let I and $\bar{G}$ be the associated incidence structure and the embedded graph. Suppose that G is not 3-connected. Then, as shown in Fig. 5(a), there exist two vertices v, v' and two faces f, f' such that v and v' are on both of the boundary of f and that of f', and that f, f', v, v' altogether separate the remaining area of the sphere into two connected areas. Let V_1 and F_1 be the vertices and the faces belonging to one of the connected areas, and let V_2 and F_2 be the set of the remaining vertices and that of the remaining faces. For $i = 1$ and 2, let E_i be the set of edges in the subgraph of G induced by the vertex set $V_i \cup \{v, v'\}$.

The bipartite graph associated with I has the structure as shown in Fig. 5(b), that is, the four nodes v, v', f, f' form $K_{2,2}$, and there is no arc between V_1 and F_2 or between V_2 and F_1. We decompose $I = (V, F, R)$ into $I_1 = (V_1 \cup \{v, v'\}, F_1 \cup \{f, f'\}, R_1)$ and

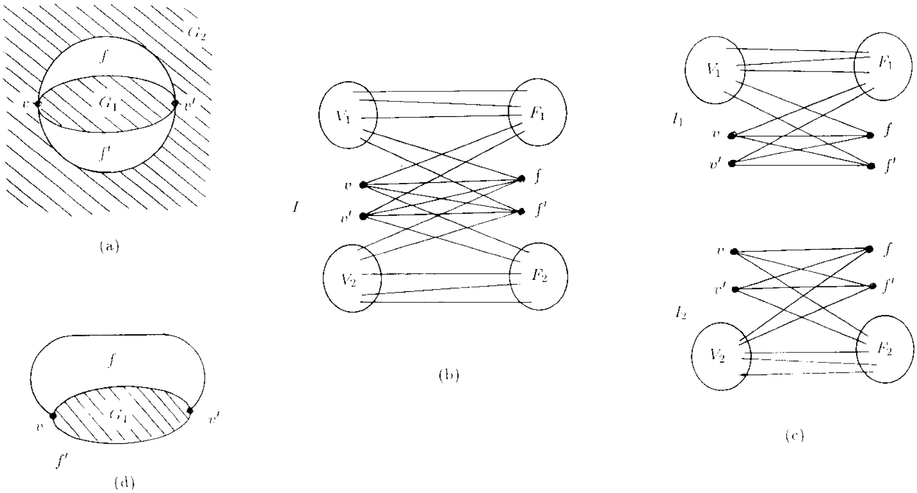

Fig. 5. Decomposition of the embedded graph into two parts with exactly two common faces and two common vertices.

$I_2 = (V_2 \cup \{v, v'\}, F_2 \cup \{f, f'\}, R_2)$ where $R_i = R \cap ((V_i \cup \{v, v'\}) \times (F_i \cup \{f, f'\}))$ $(i = 1, 2)$, as shown in Fig. 5(c).

In connection with this decomposition of I, the embedded graph $\bar{G}$ is also decomposed.

Let the resulting 3-connected components be $G_i = (V_i, E_i)$ and let the associated incidence structures and the embedded graphs be $I_i = (V_i, F_i, R_i)$ and $\bar{G}_i = (V_i, E_i, F_i)$, $i = 1, 2, \ldots, k$. Now we are ready to construct a resolvable sequence of P. We first choose one component, say G_1, and construct a resolvable sequence S_1; S_1 always exists because of Theorem 2.

Next, suppose that we have already constructed a resolvable sequence $S_1 \circ S_2 \circ \cdots \circ S_i$ of $G_1 \cup G_2 \cup \cdots \cup G_i$. We choose one of the remaining components, say G_{i+1}, whose incidence structure I_{i+1} has at least one face in common with one of $I_1, I_2, \ldots, I_i$ (here, if necessary, we rename the component numbers).

Consider the embedded graph $\bar{G}_{i+1} = (V_{i+1}, E_{i+1}, F_{i+1})$. Some of the faces in F_{i+1} are also contained in some other components $I_1, I_2, \ldots, I_{i-1}, I_{i+1}, \ldots, I_k$. We remove those faces. Then the remaining part of $\bar{G}_{i+1}$ forms a connected region possibly with holes. Note that $\bar{G}_{i+1}$ is embedded on the sphere, and hence there is no distinction between "inner holes" and the "outer hole."

Since I_j have been chosen in such a way that F_j and $F_1 \cup F_2 \cup \cdots \cup F_{j-1}$ have at least one common face $(j = 2, 3, \ldots, i)$, all the components $G_1, G_2, \ldots, G_i$ lie in one and the same hole of $\bar{G}_{i+1}$. Therefore, the vertices and the faces that are common in I_{i+1} and $S_1 \circ S_2 \circ \cdots \circ S_i$ are (i) only one face (this happens when a graph is decomposed into connected components), or (ii) one vertex and one face forming $K_{1,1}$ (this happens when a connected component is decomposed into 2-connected components), or (iii) two vertices and two faces forming $K_{2,2}$ (in the case shown in Fig. 5), or (iv) a subset of them. Therefore, Lemma 2 guarantees that there exists resolvable sequence $\hat{S}_{i+1}$ of I_{i+1}

such that elements also included in $S_1 \circ S_2 \circ \cdots \circ S_i$ are at the head of $\hat{S}_{i+1}$. Let S_{i+1} be the sequence that is obtained from $\hat{S}_{i+1}$ by removing those common elements. Then the concatenation $S_1 \circ S_2 \circ \cdots \circ S_i \circ S_{i+1}$ is a resolvable sequence of $G_1 \cup G_2 \cup \cdots \cup G_i \cup G_{i+1}$.

Repeating this procedure, we finally construct a resolvable sequence of G. Thus, Theorem 3 has been proved.

7. Algorithms

If the skeleton is 3-connected, finding a resolvable sequence is straightforward. Given a polyhedron P, we first construct the associated bipartite graph $I = (V, F, R)$ and delete the nodes with degree 3 or less one by one, obtaining the resolvable sequence from the tail backward. This procedure runs in $O(|R| + |V \cup F|) = O(|R| + n) = O(n)$ time.

If the skeleton of P is not 3-connected, we first decompose it into 3-connected components, and next apply the above procedure. Since the decomposition of a planar graph into 3-connected components can be done in linear time [6], the total procedure requires in $O(n)$ time.

When we change the polyhedron P to P' by some operation, we in general have to reconstruct a resolvable sequence of P' from the beginning. However, for certain special types of operations, a resolvable sequence of P' can be obtained by slight modification of the resolvable representation of P. Here, we describe two such operations; they are duals of each other.

The first operation is to cut P by a plane into two parts and to remove one of them. Let P' be the polyhedron obtained by the cut operation. Let f_0 be the new face of P' generated by the cut operation, and let $v_1, v_2, \ldots, v_k$ be the vertices on the boundary of f_0. We call the operation a *nondegenerate cut* if none of $v_1, v_2, \ldots, v_k$ is a vertex of P. If the cut is nondegenerate, the new vertices $v_1, v_2, \ldots, v_k$ are generated on the middle of edges of P, and consequently these new vertices are incident to exactly three faces in P'.

Theorem 4. *Let S be a resolvable sequence of polyhedron P. Let P' be a polyhedron obtained from P by nondegenerate cut, and let f_0 and $v_1, v_2, \ldots, v_k$ be the new face and the new vertices generated by the cut. Then the sequence*

$$S' = \overline{S} \circ (f_0, v_1, v_2, \ldots, v_k)$$

is a resolvable sequence of P', where $\overline{S}$ is the sequence obtained from S by deleting the faces and the vertices not belonging to P'.

Proof. Since $\overline{S}$ is a subsequence of S, $\overline{S}$ is also a resolvable sequence. Since the new face f_0 contains no old vertices, $\overline{S} \circ (f_0)$ is a resolvable sequence. Since the cutting is nondegenerate, each of $v_1, v_2, \ldots, v_k$ is incident to exactly three faces. Hence, S' is a resolvable sequence of P'. □

The other operation we consider is the following. Let P be a convex polyhedron and let v_0 be a point outside P. Let $P' = \mathrm{CH}(P \cup \{v_0\})$, where $\mathrm{CH}(X)$ denotes a convex hull of the set X of points. The construction of P' from P and v_0 is called a *pull operation*; note that intuitively P' is the result of pulling a point of elastic surface of P and moving it to v_0. We say that the pull operation is *nondegenerate* if v_0 is not coplanar of any face of P.

Theorem 5. *Let S be a resolvable sequence of convex polyhedron P. Let P' be a polyhedron obtained from P and v_0 by a nondegenerate pull operation, and let $f_1, f_2, \ldots, f_k$ be the new faces generated by the operation. The sequence*

$$S' = \overline{S} \circ (v_0, f_1, f_2, \ldots, f_k)$$

is a resolvable sequence of P', where $\overline{S}$ is the sequence obtained from S by deleting the faces and vertices not belonging to P'.

The proof is similar to that of Theorem 4.

8. Concluding Remarks

We proposed a new method for representing polyhedra, called resolvable representation, in which the numerical part is defined by a step-by-step manner according to a special sequence of vertices and faces, so that numerical errors do not violate the symbolic part of the representation. We showed that any simply connected polyhedron has a resolvable representation, and constructed an algorithm for finding it.

In general, the resolvable sequence is not unique; we have large freedom in the choice of the resolvable sequence of a given polyhedron. Different resolvable sequences define different geometric shapes for the the same numerical error. Hence, the next problem is to analyze the sensitivity, and to find the resolvable sequence that is least sensitive to numerical errors. Other problems for the future include (1) characterizing the resolvable polyhedra that are not homeomorphic to the ball, (2) constructing an algorithm for finding resolvable sequences for such polyhedra, and (3) generalizing the concept of "resolvable representation" to curved surface objects.

If polyhedron P is not resolvable, we can modify it into a resolvable polyhedron by inserting face diagonals (note that if all the faces are decomposed into triangles by inserting diagonals, the polyhedron has a resolvable sequence). Hence still another future problem is to find the minimum set of diagonals whose insertion makes the polyhedron resolvable.

Acknowledgment

The author expresses his thanks to the anonymous referees whose comments helped in improving this paper.

References

1. S. Fortune, Stable maintenance of point-set triangulations in two dimensions, *Proceedings of the* 30*th IEEE Annual Symposium on Foundations of Computer Science*, Research Triangle Park, 1989, pp. 494–499.
2. D. H. Greene and F. Yao, Finite-resolution computational geometry, *Proceedings of the* 27*th IEEE Annual Symposium on Foundations of Computer Science*, Toronto, 1986, pp. 143–152.
3. C. M. Hoffmann, The problem of accuracy and robustness in geometric computation, *Computer*, **22**(3) (March 1988), 31–41.
4. C. M. Hoffmann, *Geometric and Solid Modeling—An Introduction*, Morgan Kaufmann, San Mateo, California, 1989.
5. J. E. Hopcroft and P. J. Kahn, A paradigm for robust geometric algorithms, Technical Report TR 89-1044, Department of Computer Science, Cornell University, Ithaca, New York, 1989.
6. J. E. Hopcroft and R. E. Tarjan, Dividing a graph into triconnected components. *SIAM Journal on Computing*, **2** (1973), 135–158.
7. L. A. Lyusternik (T. J. Smith, trans.), *Convex Figures and Polyhedra*, Dover, New York, 1963.
8. M. Mäntylä, *An Introduction to Solid Modeling*, Computer Science Press, Rockville, Maryland, 1988.
9. V. Milenkovic, Verifiable implementations of geometric algorithms using finite precision arithmetic, *Artificial Intelligence*, **37** (1988), 377–401.
10. E. Steinitz, *Polyeder und Raumeinteilungen*, Encyklopädie der mathematischen Wissenschaften, Band III, Teil 1, 2, Hälfte, IIIAB12, Akademie der Wissenschaften, Berlin, 1916, pp. 1–139.
11. K. Sugihara, *Machine Interpretation of Line Drawings*, MIT Press, Cambridge, Massachusetts, 1986.
12. K. Sugihara and M. Iri, A solid modelling system free from topological inconsistency, *Journal of Information Processing*, **12** (1989), 380–393.
13. K. Sugihara and M. Iri, Construction of the Voronoi diagram for "one million" generators in single-precision arithmetic, *Proceedings of the IEEE*, **80**(9) (September 1992), 1471–1484.

Received January 21, 1997, *and in revised form April* 29, 1998.