Improved Bounds for Planar *k*-Sets and Related Problems*

T. K. Dey

Department of Computer Science and Engineering,
Indian Institute of Technology, Kharagpur,
India 721302
dey@cse.iitkgp.ernet.in

Abstract. We prove an $O(n(k + 1)^{1/3})$ upper bound for planar *k*-sets. This is the first considerable improvement on this bound after its early solution approximately 27 years ago. Our proof technique also applies to improve the current bounds on the combinatorial complexities of *k*-levels in the arrangement of line segments, *k* convex polygons in the union of *n* lines, parametric minimum spanning trees, and parametric matroids in general.

1. Introduction

The problem of determining the optimum asymptotic bound on the number of *k*-sets is one of the most tantalizing open problems in combinatorial geometry. Due to its importance in analyzing geometric algorithms [8], [9], [19], the problem has also caught the attention of computational geometers [5], [14], [18], [27], [29]. Given a set P of n points in $\Re^d$, a *k*-set is a subset $P' \subseteq P$ such that $P' = P \cap H$ for a half-space H, and $|P'| = k$ where $0 \leq k \leq n$. A close to optimal solution for the problem remains elusive even in $\Re^2$. In spite of several attempts, no considerable improvement could be made from its early bound of $O(n(k+1)^{1/2})$ given by [20] and [24]. Several proofs exist for this well-known upper bound [3], [5], [17], [29] which is quite distant from the best-known lower bound of $\Omega(n \log(k + 1))$ [17]. Pach et al. made the first dent in this upper bound improving it to $O(n(k+1)^{1/2}/\log^*(k + 1))$.[1] Even such a small improvement in $\Re^2$ was

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[1] log* *k* is the number of times the logarithm has to be applied to reduce *k* to a constant.

a distinguished result [25]. Recently Agarwal et al. [3] attacked the problem in the dual setting with a fresh look. Although they could not improve the worst-case upper bound, several new approaches were put forward to estimate the complexity of k-levels in the arrangement of n lines. By a well-known duality these results also apply to k-sets. One of the approaches of [3], based on "convex chains," inspired our proof for the new upper bound of $O(n(k+1)^{1/3})$ in $\Re^2$.

Our proof technique is surprisingly simple. It uses the concept of *crossings* in geometric graphs [1] which was first used by us to prove an $O(n^{8/3})$ bound on three-dimensional k-sets [14]. Crossings in geometric graphs have been successfully used for many problems in combinatorial geometry. See, for example, [12], [15], [16], [28]. It is expected that our approach would open up new avenues to solve the d-dimensional k-set problem, which remains largely unsolved for $d > 3$. The only nontrivial bound known for $d > 3$ is insignificantly better than the trivial bound [4], [9], [30]. In spite of the miserable state of the problem, an exact asymptotic bound is known for the number of i-sets summed over all $i \leq k$. Alon and Győri [2] showed that this number is $\Theta(nk)$ in $\Re^2$. Clarkson and Shor [11] generalized the bound to $\Theta(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil})$ for $\Re^d$.

Our proof technique also applies to establish a new $O(nk^{1/3} + n^{2/3}k^{2/3})$ complexity bound for k convex polygons whose edges are nonoverlapping and lie in the union of n lines. A number of other results follow from this bound. An optimal $\Theta(nk^{1/3})$ bound on the complexity of n-element parametric matroids with rank k follows due to a result by Eppstein [21]. As an immediate consequence, we obtain an $O(EV^{1/3})$ bound on the number of parametric minimum spanning trees of a graph with E edges and V vertices whose edge weights vary linearly with time. A new $O(n^{4/3})$ bound for k-levels in arrangement with n line segments can also be derived from the aforesaid bound.

The paper is organized as follows. We develop major tools for our proof in Section 2. A new bound for k-sets in $\Re^2$ is proved in Section 3. Section 4 describes the application of our proof technique to other related problems. Finally, we conclude in Section 5.

2. Convex Chains

Let P be a set of points in general position in $\Re^2$. This assumption of general position is safe since the number of k-sets is maximized for point sets in general position [17]. A *k-set edge* is a line segment connecting two points p, q in P whose supporting line contains exactly k points of P on one side. It is known that the number of $(k+1)$-sets equals the number of k-set edges [2]. Orient each k-set edge from left to right and let E_k be the set of oriented k-set edges whose supporting oriented line has exactly k points on its left-hand side. Consider the directed graph $G_k = (P, E_k)$. Without loss of generality assume that the cardinality of E_k is at least as large as half the number of k-set edges. Otherwise, the entire analysis can be performed for the rest of the k-set edges whose supporting oriented lines have exactly k points on their right-hand side. Let $\ell(e)$ and $r(e)$ denote the left and right endpoint of an edge $e \in E_k$. By a suitable rotation we can assume that no edge in E_k is vertical.

The edges of G_k incident to a vertex $p \in P$ have some nice properties that are crucial for our analysis. By our construction, p is the right endpoint of all incoming edges to it,
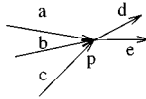
Fig. 1. Lovász's property: $s(b) > s(e) > s(a)$ and $s(d) > s(b) > s(e)$.

and it is the left endpoint of all outgoing edges from it. The *slope* of an edge e is denoted $s(e)$. The next property, discovered by Lovász, appears in [20].

Property 1. Let a, b be two incoming edges to p where $s(b) > s(a)$. There must exist an outgoing edge e from p such that $s(b) > s(e) > s(a)$. Similarly, let d, e be two outgoing edges from p where $s(d) > s(e)$. There must exist an incoming edge b to p such that $s(d) > s(b) > s(e)$. Refer to Fig. 1 for an illustration.

We wish to partition the edge set E_k such that each partition forms a convex chain. Define a relation R on E_k as follows. For an incoming edge e and an outgoing edge f incident with the same vertex, we say $e R f$ if and only if $s(e) > s(f)$ and there does not exist any outgoing edge f' with $s(e) > s(f') > s(f)$. For example, in Fig. 2, $a R b$ is true but $a R c$ is not true. First, we show a property of the relation R.

Lemma 2.1. *There is no edge f such that $e R f$ and $g R f$ where $e \neq g$.*

Proof. Suppose such an f exists. Let $s(e) > s(g)$. By definition $s(e) > s(f)$ and $s(g) > s(f)$. By Property 1 there exists an outgoing edge f' where $s(e) > s(f') > s(g)$. Hence $s(e) > s(f') > s(g) > s(f)$. But this violates that $e R f$. $\square$

Let R^* denote the reflexive, symmetric, and transitive closure of R. The equivalence relation R^* partitions E_k. Each class of this partition forms a chain of nonoverlapping directed edges going from left to right due to Lemma 2.1 and the definition of R. Further, each such chain is convex since we turn only right at each vertex according to the definition of R. See Fig. 2 for an illustration. We prove that there are at most $k + 1$ such convex chains.

Lemma 2.2. *Let $C_1, C_2, \ldots, C_j$ be the convex chains obtained by partition of E_k with R^*. Each C_i has a unique leftmost endpoint which is one of the $k + 1$ leftmost points of P.*

Proof. Suppose the leftmost endpoint of C_i is p_m where p_m is the mth point from the
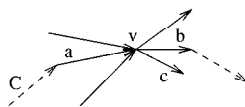
Fig. 2. Convex chain C going through the vertex v where $a R b$.

left with $m > k + 1$. Further let f be the edge of C_i with $\ell(f) = p_m$. Since C_i ends with p_m on the left, there is no edge e with $r(e) = p_m$ and $s(e) > s(f)$. Consider rotating f anticlockwise around p_m. At the beginning of this rotation its supporting oriented line ℓ_f has exactly k points on its left. When it is rotated up to the vertical position, it has exactly $m - 1 > k$ points on its left. This means that its supporting line has gained at least one point on its left during this rotation. Observe that ℓ_f gains a point to its left only if the left segment going from p_m to $x = -\infty$ hits a point. Also the number of points on the left of ℓ_f changes in every step by at most one. This means there exists a point x such that the oriented edge e with $\ell(e) = x$ and $r(e) = p_m$ must be a k-set edge. Further, $s(e) > s(f)$ which leads to a contradiction.

To prove that each left endpoint of convex chains is unique, assume C_{i_1}, C_{i_2} have a common left endpoint p. By Property 1 there is an edge e with $r(e) = p$ and that has a slope in between the slopes of the edges of C_{i_1}, C_{i_2} incident with p. By our construction, both C_{i_1} and C_{i_2} cannot have p as their left endpoint in that case. $\qquad\square$

Corollary 2.3. *There are at most $k + 1$ convex chains partitioning E_k.*

3. New k-Set Bound

Our goal is to bound the number of pairs of edges of E_k that intersect each other in their interiors. These intersections are called *crossings*. An upper bound on this crossing number, together with a lower bound [1], gives the new k-set bound. Let e and f be two edges in E_k that cross. Obviously, the convex chains, say, C_i and C_j that contain e and f, respectively, cross at $e \cap f$. Each of these crossings is uniquely charged to a common tangent to C_i and C_j according to the following charging scheme.

A *common tangent* to a pair of chains C_i, C_j is a line segment that connects two vertices, one from each of C_i and C_j and has the rest of the C_i and C_j strictly below its supporting line. Let C_i and C_j intersect at $x_1, x_2, \ldots, x_m$. Consider the upper hull of the vertices of C_i and C_j together. The vertical line passing through each x_ℓ, $1 \le \ell \le m$, intersect a unique edge of this upper hull. This edge, a common tangent to C_i and C_j, is charged for x_ℓ. It is a simple geometric fact that two such crossing points between C_i and C_j cannot charge the same common tangent. See Fig. 3. Next, we show that even two crossings from different pairs of chains cannot charge the same tangent.

Lemma 3.1. *Each common tangent is charged only once for crossings over all pairs of chains.*
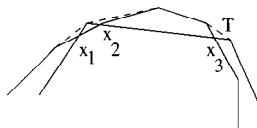
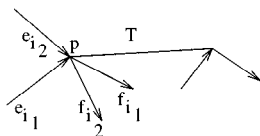**Fig. 3.** Common tangents for a pair of convex chains.

Fig. 4. Illustration for Lemma 3.1.

Proof. Let us assume on the contrary that a tangent T is charged for two pairs of chains, C_{i_1}, C_{j_1} and C_{i_2}, C_{j_2}. Let p be an endpoint of T which is not an endpoint of one chain, say C_{i_1}. Such a p must exist, otherwise Lemma 2.2 is violated. Let e_{i_1} and f_{i_1} be the incoming and outgoing edge of C_{i_1} incident with p. Consider the outgoing edge of C_{i_2} incident with p. Let this edge be f_{i_2}. See Fig. 4. Since T is a tangent to C_{i_1} and C_{i_2}, it must be true that either $s(e_{i_1}) > s(f_{i_2}) > s(f_{i_1})$ or $s(e_{i_1}) > s(f_{i_1}) > s(f_{i_2})$. The first possibility is contradicted by the fact that $e_{i_1} R f_{i_1}$. For the second possibility to be realized there must be an incoming edge e_{i_2} of C_{i_2} such that $s(f_{i_1}) > s(e_{i_2}) > s(f_{i_2})$ due to Property 1. However, in that case T cannot be tangent to C_{i_2}. □

Lemma 3.2. *There are at most $n(k+1)$ common tangents that are charged.*

Proof. Each vertex p of G_k occurs at most once as the left endpoint of a tangent to each convex chain not containing p. Since there are at most $k+1$ such chains, the claim follows. □

Comment. In fact, each common tangent is an ℓ-set edge for $\ell < k$. This can be proved using an argument similar to that used to prove Lemma 2.2. The total number of such edges is $\Theta(nk)$ due to a result in [2].

Now we are ready to prove the main theorem.

Theorem 3.3. *The number of $(k+1)$-sets that are possible with n points in $\Re^2$ is at most $6.48n(k+2)^{1/3}$.*

Proof. Each crossing between two edges of E_k appears as a crossing between the two convex chains containing those two edges. By Lemmas 3.1 and 3.2 this crossing number cannot be more than $n(k+1)$. Let $t = |E_k|$. For $t > 4n$, we must have at least $\Omega(t^2/n^3)$ pairs of edges intersecting in their interiors according to the result of Ajtai et al. [1]. The best constant for this result is given by Pach and Tóth [26] who show a $(1/33.75)t^3/n^2 - 0.9n$ lower bound on the crossing number for all values of t. Applying this result we get $(1/33.75)t^3/n^2 - 0.9n < n(k+1)$. Simple algebra gives $t < 3.24n(k+2)^{1/3}$. The worst-case number of k-sets is not more than $2t$ and hence the bound follows. □

4.   Related Problems

A point set P can be mapped to a set of lines through duality. This duality maps a point $p = (a, b)$ to a line $p^* : y = ax - b$ and a line $\ell : y = ax - b$ to a point $\ell^* = (a, b)$. The set of dual lines, denoted P^*, form a line arrangement $\mathcal{A}(P^*)$. A *k-level* in $\mathcal{A}(P^*)$ is defined as the closure of all points on given lines that have exactly k lines strictly below them. It is known that the worst-case complexity of the k-levels is within a constant factor of the worst-case number of k-sets in point sets. Hence our new bound on k-sets also provides the same upper bound on the complexity of k-levels. In fact, the entire proof can be carried out in this dual setting using properties of k-levels. Such a proof is provided in [13].

4.1.   *Convex Polygons and Matroid Optimization*

Consider a set of k convex polygons whose edges are nonoverlapping and are drawn from n lines. The complexity of these polygons is the total number of vertices they have altogether. If they are interior-wise disjoint, an optimal $\Theta(n^{2/3}k^{2/3} + n)$ bound is known [10], [23]. However, these analysis techniques fail if the polygons overlap in their interiors. Our proof technique can be used to establish an optimal $\Theta(nk^{1/3} + n^{2/3}k^{2/3})$ bound for this case.

First, we split each convex polygon into an upper chain and a lower chain. The upper chain consists of all points of the boundary that do not have any point of the polygon strictly above it. Similarly define the lower chains. Without loss of generality, we carry out the analysis for the upper chains only. The leftmost and rightmost edges of all upper chains are extended along their supporting lines to $x = -\infty$ and $x = +\infty$, respectively. With this modification, each convex upper chain is mapped in the dual to a convex chain passing through dual points. Hence we have k convex chains whose vertices are drawn from n points. Since the polygons in the primal have nonoverlapping edges, there is no edge g incident with a vertex p such that $s(e) > s(g) > s(f)$ where e, f are two edges of a convex chain passing through p. Also, for the same reason, the convex chains are edge disjoint. However, the convex chains may not have unique endpoints now. Due to all these properties, Lemma 3.1 remains valid except for the fact that a tangent T may be charged more than once only if it connects points that are endpoints of many convex chains. In that case T is charged for each pair of chains that have an endpoint coinciding with an endpoint of T. This count cannot be more than $\binom{k}{2} = O(k^2)$ altogether. Thus the total crossings among all convex chains is $O(nk + k^2)$. Using this in combination with the lower bound result on crossing [1], we obtain the desired $O(nk^{1/3} + n^{2/3}k^{2/3})$ complexity bound for convex chains. This bound is tight since, for $k < n$, the first term $nk^{1/3}$ dominates and a matching lower bound is proved in [21]. For $k > n$, the second term $n^{2/3}k^{2/3}$ dominates and a matching lower bound is established by the many-faces result of [10].

Theorem 4.1.   *A set of k convex polygons whose edges are nonoverlapping and lie in the union of n lines have $\Theta(nk^{1/3} + n^{2/3}k^{2/3})$ edges.*

In [21] Eppstein showed that an upper bound on the complexity of the class of the aforesaid convex polygons also provides an upper bound on the complexity of general parametric matroid optimization problems. He showed an $\Omega(nk^{1/3})$ lower bound for the general n-element parametric matroid optimization problem with rank k. Theorem 4.1 establishes a tight upper bound for it. An immediate implication of this result is the case of parametric minimum spanning trees of a graph with V vertices and E edges where the edge weights vary linearly with time. The previous $O(EV^{1/2})$ bound of Gusfield [22] is improved to $O(EV^{1/3})$ by our result.

4.2. Complexity of *j* Consecutive Levels

Let $L_k, L_{k-1}, \ldots, L_{k-j+1}$ be $j > 0$ consecutive levels in an arrangement of n lines. We are interested in determining the complexity of these j levels altogether. By duality, this complexity is within a constant factor of the total number of ℓ-sets in the set of dual points where $k - j + 1 \geq \ell \geq k$. Consider the convex chains partitioning E_ℓ for each ℓ where $k - j + 1 \geq \ell \geq k$. We use the proof technique of Sections 2 and 3 on these sets of $O(jk)$ convex chains. Since each common tangent is an m-set edge for $m < k$, we have $O(nk)$ tangents that are charged. Further, we argue that each tangent is charged at most $O(j^2)$ times. A tangent T cannot be charged for two pairs (C_1, C_2) and (C_3, C_4) where either C_1, C_3 (passing through the left endpoint of T) or C_2, C_4 (passing through the right endpoint of T) come from the partition of the same set E_ℓ for some ℓ. This is due to Lemma 3.1. This only means that at most $\binom{j}{2}$ different pairs can charge T. Now setting the inequality $t^3/n^2 < c \cdot nj^2(k + 1)$ for some appropriate constant $c > 0$, we obtain $t = O(n(k + 1)^{1/3} j^{2/3})$.

Theorem 4.2. *There are at most $O(n(k + 1)^{1/3} j^{2/3})$ ℓ-sets summed over $k \geq \ell \geq (k - j + 1)$.*

4.3. *k*-Levels in Arrangement of Line Segments

Let S be a set of n line segments in $\mathfrak{R}^2$ and let $\mathcal{A}(S)$ denote the corresponding arrangement. For any k, where $0 \leq k \leq n - 1$, the k-level in $\mathcal{A}(S)$ is defined as the closure of all points on given line segments that have exactly k line segments strictly below them. Notice that the k-level in this case may have discontinuities. These discontinuities are caused by the endpoints of the line segments where the k-level jumps vertically up or down. See Fig. 5 for an illustration. It is easily observed that the number of such discontinuities is at most $2n$. The technique of [3] shows an $O(n^{3/2})$ bound on the complexity of the k-level. We employ the technique of Section 4.1 to improve this bound to $O(n^{4/3})$. To apply this technique we consider a set of $O(n)$ convex chains[2] as defined in [3]. For any k, where $0 \leq k \leq n - 1$, let V_k denote the set of vertices of $\mathcal{A}(S)$ in the interior of the k-level that have exactly k segments strictly below them. We assume $V_{-1} = V_n = \emptyset$.

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[2] In [3], these chains are called concave.
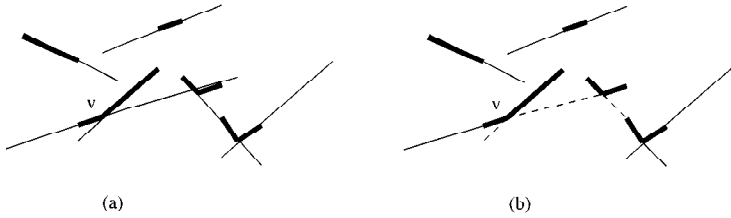
Fig. 5.   (a) First level in a line segment arrangement. (b) A convex chain is shown with broken lines.

It is known that the set of vertices of the k-level where two segments intersect either belong to V_k or V_{k-1}. The k-level, when traversed from left to right, makes a left turn at the vertices of V_{k-1}, and it makes a right turn at the vertices of V_k. For example, in Fig. 5, v is a vertex in V_0 where the 1-level makes a left turn. We count the vertices in V_{k-1}. An upper bound on $|V_{k-1}|$ also gives an upper bound on $|V_k|$ since the entire analysis can be done for any value of k where $0 \leq k \leq n - 1$.

Let $L_k = \{e\}$ denote the set of edges in the union of the segments in S satisfying the following properties:

- e is incident with a vertex in V_{k-1}. The other endpoint of e comes either from V_{k-1}, or from the set of vertices resulting from discontinuities of the k-level, or from the set of endpoints of the given line segments.
- The interior of e does not intersect the k-level.

The edges of L_k lie below the k-level and cover all vertices of V_{k-1}. For an illustration, see Fig. 5(b). See [3] for details. We define a relation R on L_k as follows. We say $e \, R \, f$ for any two edges e, f in L_k if and only if e and f share an endpoint. The reflexive, symmetric, and transitive closure R^* of R partitions L_k into a set of convex chains. This is because each pair of edges e, f, where $e \, R \, f$, belong to a chain that turns right at the vertex $v = e \cap f$. See Fig. 5(b) for an illustration. Several interesting properties of these convex chains are observed in [3]. For example, all vertices of a chain lie on the lower envelope of the lines supporting its edges.

We are now ready to apply the result of Section 4.1. There are at most $O(n)$ convex chains that cover all vertices of V_{k-1} since the endpoints of these chains are defined by the endpoints of the given line segments or the discontinuities of the k-level. Extend the leftmost and rightmost edges of all chains on their supporting lines to $x = -\infty$ and $x = +\infty$, respectively. Let $\mathcal{L}$ denote the set of n lines that support the edges in S. We have at most $O(n)$ convex chains in the union of the lines in $\mathcal{L}$ whose edges are nonoverlapping. Applying the arguments for the proof of Theorem 4.1 in Section 4.1 we immediately have the following theorem.

Theorem 4.3.   *The complexity of the k-level in line segment arrangement is $O(n^{4/3})$.*

The new bound on k-levels in line segment arrangement improves the current best bound on the k-levels in the arrangement of triangles in $\mathfrak{R}^3$. This follows from a result

of [3]. Plugging in our new bound into the analysis of [3], an $O(n^{25/9})$ bound can be established on the complexity of the *k*-levels in arrangements of triangles in $\Re^3$.

5. Conclusions

In this paper we provide a considerable improvement of the upper bound of planar *k*-sets which has defied all such attempts so far, except for a small improvement in [25] by a factor of $\log^*(k + 1)$. The technique is further employed to improve the current best bounds of several other related problems. It remains to be seen if the technique can be used in higher dimensions, albeit with necessary modifications. The generalization of the result of [1] exists [12], [15]. However, the concept of convex chains do not generalize in higher dimensions in a straightforward manner. The author believes that the technique developed in this paper would make further inroads into the challenge of *k*-set problems and probably into other related combinatorial problems.

Acknowledgments

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