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Note on the Erdős–Szekeres Theorem*

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Abstract. Let $g(n)$ denote the least integer such that among any $g(n)$ points in general position in the plane there are always n in convex position. In 1935, P. Erdős and G. Szekeres showed that $g(n)$ exists and $2^{n-2}+1 \le g(n) \le \binom{2n-4}{n-2}+1$. Recently, the upper bound has been slightly improved by Chung and Graham and by Kleitman and Pachter. In this paper we further improve the upper bound to

$$g(n) \le \binom{2n-5}{n-2}+2.$$

In 1933, Esther Klein raised the following question. Is it true that for every n there is a least number $g(n)$ such that among any $g(n)$ points in general position in the plane there are always n in convex position?

This question was answered in the affirmative in a classical paper by Erdős and Szekeres [ES1]. In fact, they showed [ES1] and [ES2] that

$$2^{n-2}+1 \le g(n) \le \binom{2n-4}{n-2}+1.$$

The lower bound, $2^{n-2}+1$, is sharp for $n = 2, 3, 4, 5$ and has been conjectured to be

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sharp for all n. However, the upper bound, $\binom{2n-4}{n-2}+1 \approx c(4^n/\sqrt{n})$, was not improved for 60 years. Recently, Chung and Graham [CG] managed to improve it by 1. Shortly after, Kleitman and Pachter [KP] showed that $g(n) \le \binom{2n-4}{n-2}+7-2n$.

Inspired by these results, in this paper we get a further improvement, roughly by a factor of 2.

Theorem. *Any set of $\binom{2n-5}{n-2}+2$ points in general position in the plane contains n points in convex position.*

In other words, $g(n) \le \binom{2n-5}{n-2}+2$. Since $2\binom{2n-5}{n-2} = \binom{2n-4}{n-2}$, our upper bound is about half of the original bound of Erdős and Szekeres. In the original proof, Erdős and Szekeres were looking for special convex n-gons, namely for n-caps and n-cups.

Definition. The points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, $x_1 < x_2 < \cdots < x_n$, form an *n-cap* if

$$\frac{y_2 - y_1}{x_2 - x_1} > \frac{y_3 - y_2}{x_3 - x_2} > \cdots > \frac{y_n - y_{n-1}}{x_n - x_{n-1}}.$$

Similarly, they form an *n-cup* if

$$\frac{y_2 - y_1}{x_2 - x_1} < \frac{y_3 - y_2}{x_3 - x_2} < \cdots < \frac{y_n - y_{n-1}}{x_n - x_{n-1}}.$$

(See Fig. 1.)

Lemma [ES1]. *Let $f(n, m)$ be the least integer such that any set of $f(n, m)$ points in general position in the plane contains either an n-cap or an m-cup. Then*

$$f(n,m) = \binom{n+m-4}{n-2} + 1.$$

Proof of Theorem. Let P be a set of points in general position in the plane and suppose that P does not contain n points in convex position. Let a be a vertex of the convex hull of P. Let b be a point outside the convex hull of P such that none of the lines determined by the points of $P\setminus\{a\}$ intersects the segment $\overline{ab}$. Finally, let ℓ be a line through b which avoids the convex hull of P (see Fig. 2).

Consider a projective transformation T which maps the line ℓ to the line at infinity, and maps the segment $\overline{ab}$ to the vertical half-line $v^-(a')$, emanating downward from

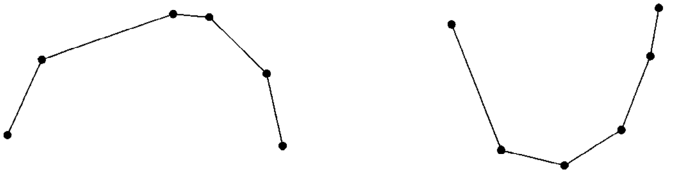

Fig. 1. A 6-cap and a 6-cup.

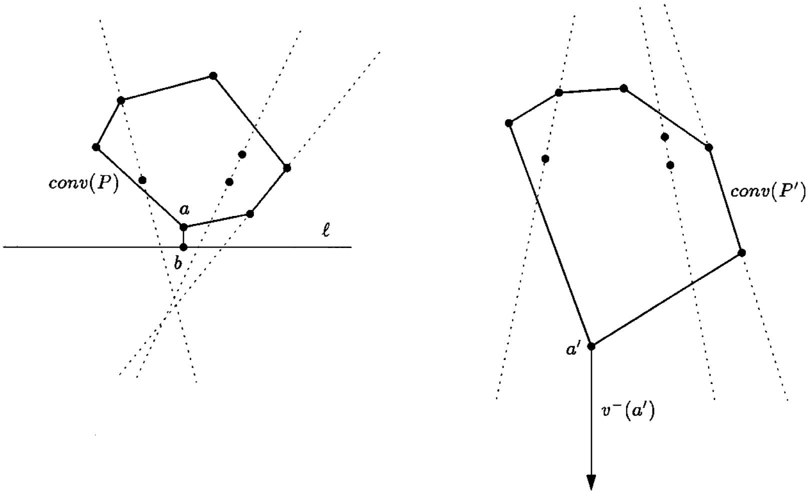

Fig. 2. The set P and its image $P' = T(P)$.

$a' = T(a)$. We get a point set $P' = T(P)$ from P. Since ℓ avoided the convex hull of P, the transformation T does not change convexity on the points of P, that is, any subset of P is in convex position if and only if the corresponding points of P' are in convex position. So the assumption also holds for P', no n points of P' are in convex position. By the choice of the point b, none of the lines determined by the points of $P' \setminus \{a'\}$ intersects $v^-(a')$. Therefore, any m-cap in the set $Q' = P' \setminus \{a'\}$ can be extended by a' to a convex $(m+1)$-gon.

Since no n points of P' are in convex position, Q' cannot contain any n-cup or $(n-1)$-cap. Therefore, by the lemma,

$$|Q'| \leq f(n, n-1) - 1 = \binom{2n-5}{n-2}, \qquad |P| \leq \binom{2n-5}{n-2} + 1,$$

and the theorem follows. □

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