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A Helly Type Conjecture*

M. Katchalski and D. Nashtir

Mathematics, Technion,
Haifa 32000, Israel
meirk@techunix.technion.ac.il

Abstract. A family of sets is Π^n, or n-pierceable, if there exists a set of n points such that each member of the family contains at least one of them. It is Π^n_k if every subfamily of size k or less is Π^n. Helly's theorem is one of the fundamental results in Combinatorial Geometry. It asserts, in the special case of finite families of convex sets in the plane, that Π^1_3 implies Π^1. However, there is no k such that Π^2_k implies 2-pierceability for all finite families of convex sets in the plane. It is therefore natural to propose the following:

Conjecture. *There exists a k_0 such that, for all planar finite families of convex sets, $\Pi^2_{k_0}$ implies Π^3.*

Proofs of this conjecture for restricted families of convex sets are discussed.

1. Introduction

A family of sets is Π^n, or n-pierceable if there exists a set of n points such that each member of the family contains at least one of them. It is Π^n_k if every subfamily of size k or less is Π^n.

Helly's theorem is one of the fundamental results in Combinatorial Geometry. It asserts that for finite families of convex sets in the plane Π^1_3 implies Π^1. Consult the excellent surveys [3] and [4] for references to Helly's theorem and its many relatives and applications.

Does Π^2_k imply Π^2 for finite planar families of convex sets? For rectangles, with sides parallel to the axis, Π^2_5 implies Π^2, see [2], and for homothetic triangles Π^2_9 implies Π^2, see [8] and [9]. However, even for translates of a symmetric convex hexagon there is no k_0 such that $\Pi^2_{k_0}$ implies Π^2, see [8] and [9]. It seems natural to propose the following:

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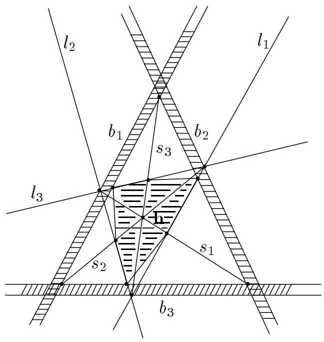

Fig. 1. Π_5^2 does not imply Π^3.

Conjecture 1. *There exists a k_0 such that, for all finite planar families of convex sets, $\Pi_{k_0}^2$ implies Π^3.*

Figure 1 shows that Π_5^2 does not imply Π^3 even for planar families of ten convex sets. This counterexample consists of a hexagon h, three segments s_1, s_2, s_3, three strips b_1, b_2, b_3, and three lines l_1, l_2, l_3.

The existence of an integer m_0 such that Π_5^2 implies Π^{m_0} follows from a recent result of Alon and Kleitman [1]. They settled a conjecture of Debrunner and Hadwiger and proved, among other things, the existence of an integer m_0 such that every finite planar family of convex sets is Π^{m_0} provided it has the property that every five membered subfamily contains three members with nonempty intersection. (However, m_0 is of the order 10^3.) Note that Π_4^2 does not imply Π^s for every finite planar family of convex sets, no matter how large s is: consider families of $2s+1$ segments such that any two intersect but no three intersect.

Here we prove Conjecture 1 for certain planar families of convex sets.

First, some terminology:

Let x^- and x^+ denote half-planes bounded by the line x with $x^+ \cap x^- = x$.

Two half-planes are *related* if one of them is a translate of the other. Related half-planes are ordered by inclusion so that $x^+ < y^+$ implies that x^+ is contained in y^+ and $x^+ \neq y^+$.

A half-plane x^+ (x^-) supports a set C if it contains it and $x \cap C \neq \emptyset$.

Let $x^+(C)$ denote the half-plane related to x^+ that supports C.

The convex polygon A is *related* to the convex m-gon $K = \bigcap_{i=1}^m k_i^+$, where $k_1^+, \ldots, k_m^+$ are the m half-planes whose intersection is equal to K, if A is the intersection of half-planes $a_1^+, \ldots, a_l$, each of which is a translate of one of the k_i^+'s. The family $\mathcal{A}$ is *related* to K if each $A \in \mathcal{A}$ is related to K. See Fig. 2 for seven convex polygons related to a quadrilateral satisfying Π_4^2 but not Π^3. Note that the concept "related" was introduced and used by Grünbaum.

For a family of sets $\mathcal{F}$ and a set s let $\bar{\mathcal{F}}(s) = \{F \in \mathcal{F} : F \cap s = \emptyset\}$ and for a point x let $\bar{\mathcal{F}}(x) = \bar{\mathcal{F}}(\{x\})$.

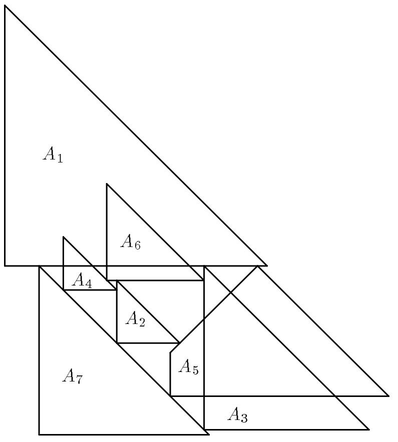

Fig. 2. $\Pi^2_4 \not\Rightarrow \Pi^3$ for a family related to a quadrilateral.

Finally, a family of triangles is *special* if there is a fixed angle α such that every triangle of the family contains an angle which is a translate of α.

The next three theorems from [10] support Conjecture 1:

Theorem 1. Π^2_{13n} *implies* Π^3 *for every finite planar family related to a convex n-gon.*

The proof is very technical and appears in [10], here we prove two related results.

Theorem 2. Π^2_8 *implies* Π^4 *for planar special families of triangles.*

Theorem 3. Π^2_{10} *implies* Π^3 *for planar families related to a convex quadrilateral.*

2. Proofs

Proof of Theorem 2. The proof relies on a theorem of Tardos on 2-intervals, see [11]. A 2-interval is a set which is the union of an interval on e and an interval on f where e and f are two fixed nonintersecting straight lines.

Tardos's Theorem. *If* $\mathcal{F}$ *is a family of* 2*-intervals and* $\mathcal{F}$ *does not contain* $k+1$ *pairwise disjoint members, then* $\mathcal{F}$ *is* Π^{2k}.

Tardos's theorem improves earlier results of Gyárfás and Lehel [6], and has been generalized to m-intervals, $m \geq 3$, by Kaiser [7]. (However, Kaiser's result is not best posible whereas Tardos's result is.) Our theorem uses Tardos's theorem only for the case $k = 2$.

Assume without loss of generality that $\mathcal{A}$ is finite, since all members of $\mathcal{A}$ are compact. Let $\mathcal{A} = \{A_i = h_i^+ \cap v_i^+ \cap d_i^+ : i \in [n] = \{1, 2, \ldots, n\}\}$ with $h_i^+ \cap v_i^+$ a translate of the

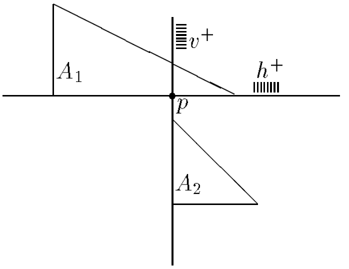

Fig. 3. The minimal half-planes h^+ and v^+ support triangles A_1 and A_2, respectively.

fixed angle α, for $i \in [n]$, with $H = \{h_i^+ : i \in [n]\}$ and $V = \{v_i^+ : i \in [n]\}$ being two families of related half-planes with $h^+ = \min H$ and $v^+ = \min V$.

Assume without loss of generality that $h^+ = h_1^+$ and $v^+ = v_2^+$ and let $p = h \cap v$, see Fig. 3.

It is easy to see that, for $A \in \mathcal{A}$:

1. If $x \in h^+ \cap A (x \in v^+ \cap A)$ and if x' is the projection of x on h (on v) along a translate of v (of h), then x' pierces A.
2. If $A \cap h^+ \cap v^+ \neq \emptyset$, then $p \in A$.
   Note that 2 follows from 1.

If $\bar{\mathcal{A}}(p) = \{A \in \mathcal{A} : p \notin A\}$ is Π^1, then $\mathcal{A}$ is Π^2, so assume that $\bar{\mathcal{A}}(p)$ is not Π^1 and apply Helly's theorem to obtain a subfamily $\mathcal{C} \subset \bar{\mathcal{A}}(p)$ with $|\mathcal{C}| \leq 3$ and $\cap\mathcal{C} = \emptyset$. Let $R_h = h \backslash v^+$ and $R_v = v \backslash h^+$ and let $\mathcal{A}' = \{A \cap (R_v \cup R_h) : A \in \mathcal{A}\}$. See Fig. 4.

$\mathcal{A}'$ is a family of 2-intervals since R_h and R_v are disjoint. By the Tardos theorem it suffices to prove that $\mathcal{A}'$ does not contain three pairwise disjoint 2-intervals.

Assume otherwise that $\mathcal{D}' \subset \mathcal{A}'$ is a subfamily of three pairwise disjoint 2-intervals and let $\mathcal{D} = \{A \in \mathcal{A} : A' \in \mathcal{D}'\}$. The subfamily $\mathcal{T} = \mathcal{D} \cup \mathcal{C} \cup \{A_1, A_2\}$ is Π^2 since $|\mathcal{T}| \leq 3 + 3 + 2 = 8$.

Let $\{X, Y\}$ pierce $\mathcal{T}$. If one of the two points pierces $\{A_1, A_2\}$, then it is in $h^+ \cap v^+$. So assume without loss of generality that by condition 2 above it is p. Then the other point pierces $\mathcal{C} \subset \bar{\mathcal{A}}(p)$—a contradiction since $\bigcap \mathcal{C} = \emptyset$.

Assume therefore without loss of generality that $\{X, Y\} \cap h^+ \cap v^+ = \emptyset$ and that

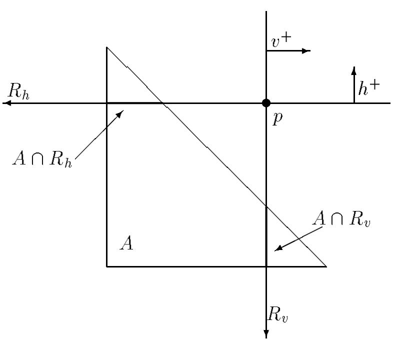

Fig. 4. The 2-interval $A' = A \cap (R_h \cup R_v)$ that corresponds to $A \in \mathcal{A}$.

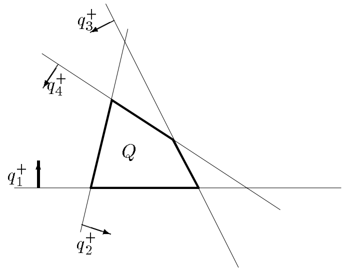

Fig. 5. A quadrilateral Q.

$X \in A_1$, $Y \in A_2$. If X' (resp. Y') is the projection of X (resp. of Y) on h (resp. on v) along a translate of v (resp. of h) then by condition 1 above $\{X', Y'\}$ pierce $\mathcal{T}$ and therefore $\mathcal{D}$. Together with $\{X', Y'\} \subset R_h \cup R_v$ this implies that $\mathcal{D}'$ is Π^2, so $\mathcal{D}'$ cannot consist of three pairwise disjoint intervals, a contradiction. □

Proof of Theorem 3. Assume that the family $\mathcal{F}$ is Π^2_{10} and is related to the convex quadrilateral $Q = \bigcap_{i=1}^4 q_i^+$, that no two sides of Q are parallel (make small changes in Q and members of $\mathcal{F}$ if necessary), and that both q_3 and q_4 intersect the two open rays $q_1 \backslash q_2^-$ and $q_2 \backslash q_1^-$ as in Fig. 5.

Note that $F = \bigcap_{i=1}^4 q_i^+(F)$ for $F \in \mathcal{F}$ where $q_i^+(F)$ is the minimal translate of q_i^+ containing F, for $1 \le i \le 4$. Assume without loss of generality that $q_i^+(A_i)$ is the minimum of $\{q_i^+(F) : F \in \mathcal{F}\}$ for $i = 1, 2, 3, 4$ (recall that related half-planes are ordered by inclusion). Let $\mathcal{A} = \{A_1, A_2, A_3, A_4\} \subseteq \mathcal{F}$ and $l_i^+ = q_i^+(A_i)$. Note that if $\bigcap_{i=1}^4 x_i^+ = \emptyset$ with x_i^+ related to q_i^+, then either $x_1^+ \cap x_2^+ \cap x_3^+ = \emptyset$ or $x_1^+ \cap x_2^+ \cap x_4^+ = \emptyset$.

It follows that if $\bigcap \mathcal{B} = \emptyset$ for $\mathcal{B} \subseteq \mathcal{F}$, then $\bigcap \mathcal{C} = \emptyset$ for a subfamily $\mathcal{C}$ of $\mathcal{B}$ of size at most 3, by Helly's theorem. Also if $A_i \in \mathcal{B}$ for $i = 1$ or 2 it can be assumed without loss of generality that $A_i \in \mathcal{C}$ and if $\{A_3, A_4\} \subset \mathcal{B}$ it can be assumed without loss of generality that either A_3 or A_4 belongs to $\mathcal{C}$.

Let $a_{ij} = l_i \cap l_j$ for $1 \le i < j \le 4$. If $l_1^+ \cap l_2^+ \cap l_3^+ \ne \emptyset$ and $l_1^+ \cap l_2^+ \cap l_4^+ \ne \emptyset$ then $a_{12} \in \bigcap \mathcal{F}$ and $\mathcal{F}$ is Π^1 and therefore Π^2. Assume therefore without loss of generality that $l_1^+ \cap l_2^+ \cap l_3^+ = \emptyset$. See Fig. 6.

There are three cases to consider:

Case 1. l_4^+ contains $\{a_{13}, a_{23}\}$.
Case 2. l_3^+ contains $\{a_{14}, a_{24}\}$.
Case 3. l_3^+ contains a_{14} but not a_{24} (l_3^+ contains a_{24} but not a_{14} is equivalent).

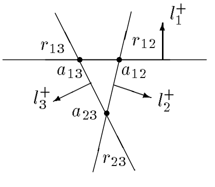

Fig. 6. $l_1^+ \cap l_2^+ \cap l_3^+ = \emptyset$.

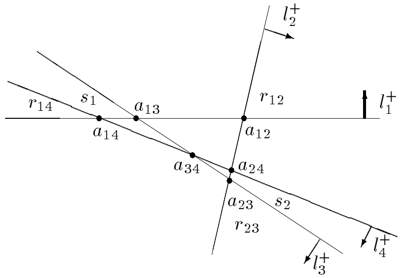

Fig. 7. l_3^+ contains a_{14} but not a_{24}.

We deal only with Case 3 as the other cases are even simpler. Let

$$s_1 = l_3^+ \cap l_1^+ \cap l_4^- \quad \text{and} \quad s_2 = l_2^+ \cap l_4^+ \cap l_3^-,$$
$$r_{ij} = l_i^+ \cap l_j^+ \qquad \text{for} \quad i \le i < j \le 4.$$

See Fig. 7.

Note that if $F \in \mathcal{F}$ meets both s_1 and s_2, then it is pierced by a_{34}. It follows that if $\bar{\mathcal{F}}(s_1)$ and $\bar{\mathcal{F}}(s_2)$ are both Π^1, then F is Π^3. So assume without loss of generality that $\bar{\mathcal{F}}(s_1)$ is not Π^1 and that $\mathcal{C}_{s_1} \subseteq \bar{\mathcal{F}}(s_1)$ with $|\mathcal{C}_{s_1}| \le 3$ and $A_2 \in \mathcal{C}_{s_1}$ (since $A_2 \in \bar{\mathcal{F}}(s_1)$).

If $\bar{\mathcal{F}}(a_{23})$ or $\bar{\mathcal{F}}(a_{12})$ are Π^1, then $\mathcal{F}$ is Π^2. So assume that both $\bar{\mathcal{F}}(a_{23})$ and $\bar{\mathcal{F}}(a_{12})$ are not Π^1. We show this to be impossible:

Otherwise let $\mathcal{C}_{23}$, $\mathcal{C}_{12}$ be subfamilies of $\bar{\mathcal{F}}(a_{23})$ and $\bar{\mathcal{F}}(a_{13})$ respectively, of size at most 3, which are both not Π^1 and with $A_1 \in \mathcal{C}_{23}$ and $\mathcal{C}_{12}$ containing one of the sets A_3 or A_4. Let $\mathcal{W} = \mathcal{A} \cup \mathcal{C}_{s_1} \cup \mathcal{C}_{12} \cup \mathcal{C}_{23}$. Then $|\mathcal{W}| \le 4 + 2 + 2 + 2 = 10$, so that $\mathcal{W}$ is Π^2 and let the 2-set B pierce $\mathcal{W}$. If a point of B pierces $A_1 \cap A_2$ or $A_2 \cap A_3$, then that point can be assumed to be a_{12} or a_{23} and then one of the families $\mathcal{C}_{12}$ or $\mathcal{C}_{23}$ is Π^1, a contradiction.

So a point of B pierces $A_1 \cap A_4$ or $A_1 \cap A_3$ and is therefore in $s_1 \cup r_{14}$. However, such a point, if it is in r_{14}, can be replaced by a_{14} which is in s_1. So assume without loss of generality that a point of B is in s_1, implying $\mathcal{C}_{s_1}$ is Π^1, a contradiction. □

3. Additional Results and Conjectures

The following results and conjectures are taken from [10].

Theorem 4. *For homothetic triangles in the plane Π_6^2 implies Π^3 and Π_{31}^3 implies Π^4.*

(In [9] it is shown that for no k does Π_k^3 imply Π^3 for all planar families of homothetic triangles.)

Conjecture 2. *For any integer $m \ge 2$ there exists an integer $k(m)$ such that $\Pi_{k(m)}^m$ implies Π^{m+1} for every finite planar family of convex sets.*

For $k = 2$ this is Conjecture 1. We could not resolve the conjecture even for homothetic triangles and $m \geq 4$.

Theorem 5. *For families of homothetic simplices in* $\mathbf{R}^3$, Π^2_{25} *implies* Π^3, *and there is no* s *such that* Π^2_s *implies* Π^2 *for all families of simplices in* $\mathbf{R}^3$.

We conclude with a conjecture of Grünbaum mentioned also in [4].

Conjecture 3. *A planar family of translates of a convex compact set is* Π^3 *provided that any two of its members intersect*.

See [5] where the conjecture is proved for translates of a compact symmetric convex set.

Acknowledgment

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