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Counterexamples to the Strong d-Step Conjecture for $d \geq 5$*

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Abstract. A *Dantzig figure* is a triple (P, x, y) in which P is a simple d-polytope with precisely $2d$ facets, x and y are vertices of P, and each facet is incident to x or y but not both. The famous *d-step conjecture* of linear programming is equivalent to the claim that always $\#^d P(x, y) \geq 1$, where $\#^d P(x, y)$ denotes the number of paths that connect x to y by using precisely d edges of P. The recently formulated *strong d-step conjecture* makes a still stronger claim—namely, that always $\#^d P(x, y) \geq 2^{d-1}$. It is shown here that the strong d-step conjecture holds for $d \leq 4$, but fails for $d \geq 5$.

Introduction

A path formed from k edges of a graph is here called a *k-path*. When x and y are vertices of a polytope P, $\delta_P(x, y)$ denotes the *distance* from x to y in P's graph; thus $\delta_P(x, y)$ is the smallest k such that x and y are joined by a k-path. The maximum of $\delta_P(x, y)$, as x and y range over all vertices of P, is called the *diameter* of P and is denoted by $\delta(P)$. For each $n > d$, $\Delta(d, n)$ denotes the maximum of $\delta(P)$ as P ranges over all convex d-polytopes that have precisely n *facets* ($(d-1)$-faces). In the geometric form reported by Dantzig [D1], [D2], the *d-step conjecture* of linear programming (first formulated by W. M. Hirsch) asserts that $\Delta(d, 2d) = d$, and the formally stronger *Hirsch conjecture* asserts that $\Delta(d, n) \leq n - d$ for all d and all $n > d$.

A d-polytope is called *simple* if each of its vertices is incident to precisely d edges, or, equivalently, to precisely d facets. We use the term *(d, n)-polytope* to refer to a simple d-polytope that has precisely n facets. Two vertices of a polytope will be called *estranged* iff they do not share a facet. In the course of showing that the d-step conjecture and the Hirsch conjecture are equivalent (though not necessarily on a dimension-for-dimension

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basis), Klee and Walkup [KW] introduced the notion of a d-dimensional *Dantzig figure*, this being a triple (P, x, y) such that P is a $(d, 2d)$-polytope and x and y are estranged vertices of P.

When x and y are vertices of a polytope P, we use $\#^k P(x, y)$ to denote the number of k-paths from x to y in P. As was shown in [KW], the d-step conjecture is equivalent to the claim that $\#^d P(x, y) \geq 1$ for each d-dimensional Dantzig figure (P, x, y). Using this equivalence, the d-step conjecture was proved in [KW] for $d \leq 5$, but it is still open for all $d \geq 6$. In [LPR], Lagarias *et al*. observed that for each d-dimensional Dantzig figure (P, x, y), $\#^d P(x, y) \leq d!$, and they formulated what they called the *strong d-step conjecture*, asserting that $\#^d P(x, y) \geq 2^{d-1}$. They verified this conjecture for $d \leq 3$ and they produced extensive numerical evidence in its favor for $4 \leq d \leq 15$. Subsequently, Lagarias and Prabhu [LP] showed for each d, that if r is either the minimum $(d^2 - d + 2)$ or the maximum number of vertices that a $(d, 2d)$-polytope can have, then there exists a d-dimensional Dantzig figure (P, x, y) such that $\#^d P(x, y) = 2^{d-1}$ and P has precisely r vertices.

This paper shows that the strong d-step conjecture is correct when $d = 4$ but fails for all $d \geq 5$. The proof for $d = 4$ is a routine computation based on the Grünbaum–Sreedharan catalog [GS] of the 37 combinatorial types of simple 4-polytopes with 8 facets. The disproof for $d \geq 5$ starts with a (4, 9) dual-neighborly polytope of diameter 5 that was first constructed in [KW], and then applies the wedging operation of [KW] to show that for each $d \geq 5$ there exists a d-dimensional Dantzig figure (P, x, y) for which $\#^d P(x, y) = 3 \cdot 2^{d-3} < 2^{d-1}$. (In the constructed examples, the number of vertices is $d^2 + 9d - 28$.)

As general references on the combinatorial structure of polytopes, the books by Grünbaum [G] and Ziegler [Z] are recommended. Both discuss the d-step conjecture.

1. Computational Procedure

The following procedure finds, for each estranged pair of vertices of a simple d-polytope P, the number of d-paths that join the two vertices.

(0) (Input.) For a simple d-polytope P with n facets and m vertices, let M denote the $n \times m$ facet-versus-vertex incidence matrix of P. The ith row of M tells which vertices are incident to facet i. The jth column of M tells which facets are incident to vertex j.

(1) $S := M^T M$. (S is an $m \times m$ matrix (s_{ij}) in which s_{ij} is the number of facets shared by vertex i and vertex j.)

(2) $B := (s_{ij} \stackrel{?}{=} 0)$, an $m \times m$, 0–1 matrix (b_{ij}) in which the 1 entries correspond to pairs of vertices that are estranged. If $B = 0$, there are no estranged pairs and the computation halts.

(3) $A := (s_{ij} \stackrel{?}{=} d - 1)$, the $m \times m$ adjacency matrix of the graph formed by P's vertices and edges.

(4) (Output.) $N := A^d \circ B$, in which $\circ$ denotes the *Hadamard* (entry-by-entry) product. The (i, j) entry of A^d is the number of walks of length d from vertex i to vertex j. However, when two vertices x and y of a simple d-polytope P are estranged, they cannot be connected by a walk of length less than d, and hence each walk of length d

from x to y must, in fact, be a d-path. Thus the matrix N tells, for each estranged pair of vertices (x, y), the number $\#^d P(x, y)$ of d-paths that connect the two vertices.

2. Proof for $d \leq 4$

2.1. Theorem. *The strong d-step conjecture is correct for $d \leq 4$.*

Proof. The strong d-step conjecture is obvious for $d = 2$, and [LPR] noted that it also holds for $d = 3$. Verification for $d = 3$ is almost immediate, because there are only two different combinatorial types of (3, 6) polytopes. The first is the 3-cube I^3, for which $\#^3 I^3(x, y) = 6$. (In general, I^d has 2^{d-1} estranged pairs (x, y), and $\#^d I^d(x, y) = d!$ for each such pair.) The second $(3, 6)$-polytope Q is combinatorially equivalent both to a triangular prism truncated at one vertex and to the wedge over a pentagon with an edge as foot. In Q there are two estranged pairs (x, y), and $\#^3 Q(x, y) = 4$ for each of them.

To verify the strong d-step conjecture for $d = 4$, we use the complete catalog of simplicial 4-polytopes with eight vertices that was published in 1967 by Grünbaum and Sreedharan [GS], correcting a 1909 list of Brückner [Br]. With the aid of the usual polarity, this may also be regarded as a catalog of simple 4-polytopes with eight facets. There are 37 different combinatorial types. In terms of the indexing of [GS], the procedure described in Section 1 yields the information that is listed below concerning the numbers of d-paths connecting estranged pairs of vertices.

The indices in parentheses are the identification numbers used in [GS]. An "na" indicates that the polytope in question has no estranged pairs. Polytope number (34) is the 4-cube, in which there are eight estranged pairs and each pair is connected by twenty-four 4-paths. In polytope number (25) there are four estranged pairs, with one such pair connected by eight 4-paths, another pair connected by ten 4-paths, and two pairs for each of which there are eleven 4-paths. The other data are interpreted similarly.

(1) na; (2) na; (3) na; (4) na; (5) 8_2; (6) 8_2; (7) 12_2; (8) $8_1, 10_1$; (9) 10_2; (10) na; (11) 8_2; (12) 8_2; (13) na; (14) 8_2; (15) 8_4; (16) 12_2; (17) 16_4; (18) 10_4; (19) 13_2; (20) 10_2; (21) 13_4; (22) $12_2, 14_2$; (23) 11_2; (24) 10_2; (25) $8_1, 10_1, 11_2$; (26) 18_6; (27) $14_2, 15_2$; (28) $12_2, 13_2$; (29) $8_1, 12_4, 14_1$; (30) $10_2, 12_2$; (31) $8_1, 10_1, 11_1, 12_1$; (32) 12_2; (33) $8_4, 11_2$; (34) 24_8; (35) $8_4, 12_2$; (36) $8_1, 9_2, 12_1$; (37) $8_2, 9_2$.

Note that for each of the 37 polytopes, each estranged pair is connected by at least eight 4-paths. This proves the strong d-step conjecture for $d = 4$. □

3. Wedging and Truncation

Suppose that P is a d-polytope in $\mathbb{R}^d$, and that F is a face of P. In the terminology of [KW], a *wedge* over P with *foot* F is a $(d + 1)$-polytope $\omega_F(P)$ that is formed by intersecting the "cylinder" $C = P \times [0, \infty[$ with a closed half-space J in $\mathbb{R}^{d+1}$ such that the intersection $J \cap C$ is bounded and has nonempty interior, and the bounding hyperplane H of J is such that $H \cap (\mathbb{R}^d \times \{0\}) = F \times \{0\}$. The boundary complex of $\omega_F(P)$ is combinatorially equivalent to the complex formed from the boundary complex of the prism $P \times [0, 1]$ by identifying $\{p\} \times [0, 1]$ with $(p, 0)$ for each point p of F. Henceforth, we specialize to the case in which F is a facet of P. Then, in effect, the identification process replaces the facet (d-face) $F \times [0, 1]$ of the prism by a ridge (($d - 1$)-face) R that is a copy of F. In the wedge $\omega_F(P)$, there are two facets that contain the ridge R, and each of these facets is combinatorially equivalent to P. We shall denote these facets by B $(= P \times \{0\})$ and T $(= P \times \{1\})$ and call them the *base* and the *top* of the wedge; thus $R = B \cap T$. Since each vertex of $\omega_F(P)$ is incident to T or B, it corresponds naturally to a vertex in P. Each vertex $v \in F$ has a unique natural image in the ridge R in $\omega_F(P)$. Each vertex $v \in P \backslash F$ has a natural image in the base B and a second natural image in the top T; we denote these images by v_b $(= v \times \{0\})$ and v^t $(= v \times \{1\})$, respectively. If P is a (d, n)-polytope and F is a facet of P, then the wedge $\omega_F(P)$ is a $(d + 1, n + 1)$-polytope.

To derive the incidence matrix for $\omega_F(P)$ from the incidence matrix $M(P)$ of P, we first determine the index of F: $f_i = F$. Recall that the rows of M correspond to facets and the columns to vertices. Let C_i be the submatrix of $M(P)$ consisting of the columns that correspond to vertices not incident to f_i, and let E_i be a matrix of the same dimensions as C_i $(n \times (f_0(P) - f_0(F)))$ in which all entries are zero, except those in the ith row which are all ones. Then

$$M(\omega_{f_i}(P)) = \begin{pmatrix} C_i + E_i : M(P) \\ \langle 0 \rangle \quad : \quad \langle 1 \rangle \end{pmatrix}.$$

With $M(\omega_F(P))$ so constructed, we have the base $B = f_i$, and the new row is the top $T = f_{n+1}$. The vertices of the foot are indicated precisely by the columns that have 1's in both of these rows.

When F is any face of a d-polytope P, and x and y are vertices of P, we denote by $\#P(x, y)$ the number of shortest paths from x to y in P, and by $\#P(x, F, y)$ the number of shortest paths from x to y that visit F. Note that this differs from the practice of [LPR] and [LP], who use $\#P(x, y)$ to denote the number of d-paths from x to y in a d-dimensional Dantzig figure (P, x, y). (For that specialized purpose, we have used the notation $\#^d P(x, y)$.)

Let $W = \omega_F(P)$. Since the facets B and T are combinatorially equivalent to P, each vertex v of P has two natural images in W, and we denote these by v_b and v^t; if v is incident to F, then these two images coincide: $v_b = v^t = v$. Since a vertex w of W is incident to at least one of B or T, w has a natural image in P, which we denote by $\overline{w}$. Thus $\overline{v_b} = \overline{v^t} = v$ for each vertex v of P.

From these maps of vertices, we obtain for each path in W a unique natural image in P. Let $[w_0, w_1, \ldots, w_m]$ be a path in W. For each i, $[w_i, w_{i+1}]$ is an edge in W, so either $[\overline{w_i}, \overline{w_{i+1}}]$ is an edge of P or $\overline{w_i} = \overline{w_{i+1}}$ (i.e., $\{w_i, w_{i+1}\} = \{v_b, v^t\}$ for some vertex v of P). In the latter case, we say that $[w_i, w_{i+1}]$ is a *vertical* edge. The natural image of a vertical edge in W is a vertex in P. The natural image of $[w_0, w_1, \ldots, w_m]$ is $[\overline{w_0}, \overline{w_1}, \ldots, \overline{w_m}]$, to which sequence of vertices we apply the contraction that replaces v, v by v. In effect, we eliminate the vertical edges and map the remaining edges to their natural images in P.

The natural image of an m-path in W is a k-path in P with $k = m - e$, e the number of vertical edges in the m-path. For a path ρ in P and fixed images w_0 and w_m of its endpoints in W, we define the *tight natural images* of ρ from w_0 to w_m to be those paths of minimal length among all the paths from w_0 to w_m in W whose natural image is ρ. For shortest paths, we have the following result.

3.1. Wedging Lemmas. *Suppose x and y are vertices and F is a facet of the (d, n)-polytope P. Then the wedge $W = \omega_F(P)$ is a $(d+1, n+1)$-polytope.*

(1) Case (i). *If no shortest path from x to y visits F, then*

$$\delta_W(x_b, y^t) = \delta_P(x, y) + 1,$$

and each shortest path from x to y in P corresponds naturally to $\delta_P(x, y) + 1$ shortest paths from x_b to y^t in W. Further,

$$\#W(x_b, x^t, y^t) = \#P(x, y),$$

and for each neighbor v of x in P

$$\#W(x_b, v_b, y^t) = \delta_P(x, y) \cdot \#P(x, v, y) + \sum_{\rho} 2^{r_\rho - 1},$$

the sum being taken over all $(\delta_P(x, y) + 1)$-paths ρ from x to y via v which visit F r_ρ (> 0) times.

(2) Case (ii). *If some shortest path from x to y visits F, then*

$$\delta_W(x_b, y^t) = \delta_P(x, y),$$

and each shortest path in P from x to y that visits F r times corresponds naturally to 2^{r-1} shortest paths from x_b to y^t in W.

If every shortest path in P from x to y that visits F does so only once, then the shortest paths from x to y are in natural one-to-one correspondence with the shortest paths in W from x_b to y^t. Under this nonrevisiting assumption,

$$\#W(x_b, y^t) = \#P(x, F, y).$$

If v is a neighbor of x in P, then

$$\#W(x_b, v_b, y^t) = \#P(v, F, y),$$

and

$$\#W(x_b, x^t, y^t) = 0.$$

Proof. Let $[x = v_0, v_1, \ldots, v_m = y]$ be an m-path from x to y in P which does not visit F. Then $[x_b = v_{0b}, \ldots, v_{ib}, v_i^t, \ldots, v_m^t = y^t]$ is an $(m+1)$-path from x_b to y^t in W, for each $0 \leq i \leq m$. These $m+1$ distinct paths are the shortest paths in W for which the natural image in P is the given path. Moving from the base to the top requires the addition of a vertical edge somewhere in the path.

Now suppose that in the m-path $[x = v_0, v_1, \ldots, v_m = y]$, v_i is incident to F. Then $[x_b = v_{0b}, \ldots, v_{(i-1)b}, v_i, v_{(i+1)}^t, \ldots, v_m^t = y^t]$ is an m-path from x_b to y^t in W. Moving from the base to the top requires no additional edge.

For an m-path from x to y in P which visits F, its tight natural images from x_b to y^t in W necessarily enter the first visit to F from the base and leave the last visit to F on the top. After visiting F the first time and before visiting F the last time, any choice of base or top between visits to F yields a tight natural image from x_b to y^t. There are 2^{r-1} ways of choosing whether the natural image in W of each of the $r-1$ sequences of vertices between visits to F is in the base or top. Thus a path in P which visits F r times has 2^{r-1} distinct tight natural images from x_b to y^t in W.

Now let $m = \delta_P(x, y)$, and consider the set of shortest paths from x to y in P. Those which do not visit F have $m+1$ tight natural images from x_b to y^t in W, each of length $m+1$. Those which visit F r times ($r > 0$) have 2^{r-1} tight natural images from x_b to y^t, each of length m.

In the case that none of the shortest paths from x to y in P visits F, we have established all the claims except the specific counts of shortest paths from x_b to y^t incident to given neighbors. Let v be a neighbor of x in P. Any shortest path from x to y via v consists of the edge $[x, v]$ prepended to a shortest path from v to y. Necessarily, $\delta_P(v, y) = \delta_P(x, y) - 1$, and each of the $\delta_P(x, y)$ tight natural images of a shortest path from v_b to y^t can be prepended to a shortest path from x_b to y^t. We have accounted for all the shortest paths from x_b to y^t via v_b which do not visit F. However, an $(m+1)$-path from x to y via v and visiting F r times has 2^{r-1} tight natural images from x_b to y^t in W, each of length $m+1$; hence each of these images will be a shortest path from x_b to y^t. We summarize this accounting in

$$\#W(x_b, v_b, y^t) = m \cdot \#P(x, v, y) + \sum_{\rho} 2^{r_\rho - 1}.$$

An $(m+1)$-path from x_b to y^t via x^t consists of the initial edge $[x_b, x^t]$ followed by an m-path ρ from x^t to y^t. Since none of the m-paths from x to y in P visits F, ρ must lie entirely in T, and so $\overline{\rho}$ is an m-path in P from x to y. On the other hand, for every m-path β from x to y in P, the tight natural image $[x_b, \beta^t]$ is an $(m+1)$-path from x_b to y^t in W. From this natural one-to-one correspondence, we have

$$\#W(x_b, x^t, y^t) = \#P(x, y).$$

We now address case (ii), in which some shortest m-path from x to y visits F. No path from x_b to y^t can have length less than m, but the tight natural images of an m-path which visits F has length m; hence $\delta_W(x_b, y^t) = \delta_P(x, y)$, and as observed above, an m-path in P which visits F r times has 2^{r-1} tight natural images in W, each of length m. For any path from x to y in P which does not visit F, the tight natural images from x_b to y^t are of length $m+1$ and so are not shortest paths. Summing over all shortest paths ρ from x to y in P which visit F r_ρ times, we have

$$\#W(x_b, y^t) = \sum_{\rho} 2^{r_\rho - 1}.$$

We now assume further that the shortest paths from x to y which visit F do so only once ($r = 1$). Under this assumption, each shortest path from x to y which visits F has a unique tight natural image from x_b to y^t in W. Hence, for each neighbor v of x in P,

$$\#W(x_b, v_b, y^t) = \#P(v, F, y),$$

and we can rewrite the above sum

$$\#W(x_b, y^t) = \#P(x, F, y).$$

To see finally that $\#W(x_b, x^t, y^t) = 0$, we can either observe that no shortest paths from x_b to y^t are left uncounted, or we could observe that an m-path from x_b to y^t via x^t would have as its natural image in P a path from x to y of length less than m. □

When a simple d-polytope P and two vertices x and y of P are fixed, we define a function γ_x on the neighbors of x in P by setting $\gamma_x(v) = \#P(x, v, y)$ for each neighbor v. We can list γ_x as a d-vector since P is simple:

$$\gamma_x = (\#P(x, v_1, y), \ldots, \#P(x, v_d, y)).$$

The conclusion of the second case in the above lemma can now be written succinctly:

$$\gamma_{x_b} = (\gamma_x, 0),$$

by which we mean $\gamma_{x_b}(v_b) = \gamma_x(v)$ for neighbors v of x in P, and $\gamma_{x_b}(x^t) = 0$.

In the construction of counterexamples, we also employ the operation of truncating a (d, n)-polytope P at a vertex v. To perform the truncation geometrically, we form the intersection $\tau_v(P)$ of P with any closed half-space that misses v and whose bounding hyperplane passes strictly between v and the remaining vertices of P. Again note that since P is simple, $\tau_v(P)$ is a $(d, n+1)$-polytope with new facet $\tau(v)$ and $d-1$ additional vertices.

Combinatorially, the vertex v is replaced by a $(d-1)$-simplex $\Sigma(v)$ with one of its vertices on each edge incident to v. For example, if u is a neighbor of v in P, then in $\tau_v(P)$, $\sigma(u)$ is a vertex in $\Sigma(v)$ whose neighbors are the $d-1$ other vertices in $\Sigma(v)$ and u.

We form the incidence matrix for the truncated polytope $\tau_v(P)$ from that of P thus:

$$M(\tau_v(P)) = \begin{pmatrix} M(P \backslash v) : M(\Sigma(v) \backslash \tau(v)) \\ \langle 0 \rangle \quad : \quad \langle 1 \rangle \end{pmatrix}.$$

We start with a copy of $M(P)$ and remove the column corresponding to v; this is the upper-left block $M(P \backslash v)$. We take d copies of the column for v, and in each copy replace one of the d 1's by a 0 so that no two of these columns are the same; this is the upper-right block $M(\Sigma(v) \backslash \tau(v))$. Finally, we append a new row with 1's under these rightmost d columns and 0's under $M(P \backslash v)$; this new row corresponds to the facet $\tau(v)$.

We note some natural correspondences between paths on P and paths on $Q = \tau_v(P)$. Paths in Q have unique natural images in P, obtained by replacing each occurrence of a vertex in $\Sigma(v)$ with v and then applying the contraction that replaces v, v by v. For a fixed path ρ in P, we define a *tight natural image* of ρ in Q to be a path of minimal length in Q whose natural image in P is ρ. Every path in P has a unique tight natural image in Q. In particular, for distinct neighbors u and w of v in P, the paths $[u, v]$ and $[u, v, w]$ correspond, respectively, to the paths $[u, \sigma(u)]$ and $[u, \sigma(u), \sigma(w), w]$ in Q. Note that the tight natural images in Q of m-paths in P which do not visit v, except possibly as a terminal vertex, are also of length m; if an m-path in P does not terminate at v but visits v r times, then its tight natural image is an $(m + r)$-path in Q.

3.2. Truncation Lemmas. *Suppose x and v are distinct vertices in the (d, n)-polytope P, and u and w are distinct neighbors of v in P. Then $Q = \tau_v(P)$ is a $(d, n+1)$-polytope.*

(1) Case (i). *If $\delta_P(x, w) = \delta_P(x, v)$, then $\delta_Q(x, \sigma(w)) = \delta_P(x, v) + 1$,*

$$\#Q(x, w, \sigma(w)) = \#P(x, w),$$

and

$$\#Q(x, \sigma(u), \sigma(w)) = \#P(x, u, v).$$

(2) Case (ii). *If $\delta_P(x, w) = \delta_P(x, v) - 1$, then $\delta_Q(x, \sigma(w)) = \delta_P(x, v)$,*

$$\#Q(x, w, \sigma(w)) = \#P(x, w, v),$$

and

$$\#Q(x, \sigma(u), \sigma(w)) = 0.$$

(3) Case (iii). *If $\delta_P(x, w) = \delta_P(x, v) + 1$, then $\delta_Q(x, \sigma(w)) = \delta_P(x, v) + 1$,*

$$\#Q(x, w, \sigma(w)) = 0,$$

and

$$\#Q(x, \sigma(u), \sigma(w)) = \#P(x, u, v).$$

Proof. Let w be a neighbor of v in P. Since w is a neighbor of v, their distances from x differ by at most 1. For case (i) let $m = \delta_P(x, w) = \delta_P(x, v)$. Necessarily, $\#P(x, w, v) = \#P(x, v, w) = 0$. The tight natural image of any m-path in P from x to v via a neighbor $u \neq w$ is an m-path in Q from x to $\sigma(u)$, which extends to an $(m + 1)$-path from x to $\sigma(w)$. Each m-path from x to w in P can be identified with its tight natural image in Q and then extended to an $(m + 1)$-path from x to $\sigma(w)$. Thus, $\delta_Q(x, \sigma(w)) = m + 1$; moreover, we have the specific counts $\#Q(x, u, \sigma(w)) = \#Q(x, \sigma(u), \sigma(w)) = \#P(x, u, v)$, and $\#Q(x, w, \sigma(w)) = \#P(x, w)$.

In case (ii) we let $m = \delta_P(x, v) = \delta_P(x, w) + 1$. So $\#P(x, w, v) = \#P(x, w)$, and $\#P(x, v, w) = 0$. The tight natural image of any $(m - 1)$-path in P from x to w can be extended in Q to an m-path from x to $\sigma(w)$. On the other hand, for any other neighbor u of v, a path in Q from x to $\sigma(w)$ via $\sigma(u)$ has length at least $m + 1$. We conclude, in this case, that $\delta_Q(x, \sigma(w)) = m$ with $\#Q(x, w, \sigma(w)) = \#P(x, w) = \#P(x, w, v)$ and $\#Q(x, \sigma(u), \sigma(w)) = 0$.

For case (iii) we let $m = \delta_P(x, w) = \delta_P(x, v) + 1$. In this case, $\#P(x, w, v) = 0$ and $\#P(x, v, w) = \#P(x, v)$. Any m-path in P from x to w can be identified with its tight natural image in Q and then extended to an $(m + 1)$-path from x to $\sigma(w)$ via w. On the other hand, an $(m-1)$-path from x to v in P must arrive at v via a neighbor $u \neq w$, and so its tight natural image is an $(m-1)$-path in Q from x to $\sigma(u)$, which can be extended to an m-path from x to $\sigma(w)$. Thus $\delta_Q(x, \sigma(w)) = m$ with $\#Q(x, \sigma(u), \sigma(w)) = \#P(x, u, v)$ and $\#Q(x, w, \sigma(w)) = 0$. □

4. Disproof for $d = 5$

4.1. Theorem. *There is a five-dimensional Dantzig figure (P, x, y) for which $\#P(x, y) = 12$. Hence the strong d-step conjecture fails for $d = 5$.*

Proof. We produce the counterexample for $d = 5$ as the wedge over a certain $(4, 9)$-polytope Q_4 which was first constructed in [KW]. The polytope Q_4 has 9 facets and 27 vertices, and is the only $(4, 9)$-polytope of diameter 5. The combinatorial structure of Q_4 is described explicitly on p. 741 of [KK′]. With a convenient numbering of facets and vertices, Q_4's incidence matrix is as follows. The estranged vertices $x(= v_1)$ and $y(= v_{15})$ of Q_4 have $\delta_{Q_4}(x, y) = 5$, and the facet $F(= f_9)$ misses both x and y. The facet F has 12 vertices.

$$M(Q_4) = \begin{pmatrix} 110111000000100111111000000 \\ 101000111000010110000111100 \\ 111000111111000101000010000 \\ 111111000111000010100100000 \\ 001100100100101000110101011 \\ 010010010010011001001010111 \\ 000011011011111000011000010 \\ 000101101101111000000001101 \\ 000000000000000111111111111. \end{pmatrix}.$$

Let P_5 denote the wedge over Q_4 with foot F. Then F becomes a ridge in P_5, and each vertex v of $Q_4 \backslash F$ has two images in P_5: an image v_b in the base B and an image v^t in the top T, connected by an edge. There are 15 such pairs, and with the 12 vertices in F this yields a total of 42 vertices in P_5.

Following the method in Section 3, we produce the incidence matrix $M(P_5)$ from $M(Q_4)$.

$$M(P_5) = \begin{pmatrix} 110111000000100111111000000110111000000100 \\ 101000111000010110000111100101000111000010 \\ 111000111111000101000010000111000111111000 \\ 111111000111000010100100000111111000111000 \\ 001100100100101000110101011001100100100101 \\ 010010010010011001001010111010010010010011 \\ 000011011011111000011000010000011011011111 \\ 000101101101111000000001101000101101101111 \\ 111111111111111111111111111000000000000000 \\ 000000000000000111111111111111111111111111 \end{pmatrix}.$$

In this incidence matrix we have the base $B = f_9$, the top $T = f_{10}$, and the vertices $x_b = v_1$, $y_b = v_{15}$, $x^t = v_{28}$, and $y^t = v_{42}$.

When applied to $M(P_5)$, the procedure of Section 1 yields as output a 42×42 matrix $N(P_5)$ whose only nonzero entries are

$$n_{1,42} = 12, \quad n_{4,35} = 36, \quad n_{5,34} = 36, \quad n_{7,32} = 36, \quad n_{8,31} = 36, \quad n_{15,28} = 12,$$

$$n_{42,1} = 12, \quad n_{35,4} = 36, \quad n_{34,5} = 36, \quad n_{32,7} = 36, \quad n_{31,8} = 36, \quad n_{28,15} = 12.$$

Using the same notation as in Section 2, the summary statistic for P_5 is $12_2, 36_4$. That is:

- P_5 has six estranged pairs in all, each of distance 5.
- There are thirty-six shortest paths for each of four estranged pairs.
- For two of the estranged pairs, (x_b, y^t) and (x^t, y_b), there are only twelve shortest paths.

In Q_4 there are sixteen 5-paths from x to y, but only twelve of those paths visit F. From the Wedging Lemmas, as confirmed by the computational procedure, we have $\#^5 P_5(x_b, y^t) = 12$. Since (P_5, x_b, y^t) is a five-dimensional Dantzig figure, and $\#^5 P_5(x_b, y^t) < 16$, this is a counterexample to the strong 5-step conjecture. $\square$

5. Disproof for $d \geq 6$

With $M(P_5)$ as in Section 4, truncate P_5 at v_{42} to produce $\tau(P_5)$. Then

$$M(\tau(P_5)) = \begin{pmatrix} 1101110000001001111110000001101110000001000000 \\ 1010001110000101100001111001010001110000100000 \\ 1110001111110001010000100001110001111110000000 \\ 1111110001110000101001000000111111000111000000 \\ 0011001001001010001101010110011001001001001111 \\ 0100100100100110010010101110100100100100110111 \\ 0000110110111110000110000100000110110111111011 \\ 0001011011011110000000011010001011011011111101 \\ 1111111111111111111111111110000000000000000000 \\ 0000000000000011111111111111111111111111111110 \\ 0000000000000000000000000000000000000000011111 \end{pmatrix}.$$

Let P_6 be the wedge over $\tau(P_5)$ with foot f_{10}. Then

$$M(P_6) = \begin{pmatrix} 11011100000010001101110000001001111110000001101110000001000000 \\ 10100011100001001010001110000101100001111001010001110000100000 \\ 11100011111100001110001111110001010000100001110001111110000000 \\ 11111100011100001111110001110000101001000000111111000111000000 \\ 00110010010010110011001001001010001101010110011001001001001111 \\ 01001001001001110100100100100110010010101110100100100100110111 \\ 00001101101111110000110110111110000110000100000110110111111011 \\ 00010110110111110001011011011110000000011010001011011011111101 \\ 11111111111111101111111111111111111111111111100000000000000000 \\ 11111111111111110000000000000000111111111111111111111111111110 \\ 00000000000000010000000000000000000000000000000000000000011111 \\ 00000000000000001111111111111111111111111111111111111111111111 \end{pmatrix}.$$

Applying the procedure of Section 1 to this incidence matrix, we find that there are only two estranged pairs, (v_1, v_{62}) and (v_{17}, v_{16}), with summary statistic 24_2. Since the strong 6-step conjecture would require this number to be at least $32 = 2^{6-1}$, P_6 is a counterexample.

In the remainder of this section we show that the process of truncating and wedging can be repeated to produce a family of counterexamples to the strong d-step conjecture for all $d > 5$.

A triple (P, x, y^t) is a W_d-*figure* iff P is a $(d, 2d)$-polytope and is also a wedge $P = \omega_F(Q)$, with vertices $x \in B\backslash F$ and $y^t \in T\backslash F$ such that $\delta_P(x, y^t) = \delta_P(x, y_b) = d$.

For a W_d-figure (P_d, x, y^t), truncation at y^t yields a $(d, 2d+1)$-polytope Q with a vertex $z = \sigma(y_b)$ that is estranged from x, and with $\delta_Q(x, z) = d + 1$. The truncated top $\tau(T)$ is the unique facet of Q not incident to either x or z. Taking the wedge over Q with foot $\tau(T)$ yields a $(d+1, 2d+2)$-polytope P_{d+1} with only two estranged pairs (x_b, z^t) and (x^t, z_b), each at distance $d + 1$. Since (P_d, x, y^t) is a W_d-figure, we can obtain a stronger result.

Proposition 5.1. *If (P_d, x, y^t) is a W_d-figure with $\#P_d(x, y^t) = k$, and*

$$P_{d+1} = \omega_{\tau(T)}\tau_{y^t}(P_d),$$

then (P_{d+1}, x_b, z^t) is a W_{d+1}-figure with

$$\#P_{d+1}(x_b, z^t) = 2k,$$

$$\gamma_{x_b} = (2\gamma_x, 0)\,,$$

and

$$\gamma_{z^t} = \left(\gamma_{y^t}, k\right).$$

Proof. Since (P_d, x, y^t) is a W_d-figure, $P_d = \omega_F(Q)$ for some $(d-1, 2d-1)$-polytope Q with facet F, and every d-path from x to y^t visits F. The polytope P_d satisfies the first case of the Truncation Lemmas, with $v = y^t$ and $w = y_b$. Let $z = \sigma(y_b)$ in $\tau_{y^t}(P_d)$. Then from the Truncation Lemmas it follows that the collection of shortest paths from x to z is in natural bijection with the union of the collection of shortest paths in P_d from x to y^t and the collection of shortest paths in P_d from x to y_b.

Once we take the wedge over $\tau_{y^t}(P_d)$ with foot $\tau(T)$, the shortest paths from x_b to z^t are in natural bijection with shortest paths from x to z that visit $\tau(T)$. This includes all those shortest paths on $\tau_{y^t}(P_d)$ which correspond to shortest paths from x to y^t on P_d; it also includes those shortest paths on $\tau_{y^t}(P_d)$ which correspond to shortest paths from x to y_b on P_d which visit F, since $F \subset T$.

By the Wedging Lemmas there is a natural bijection between shortest paths in P_d from x to y^t and those from x to y_b which visit F. In particular,

$$\#P_d(x, y^t) = \#P_d(x, F, y_b)$$

($= k$ by assumption). Thus, from these natural correspondences, we conclude not only that

$$\#P_{d+1}(x_b, z^t) = 2k,$$

but also that

$$\gamma_{x_b} = (2\gamma_x, 0)$$

and

$$\gamma_{z^t} = (\gamma_{y^t}, k).$$

We note also that P_{d+1} is a W_{d+1}-figure. □

Corollary 5.2. *If (P_d, x, y^t) is a W_d-figure and a counterexample to the strong d-step conjecture, then with $P_{d+1} = \omega_{\tau(T)}\tau_{y^t}(P_d)$, (P_{d+1}, x_b, z^t) is a W_{d+1}-figure and a counterexample to the strong $(d+1)$-conjecture.*

Corollary 5.3. *Let Q be a $(c, 2c+1)$-polytope of diameter $c+1$ with an estranged pair (x, y) at distance $c+1$, and $\#Q(x, F, y) < 2^c$ for F the unique facet F not incident to x or y. Then $(\omega_F(Q), x_b, y^t)$ is a counterexample to the strong $(c+1)$-conjecture and is a W_{c+1}-figure.*

That is, any polytope Q with the prescribed properties serves as the seed for a family of counterexamples to the strong d-step conjecture for all $d > c$, simply by iterating the construction in Proposition 5.1 above. The Q_4 of Section 4 is such a polytope, and serves as the seed for the family of counterexamples constructed here.

For this first family of counterexamples, denoting by x the vertex x_b in every iterate P_d, γ_x has only four nonzero entries, an extreme case of a phenomenon already noted in [KK′]. Only four of the d edges incident to x occur in a shortest path from x to y^t; for large d, most choices of pivot at x will not yield a shortest path. For example in P_5, $\gamma_x = (4, 4, 2, 2, 0)$, and in P_6, $\gamma_x = (8, 8, 4, 4, 0, 0)$. In this family,

$$\gamma_x = (2^{d-3}, 2^{d-3}, 2^{d-4}, 2^{d-4}, 0, \ldots, 0),$$

and

$$\gamma_{y^t} = (0, 2, 2, 4, 4, 12, 24, \ldots, 3 \cdot 2^{d-4}).$$

Since there is only one 0 in γ_{y^t} for each iterate, the truncation-and-wedge construction is unique at y^t; that is, once we have truncated at y^t, there is a unique choice of $z \in \Sigma(y^t)$ to produce a counterexample. However, many variations of this family can be constructed by applying the truncation at x in any iterate; $z \in \Sigma(x)$ can be chosen to be $\sigma(u)$ for any of the $d-4$ neighbors u of x with $\gamma_x(u) = 0$. Although many combinatorial types of counterexamples may be produced in this way, with many γ_x and γ_{y^t}, and with small variations in the number of vertices, all such counterexamples P will have $\#^d P(x, y^t) = 3 \cdot 2^{d-3}$. In fact, except for P_5, all counterexamples constructed in these ways will have summary statistic $(3 \cdot 2^{d-3})_2$; to prove this, all we have left to show is the following.

Proposition 5.4. *If (P_d, x, y^t) is a W_d-figure, and $P_{d+1} = \omega_{\tau(T)}\tau_{y^t}(P_d)$ with $z = \sigma(y_b)$, then there are only two estranged pairs in P_{d+1}, (x_b, z^t) and (x^t, z_b).*

Proof. Since P_{d+1} is a wedge, one vertex of any estranged pair must lie in the top, the other in the base, and neither in the foot $\tau(T)$. So suppose without loss of generality that u_b and v^t are estranged vertices in P_{d+1} with u_b in the base, v^t in the top. Then in $\tau_{y^t}(P_d)$, u and v are estranged vertices, neither incident to $\tau(T)$. However, P_d is itself a wedge, so either $u \in B$ and $v = z$, or $u = z$ and $v \in B$. Since neither u nor v is incident to $\tau(T)$ in this $(d, 2d+1)$-polytope, there is only one vertex in B estranged from z, but x is estranged from z and so must be this vertex. Hence, either $u = x$ and $v = z$, or $u = z$ and $v = x$, and the result follows. □

6. Additional Comments

If (P, x, y) is a (simple) d-dimensional Dantzig figure, then the polar polytope Q is simplicial. The boundary complex of Q is a triangulated $(d-1)$-sphere with $2d$ vertices and the facets ($(d-1)$-simplices) F_x and F_y of Q that correspond to x and y do not share a vertex and hence may be called *estranged*. Under polarity, the paths (*edge-paths*) of length d from x to y in P correspond to *ridge-paths* of length d from F_x to F_y in Q. (See [KK′] for details.) The computational procedure of Section 1 applies without change to determine, for each estranged pair of facets of a triangulated $(d-1)$-manifold, the number of ridge-paths of length d joining the two facets.

In addition to the 37 different combinatorial types of simplicial 4-polytopes with 8 vertices, there are nonpolytopal triangulated 3-spheres with 8 vertices. The Brückner sphere, listed in [GS], does not have any estranged pair of facets. The Barnette sphere [Ba] has summary statistic 15_2.

In cataloging the triangulated 3-manifolds with 9 vertices, Altshuler and Steinberg [AS] found 1297 different combinatorial types. With the aid of Bokowski (as reported in [ABS]), these were found to consist of one nonsphere, 154 nonpolytopal spheres, and 1142 polytopes. A tape containing their catalog was (many years ago) sent by Steinberg to Klee, who found that all but one of those manifolds is of ridge-diameter ≤ 4. The sole exception was the simplicial 4-polytope that is dual to the simple 4-polytope Q_4 (with 9 facets and edge-diameter 5) that was used in Section 3 as the basis for our constructions.

Early in the study of the d-step conjecture, it was felt that the dual-cyclic polytopes and other dual-neighborly polytopes were the most natural candidates for counterexamples to the conjecture. However, the Hirsch conjecture was proved by [Kl] for the dual-cyclic polytopes, and Lagarias and Prabhu [LP] have proved the strong d-step conjecture for these polytopes. Both the d-step conjecture and the strong d-step conjecture are still open for more general dual-neighborly polytopes, but Kalai [K1] established a weaker form of the d-step conjecture (and of the Hirsch conjecture), showing that $\delta(P) \leq d^2(n-d)^2 \log n$ for each dual-neighborly (d, n)-polytope.

Among the $(d, 2d)$ polytopes, the minimum possible number of vertices is $d^2 - d + 2$ and the maximum is

$$2\binom{(3d-1)/2}{d} \quad \text{or} \quad \frac{4}{3}\binom{3d/2}{d}$$

according as d is odd or even. The maximum is attained by the polars of cyclic polytopes and the minimum by the polars of stacked polytopes, and the strong d-step conjecture has been verified for both of these classes by Lagarias and Prabhu [LP]. The number of vertices is relatively small for the counterexamples to the strong d-step conjecture constructed (for $d \geq 5$) in Sections 4 and 5; the number of vertices of P_d is $d^2 + 9d - 28$.

Finally, it should be mentioned that Kalai [K2], [K3], Kalai and Kleitman [KK], and Matoušek, Sharir, and Welzl [MSW] have established subexponential upper bounds on $\Delta(d, n)$, and that Frieze and Teng [FT] have shown that computing the diameter of a polytope is an $\mathbb{NP}$-hard problem.

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