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A Helly-Type Theorem for Unions of Convex Sets*

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Abstract. We prove that for any $d, k \geq 1$ there are numbers $q = q(d, k)$ and $h = h(d, k)$ such that the following holds: Let $\mathcal{K}$ be a family of subsets of the d-dimensional Euclidean space, such that the intersection of any subfamily of $\mathcal{K}$ consisting of at most q sets can be expressed as a union of at most k convex sets. Then the Helly number of $\mathcal{K}$ is at most h. We also obtain topological generalizations of some cases of this result. The main result was independently obtained by Alon and Kalai, by a different method.

1. Results and Open Questions

Let $\mathbb{R}^d$ denote the d-dimensional Euclidean space. A famous theorem of Helly, discovered in 1913, asserts that if $\mathcal{F}$ is a finite family of convex sets in $\mathbb{R}^d$ such that any $d + 1$ or fewer sets of $\mathcal{F}$ have a point in common, then also the intersection of all sets of $\mathcal{F}$ is nonempty. Over the years, a vast body of analogs and generalizations of this result has been accumulated in the literature, see [Ec] for a recent survey.

A general scheme of a Helly-type theorem is captured by the following definition. Let $\mathcal{K}$ be an arbitrary family of sets. We say that $\mathcal{K}$ has *Helly number* h (h is a natural number) if the following holds for any finite subfamily $\mathcal{F} \subseteq \mathcal{K}$: if the intersection of any subfamily of $\mathcal{F}$ consisting of $\leq h$ sets is nonempty, then $\bigcap \mathcal{F}$ (the intersection of all sets of $\mathcal{F}$) is nonempty. Thus, the family $\mathcal{C}$ of all convex sets in $\mathbb{R}^d$ has Helly number $d + 1$.

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We prove the following Helly-type result:

Theorem 1. *For any $k \geq 1$, $d \geq 1$ there are numbers $q = q(d, k)$ and $h = h(d, k)$ with the following property. Let $\mathcal{K}$ be a family of subsets of $\mathbb{R}^d$ such that the intersection of any subfamily of $\mathcal{K}$ consisting of q or fewer sets can be expressed as a union* (*not necessarily disjoint*) *of at most k convex sets. Then $\mathcal{K}$ has Helly number at most h.*

This result was independently obtained by Alon and Kalai [AKa]. Their method is more complicated than the one presented here (and quite different), on the other hand, they obtain the result as a consequence of a powerful theorem concerning piercing numbers of certain families, which they prove by a method developed by Alon and Kleitman [AK1].

The estimates of the numerical values of the Helly numbers and of the numbers $q(d, k)$ following from our proof of Theorem 1 are quite large. For instance, for unions of convex sets in the plane, the current proof gives estimates $h(2, 2) \leq 20$, $h(2, 3) \leq 90$, $h(2, 4) \leq 231$, etc. Small improvements are possible by refining the argument, but we are still probably quite far from the best values. It would be interesting to get better upper bounds and/or some nontrivial lower-bound examples.

We also prove a partial topological analog of Theorem 1. Before stating it, we recall the notion of a j-*connected* topological space. A topological space X is said to be (-1)-connected if it is nonempty. For $j = 0, 1, 2, \ldots$, X is j-connected if it is $(j-1)$-connected and moreover any continuous mapping f of the sphere S^j into X can be extended to a continuous mapping $\bar{f}\colon B^{j+1} \to X$, where B^{j+1} is the ball bounded by S^j. In particular, a 0-connected set means just path-connected and nonempty, and 1-connected means 0-connected and simply connected. Intuitively, a 1-connected subspace of $\mathbb{R}^3$ may not have any "tunnels" but may have "bubbles."

Theorem 2. *For any $d \geq 2$, $k \geq 1$ there is a number $h = h(d, k)$* (*where $h(d, 1) \leq d + 2$ for d even and $h(d, 1) \leq d + 3$ for d odd*) *with the following property. Let $\mathcal{K}$ be a family of subsets of $\mathbb{R}^d$ such that the intersection of any nonempty finite subfamily of $\mathcal{K}$ has at most k path-connected components, each of which is $(\lceil d/2 \rceil - 1)$-connected.* (*In particular, for $d = 2$ we require that any intersection of a finite subfamily of $\mathcal{K}$ has at most k path-connected components.*) *Then $\mathcal{K}$ has Helly number at most h.*

The author learned of a proof of the simple special case $d = 2$, $k = 1$ of Theorem 2 (in a slightly different context) from Nina Amenta. We suspect it was also observed by others a long time ago, but currently we have no explicit reference.

Currently we do not know whether it is sufficient to restrict the assumption to at most q-wise intersections for some bounded $q = q(d, k)$ (as can be done in Theorem 1). We believe that there might be a general result saying that if all at most q-wise intersections of sets of a family $\mathcal{K}$ in $\mathbb{R}^d$ have "topological complexity" bounded by k, then the Helly number of $\mathcal{K}$ is bounded in terms of d and k; here q would depend on k, d. A definition of "topological complexity" of a set suitable in this context is yet to be found; perhaps it could be the minimum number of simplices needed to triangulate the set, or some number derived from the homology groups.

The rest of the paper is structured as follows. In Section 2 we describe some previous work related to our result and we mention a recent new motivation for studying Helly-type problems. This part is not needed for understanding the proofs. The proofs are given in Sections 3 (Theorem 1) and 4 (Theorem 2). Section 5 presents two examples whose meaning is discussed in Section 2 below.

2. Motivation and Related Work

LP-Type Problems. Recently, a new motivation for studying Helly-type results came from geometric optimization algorithms. Sharir and Welzl [SW] defined a class of optimization problems, the so-called *LP-type problems*, which encompasses linear programming, convex programming, and other natural geometric optimization problems. They gave an efficient algorithm for solving problems in this class (provided that certain primitive operations can be implemented efficiently for the problem in question). Also several other algorithms can be applied for LP-type problems; see [G], [C], [MSW], and [Ma]. A crucial parameter in these problems is their *dimension*, and this is closely related to the Helly number of suitable set systems in $\mathbb{R}^d$. Roughly speaking, if it is desired to show that the above-mentioned algorithms work fast for an optimization problem with some set of constraints, then it is necessary to bound the Helly number of certain derived set systems by a small number. These relations have been investigated by Amenta [A1], [A2]. We believe that studying Helly-type properties is important for understanding the structure of LP-type problems and potentially also for developing and analyzing yet more-efficient algorithms for these problems, or perhaps proving lower bounds for such algorithms.

Previous Work: *Disjoint Unions*. Grünbaum and Motzkin [GM] considered Helly-type theorems for disjoint unions of convex sets. They conjectured the following:

Theorem 3. *Let $\mathcal{K}$ be a family of sets in $\mathbb{R}^d$ such that the intersection of any its subfamily of size at most k can be expressed as a disjoint union of k closed convex sets*. *Then the Helly number of $\mathcal{K}$ is at most $k(d+1)$.*

In this theorem it is important to assume that not only the members of $\mathcal{K}$, but also their $\leq k$-wise intersections can be expressed as disjoint unions of at most k closed convex sets. For instance, the family of all unions of disjoint pairs of closed convex sets has no finite Helly number. For the same reason, in Theorem 1 we need an assumption concerning $\leq q$-wise intersections of sets of $\mathcal{K}$ (rather than sets of $\mathcal{K}$ only).

Examples show that the bound $k(d+1)$ for the Helly number in Theorem 3 is the best possible in general. Grünbaum and Motzkin [GM] proved Theorem 3 for $k = 2$, the $k = 3$ case was established by Larman [L], and the general case was proved by Morris [Mo].[1] A short elegant proof of Theorem 3 was given by Amenta [A1].[2]

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[1] To which Eckhoff [Ec] remarks "However, Morris' proof ... is extremely involved and the validity of some of his arguments is, at best, doubtful."

[2] Grünbaum and Motzkin in fact conjectured a more general result in an abstract setting. Let $\mathcal{B}$ be a family of sets which is *intersectional* (that is, $B_1 \cap B_2 \in \mathcal{B}$ for any $B_1, B_2 \in \mathcal{B}$) and *nonadditive* (that is, no finite disjoint union of at least two nonempty sets of $\mathcal{B}$ belongs to $\mathcal{B}$) and has Helly number at most h. Let $[\mathcal{B}]_k$ denote the family of all disjoint unions of at most k members of $\mathcal{B}$. The conjecture of [GM] can be formulated

We note that our Theorem 1 implies that, in particular, under the assumptions of Theorem 3, the Helly number is bounded, but the proof only provides a much worse bound than the correct value $k(d+1)$. On the other hand, in our Theorem 1 the unions need not be disjoint and the convex sets considered need not be closed.

The closedness assumption in Theorem 3 is important for the value of the Helly number. We consider a family $\mathcal{K}$ in the plane such that each set as well as the intersection of each two sets is a disjoint union of two convex sets (not necessarily closed). Grünbaum and Motzkin [GM] construct such a family with Helly number 9 (instead of 6 as would be obtained with convex sets replaced by closed convex sets). On the other hand, it is still possible that the Helly number of any such $\mathcal{K}$ is bounded; Grünbaum and Motzkin conjecture that 9 is the correct value. Theorem 1 implies a weaker statement, namely that if it is assumed that the intersection of any at most three sets of $\mathcal{K}$ is a union of at most two convex sets, then the Helly number is bounded by a constant.

Depth of Intersections and n-Convexity. As was observed in [GM], if $\mathcal{K}$ is a family such that the intersection of any $\leq k$ sets of $\mathcal{K}$ is a union of at most k closed convex sets, then the intersection of *any number* of sets of $\mathcal{K}$ is a union of at most k closed convex sets. This leads to the question whether something similar holds in the situation of Theorem 1. Specifically, we might ask for which d and k there are numbers $c = c(d,k)$, $K = K(d,k)$ such that the following statement holds:

Statement 4. Suppose that $\mathcal{K}$ is a family in $\mathbb{R}^d$ such that the intersection of any at most c sets of $\mathcal{K}$ can be expressed as a union of at most k convex sets. Then the intersection of any finite subfamily of $\mathcal{K}$ is a union of at most K convex sets.

A simple example, presented as Example 9 in Section 5 below, shows that, with $K = k$ (which would be the strongest form), Statement 4 holds for no c, even in the simplest case $d = k = 2$. Example 10 then implies that, for $d \geq 4$, $k \geq 2$, Statement 4 holds with no c, K at all.

Example 9 was noted by Pavel Valtr. Later the author found out that a related question has been investigated in several previous papers (see [PS], [BK], and references therein) under the heading of *n-convexity*, and the examples were essentially known. In particular, the four-dimensional Example 10 uses almost the same construction as an example due to Perles (published in [KPS]). We thus include the examples for the reader's convenience only.

For $d = 2$ and perhaps for $d = 3$, Statement 4 might be true with large enough c and K. For $d = 2$, the results of Perles and Shelah [PS] easily imply that Statement 4 holds, after replacing "convex sets" by "closed convex sets," with $c(2,k) \leq \binom{k+1}{2}$,

as follows: *If $\mathcal{K}$ is a family of sets such that the intersection of any at most k sets of $\mathcal{K}$ belongs to $[\mathcal{B}]_k$, then the Helly number of $\mathcal{K}$ is at most kh.* The family of all closed (resp. open) convex sets in $\mathbb{R}^d$ is both intersectional and nonadditive. On the other hand, nonadditivity fails for the family of all convex sets. The $k = 2, 3$ cases were proved in the above-mentioned papers [GM] and [L], respectively; for larger k, only Morris' proof seems to be available: Amenta's proof of Theorem 3 can also be stated in an abstract setting, but different from the one just described.

$K(2, k) \leq k^6$. A forthcoming paper [MV] improves the bound k^6 to $O(k^3)$, and shows that, for $d = 2$, Statement 4 holds for all k (with suitable $c(2, k)$ and $K(2, k)$). The case $d = 3$ seems to be wide open.

Topological Helly-Type Theorems. Helly's theorem on convex sets has various topological generalizations. Helly himself gave a topological version of his theorem in [H]. A modern proof and some generalizations were given by Debrunner [D]. To state some of them, we recall a few more topological notions.

A topological space X is called a *homology cell* if it is nonempty and its (singular, reduced) homology groups of all dimensions vanish; in particular, convex sets are homology cells. More generally, X is called j*-acyclic* if all its homology groups up to dimension j vanish. We remark that a j-connected topological space is j-acyclic, and for $j \geq 2$ a j-cyclic 1-connected space is j-connected (but acyclic spaces exist which are not 1-connected).

Helly's result can be rephrased as follows (see [D]): *Let $\mathcal{F}$ be a finite family of open subsets of $\mathbb{R}^d$ such that the intersection of any at most d members of $\mathcal{F}$ is a homology cell, and the intersection of any $d + 1$ members is nonempty. Then $\bigcap \mathcal{F}$ is a homology cell; in particular, it is nonempty*. As a consequence, assuming that the intersection of any at most d members of a family $\mathcal{K}$ of open sets in $\mathbb{R}^d$ is a homology cell, we get that the Helly number of $\mathcal{K}$ is $d + 1$. Debrunner [D] shows that it is enough to assume that the j-wise intersections of sets of $\mathcal{F}$ are $(d - j)$-acyclic, $j = 1, 2, \ldots, d + 1$. He also generalizes the results to families of open subsets of an arbitrary d-manifold.

In this context, Theorem 2 is of some interest also for $k = 1$, since the topological theorems just mentioned require that the sets in the considered family are homology cells, while our result allows them to have "holes," more precisely, a nonzero homology in dimensions $\lceil d/2 \rceil$ through $d-1$. On the other hand, we assume $(\lceil d/2 \rceil - 1)$-connectedness for all intersections, which is a very strong requirement.

3. Proof for Unions of Convex Sets

In this section we prove Theorem 1.

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. For a set X and a natural number t, let $\binom{X}{t}$ denote the set of all t-element subsets of X (we sometimes call them t-sets).

We need a suitable d-dimensional generalization of the well-known fact that a planar graph with n vertices has only $O(n)$ edges. We set $b = \lceil d/2 \rceil + 1$.

Lemma 5. *Let d and $\alpha > 0$ be fixed, and let $n = n(d, \alpha)$ be sufficiently large. Let P be an n-point set in $\mathbb{R}^d$, and let $\mathcal{S} \subseteq \binom{P}{b}$ be a family of b-sets of P with $|\mathcal{S}| \geq \alpha \binom{n}{b}$. Then disjoint sets $S, S' \in \mathcal{S}$ exist such that* $\operatorname{conv}(S) \cap \operatorname{conv}(S') = \emptyset$.

We remark that, for our proof, it would be sufficient to have this lemma, e.g., with $d + 1$ instead of b, only we get somewhat worse values for $q(d, k)$ and $h(d, k)$. Such a statement is explicitly proved by Alon *et al*. [ABFK] (see also [BFL]).

For $d = 3$, Lemma 5 is a special case of the results of Dey and Edelsbrunner [DE]. Dey and Pach [DP] consider extensions of a similar method into higher dimensions; they explicitly prove the statement of the Lemma with d instead of b, and it seems that their method can be extended to yield the lemma itself.

Another proof for a general dimension can be given along the lines of Alon *et al.* [ABFK]; we only sketch the method here. Consider the hypergraph $\mathcal{H}$ with vertex set $V(\mathcal{H}) = \{(i, j); i = 1, 2, 3, j = 1, 2, \ldots, b\}$ and edge set

$$E(\mathcal{H}) = \{\{(i_1, 1), (i_2, 2), \ldots, (i_b, b)\}; i_1, i_2, \ldots, i_b \in \{1, 2, 3\}\}$$

(a complete b-partite b-uniform hypergraph). By a result of van Kampen [vK] (mentioned in [S]), the simplicial complex whose maximal simplices are the edges of $\mathcal{H}$ cannot be embedded into $\mathbb{R}^d$ so that no two vertex-disjoint simplices intersect. By the Erdös–Stone theorem, the hypergraph with vertex set P and edge set $\mathcal{S}$ contains a copy of $\mathcal{H}$, see [ABFK], and the lemma follows. $\square$

Our basic approach is to prove Theorem 1 by induction on the number of sets in the considered finite family $\mathcal{F}$. One problem here is that we only assume that the intersection of any subfamily of size $\leq q$ is "nice," i.e., is a union of k convex sets. As was mentioned in Section 2 (see Statement 4 and also Example 10 in Section 5), this does not necessarily mean that intersections of larger subfamilies must be "nice" in this sense. We circumvent this by introducing another notion of "nice," which behaves more regularly in this respect.

Definition 6. Let t, j be natural numbers, $t \geq j$, and let X be a set in $\mathbb{R}^d$. We say that X has *property* $P(t, j)$ if for any t-set $T \subseteq X$ a j-set $J \subseteq T$ exists such that the convex hull of J is contained in X.

We note that if a set X is a union of at most k convex sets and $t \geq k(j-1)+1$, then X has property $P(t, j)$ (by the pigeonhole principle). Property $P(t, j)$ is more convenient to work with for our purposes than the property "being a union of $\leq k$ convex sets," because of the following easy lemma:

Lemma 7. *Let $\mathcal{K}$ be a family of sets in $\mathbb{R}^d$ such that the intersection of any at most $\binom{t}{j}$ sets of $\mathcal{K}$ has property $P(t, j)$. Then the intersection of any subfamily of $\mathcal{K}$ has property $P(t, j)$.*

Proof. Let X be the intersection of a subfamily $\mathcal{F} \subseteq \mathcal{K}$, and suppose that property $P(t, j)$ fails for X, that is, a t-set $T \subseteq X$ can be found such that none of the convex hulls of its j-sets is fully contained in X. For each $J \in \binom{T}{j}$, fix a set $F_J \in \mathcal{F}$ with $\operatorname{conv}(J) \not\subseteq F_J$. Then property $P(t, j)$ also fails for the intersection

$$\bigcap_{J \in \binom{T}{j}} F_J. \qquad \square$$

We fix $t = k(b-1)+1$, $q = q(d,k) = \binom{t}{b}$. Then the assumption of Theorem 1 with this value of q implies that the intersection of any at most q sets of sets of $\mathcal{K}$ has property $P(t,b)$, and by Lemma 7 any intersection of sets of $\mathcal{K}$ has property $P(t,b)$. Hence Theorem 1 is proved by establishing the following:

Proposition 8. *For any $d \geq 1$, $t \geq b = \lceil d/2 \rceil + 1$ there is a number $h = h(d,t)$ with the following property. Let $\mathcal{K}$ be a family of subsets of $\mathbb{R}^d$ such that the intersection of any of its finite subfamily has property $P(t,b)$. Then $\mathcal{K}$ has Helly number at most h.*

Proof. Throughout the proof, d and t are treated as constants (in the O, Ω notation). Let h be sufficiently large so that all estimates below are valid (by inspecting the current proof and a proof of Lemma 5, a specific value for $h = h(d,t)$ can be found).

To prove that $\mathcal{K}$ has Helly number h it is enough to show that if $\mathcal{F} \subseteq \mathcal{K}$ is any subfamily of $h+1$ sets of $\mathcal{K}$ such that the intersection of each h sets of $\mathcal{F}$ is nonempty, then $\bigcap \mathcal{F} \neq \emptyset$ (the case of a larger $\mathcal{F}$ is then handled by induction on the number of elements of $\mathcal{F}$). Let the sets of $\mathcal{F}$ be numbered $F_1, F_2, \ldots, F_n$, $n = h+1$. For each i, choose a point $p_i \in \bigcap_{j \in [n]\setminus\{i\}} F_j$.

Consider a t-set $J \in \binom{[n]}{t}$. All the points p_j, $j \in J$, belong to the intersection $X_J = \bigcap_{j \in [n]\setminus J} F_j$. Since X_J has property $P(t,b)$, we can fix a b-set $K = K(J) \subseteq J$ such that $\Delta_K := \operatorname{conv}\{p_j;\, j \in K\}$ is contained in X_J (if there are more choices for $K(J)$ fix one arbitrarily). In this situation we assign the set $J \setminus K$ to the b-set $K = K(J)$ as a label.

As we fix the $K(J)$ for each t-set J, labels are assigned to b-sets. One b-set K can receive one or several labels, or no label at all. If L is one of the labels of K, then we know that the $(b-1)$-simplex Δ_K is contained in all sets F_j with $j \notin L \cup K$. We call a b-set K *good* if it has at least one label but the intersection of all its labels is empty, and call K *bad* otherwise. Thus, for a good b-set K, Δ_K is contained in all sets F_j with $j \notin K$.

Each bad b-set is assigned at most $\binom{n-b-1}{t-b-1}$ labels, and there are $\binom{n}{b}$ b-sets, thus at most $O(n^{t-1})$ labels are assigned to bad b-sets. Since $\binom{n}{t}$ labels are assigned altogether, at least $\Omega(n^t)$ labels are assigned to good b-sets. A good b-set is assigned $O(n^{t-b})$ labels, therefore there are $\Omega(n^b)$ good b-sets.

By Lemma 5, there are two disjoint good b-sets K, K' with $\Delta_K \cap \Delta_{K'} \neq \emptyset$. By the above-mentioned property of a good b-set, Δ_K is contained in all sets F_j with $j \notin K$, and similarly for $\Delta_{K'}$. Thus, $\emptyset \neq \Delta_K \cap \Delta_{K'} \subseteq \bigcap_{j=1}^{n} F_j$. $\square$

4. Proof of the Topological Result

The proof of Theorem 2 begins similarly to the proof of Proposition 8. We consider a family $\mathcal{F} = \{F_1, \ldots, F_n\}$, with $n = h+1$ sufficiently large, and points $p_1, \ldots, p_n$, with p_i belonging to all sets of $\mathcal{F}$ but possibly F_i. We set $t = k+1$, and consider a t-set $J \in \binom{[n]}{t}$. We know that all points p_j, $j \in J$, belong to the intersection $X_J = \bigcap_{i\in[n]\setminus J} F_i$, and this is a union of at most k path-connected components. Therefore, there is a pair $P(J) = \{j_1, j_2\} \subset J$ of indices such that p_{j_1} and p_{j_2} belong to a common path-connected component of X_J (in fact, there may be many pairs, so we fix one arbitrarily for each t-set J).

The t-set considered is linearly ordered, and the pair $P(J)$ can be uniquely encoded by specifying a pair of elements of the set $\{1, 2, \ldots, t\}$ (we call it the *type* of the pair $P(J)$). The number of possible types is $\binom{t}{2}$, the important fact is that it is bounded by a function of t. In this way, each $J \in \binom{[n]}{t}$ is assigned one of a bounded number of types, which can be viewed as coloring all t-sets on $[n]$. If m is a prescribed parameter and $n = n(m, t)$ is chosen sufficiently large, by Ramsey's theorem (see, e.g., [GRS]) an m-element subset $M \subseteq [n]$ exists such that all t-sets $J \in \binom{M}{t}$ have the same color. In our case this means that they all have the same type of the pair $P(J)$.

The Planar Case. First we finish the proof for the case $d = 2$, which is somewhat simpler than the case of an arbitrary dimension. Choose a fixed nonplanar graph G, say $G = K_5$, and let V be its vertex set and E its edge set. We assume that a set $M \subseteq [n]$ has been selected as above, so that all t-sets have the same type of $P(J)$, and $|M|$ is sufficiently large (as we will see, $|M| \geq 5 + 10(t-2)$ suffices). We illustrate the method for $k = 2$ (then $t = 3$) and assuming that the type of the pair is $\bullet\ \bullet\ \circ$, the other cases are quite similar. To each vertex $v \in V$ we assign a point $\varphi(v) \in M$ and to each edge $e \in E$ we assign a t-set $\Phi(e)$ of points of M, in such a way that:

(C1) For any two disjoint $e, e' \in E$, we have $\Phi(e) \cap \Phi(e') = \emptyset$, and if $e \neq e'$ share a vertex v, then $\Phi(e) \cap \Phi(e') = \{\varphi(v)\}$.
(C2) For each $e = \{u, v\} \in E$, the points $\varphi(u)$ and $\varphi(v)$ are just the points of the pair $P(\Phi(e))$.

Such an assignment, for the particular type of the pair $P(J)$ of 3-sets, may be chosen as indicated in Fig. 1 (the solid circles represent the points $\varphi(v_1), \ldots, \varphi(v_5)$, the open circles the other points of M, and the triples are marked by the segments at various levels connected to their points). For larger t and/or other types of the pairs $P(J)$, we proceed similarly: First we map the vertices of G by φ to distinct points of M, leaving large enough gaps among them. For each edge $e = \{u, v\}$, its Φ-image consists of $\varphi(u)$ and $\varphi(v)$ and $t-2$ other points of M, chosen so that they do not occur in the image of any other edge, and so that $P(\Phi(e)) = \{\varphi(u), \varphi(v)\}$.

Now by (C2), for each edge $e = \{u, v\} \in E$, the points $p_{\varphi(u)}$ and $p_{\varphi(v)}$ lie in the same path-connected component of the intersection $X_{\Phi(e)} = \bigcap_{j\in[n]\setminus\Phi(e)} F_j$, so we can choose a path $\Delta_e \subseteq X_{\Phi(e)}$ connecting $p_{\varphi(u)}$ and $p_{\varphi(v)}$. These paths together yield a drawing of K_5, so some two paths belonging to vertex-disjoint edges e, e' cross. The intersection of these paths belongs to all sets F_i with $i \notin \Phi(e)$, and also to all F_i with $i \notin \Phi(e')$. However, $\Phi(e) \cap \Phi(e') = \emptyset$ by (C1) and we are done.

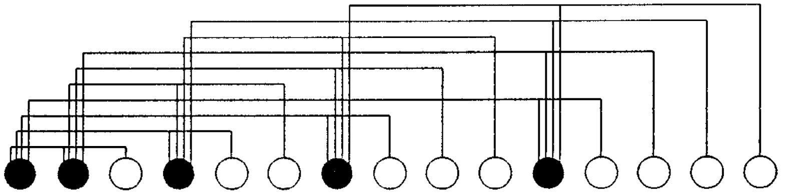

Fig. 1. The embedding Φ for the graph K_5.

Arbitrary Dimension. To complete the proof of Theorem 2 for a general dimension d, we need a finite $\lceil d/2\rceil$-dimensional simplicial complex $\mathcal{S}$, which will play the role of the nonplanar graph K_5 used in the planar case. Precisely, we use the following property: Whenever $f\colon \|\mathcal{S}\| \to \mathbb{R}^d$ is a continuous mapping, vertex-disjoint simplices $s, s' \in \mathcal{S}$ exist such that $f(\|s\|) \cap f(\|s'\|) \neq \emptyset$ (here $\|\mathcal{S}\|$ denotes the *polyhedron* of $\mathcal{S}$, which is the topological space of some geometric realization of $\mathcal{S}$, and, for a simplex $s \in \mathcal{S}$, $\|s\|$ means the closed subset of $\|\mathcal{S}\|$ corresponding to the simplex s; see, e.g., [Mu] for an introduction to simplicial complexes). We might use, e.g., the complex mentioned in the proof sketch for Lemma 5, or we may apply a more well-known result of van Kampen [vK] and Flores [F], which says that if $\mathcal{S}$ is the j-skeleton of the $(2j+2)$-dimensional simplex, then it has the required property for $d \le 2j$.

Let a suitable simplicial complex $\mathcal{S}$ be fixed. We let G be its 1-skeleton, consisting of the vertices and 1-simplices (edges) of $\mathcal{S}$. Hence G is formally a one-dimensional simplicial complex, but we may also regard it as a graph with vertex set V and edge set E consisting of all one-dimensional simplices of $\mathcal{S}$. We perform for G the construction we did for K_5 in the planar case. Thus, we have a sufficiently large set $M \subseteq [n]$ of indices such that all t-sets of M have the same type, an injective mapping $\varphi\colon V \to M$, a mapping $\Phi\colon E \to \binom{M}{t}$ satisfying (C1) and (C2), and a "drawing" of G in $\mathbb{R}^d$, that is, a continuous mapping $f\colon \|G\| \to \mathbb{R}^d$. This f satisfies $f(v) = p_{\varphi(v)}$ for all $v \in V$ and $f(\|e\|) \subseteq X_{\Phi(e)}$ for each edge $e \in E$.

We extend the mapping f continuously to the whole $\|\mathcal{S}\|$. First we extend the definition of the mapping Φ to all simplices $s \in \mathcal{S}$. $\Phi(s)$ is already defined if $s \in \mathcal{S}$ is an edge (1-simplex). If $s = \{v\} \in \mathcal{S}$ is a vertex, we simply put $\Phi(s) = \{\varphi(v)\}$, and for $s \in \mathcal{S}$ of dimension ≥ 2 we let

$$\Phi(s) = \bigcup_{e\in E;\, e\subseteq s} \Phi(e).$$

It is easy to check that this extended Φ satisfies:

(C1′) For any two vertex-disjoint simplices $s, s' \in \mathcal{S}$, we have $\Phi(s) \cap \Phi(s') = \emptyset$.

The continuous extension of the mapping f is constructed inductively. Suppose that f has already been defined at all points of each $\|s\|$, where $s \in \mathcal{S}$ is a simplex of dimension less than j (for some j, $2 \le j \le \lceil d/2\rceil$), and moreover we have, for any such s,

$$f(\|s\|) \subseteq X_{\Phi(s)}. \tag{1}$$

We define f on j-dimensional simplices. Consider a j-dimensional simplex $s \in \mathcal{S}$. Let $\|\partial s\|$ denote the portion of $\|s\|$ corresponding to proper faces of s. This set is homeomorphic to the $(j-1)$-sphere S^{j-1}, and $f(\|\partial s\|)$ is contained in $X_{\Phi(s)}$, as follows from (1) applied on the $(j-1)$-faces of s. Since $\|\partial s\|$ is path-connected, its image is contained in a single path-connected component of $X_{\Phi(s)}$. By our assumptions, this component is $(j-1)$-connected, so we can extend f continuously to the relative interior of $\|s\|$, in such a way that the image is still contained in $X_{\Phi(s)}$. This finishes the induction step.

Having defined f on the whole $\|\mathcal{S}\|$, by the choice of $\mathcal{S}$ we know there are two vertex-disjoint faces $s, s' \in \mathcal{S}$ with $f(\|s\|) \cap f(\|s'\|) \neq \emptyset$. By (1), we have $f(\|s\|) \cap f(\|s'\|) \subseteq X_{\Phi(s)} \cap X_{\Phi(s')} = X_{\Phi(s)\cap\Phi(s')} = X_{\emptyset} = \bigcap_{[n]} F_i$, by (C1′). This concludes the proof. $\square$

5. Examples

In this section we present examples related to Statement 4 discussed in Section 2.

Example 9. For any odd integer n, sets $F_1, \ldots, F_n \subseteq \mathbb{R}^2$ exist such that the intersection of any at most $n-1$ of them is a union of two convex sets, while $\bigcap_{i=1}^n F_i$ cannot be expressed as a union of fewer than three convex sets.

Proof. First, let C be a regular convex n-gon and let $v_1, \ldots, v_n$ denote its vertices numbered along the circumference. We let F_i be C minus the relative interior of the edge $v_i v_{i+1}$ (v_{n+1} meaning v_1). If the intersection $\bigcap_{i=1}^n F_i$ were a union of two convex sets, there would be two consecutive vertices v_i, v_{i+1} belonging to the same convex set, but this is impossible, since the interior of the edge $v_i v_{i+1}$ is missing from the intersection. On the other hand, for any intersection X of fewer than n of the F_i, there is one edge, say $v_n v_1$, which is contained in X. Let A_1 be the set X minus all vertices v_i with i odd, and let A_2 be X minus all vertices v_i with i even. Then we have $X = A_1 \cup A_2$ and it is easily checked that A_1 and A_2 are convex.

The example can be easily modified so that sets F_i are closed (or open). Consider the midpoint m_i of the edge $v_i v_{i+1}$. Move m_i a little bit toward the center of C, obtaining a point $\bar{m}_i$, and let F_i be C minus the interior of the triangle $v_i \bar{m}_i v_{i+1}$. Figure 2 illustrates the construction for $n = 5$: the left part shows the set F_1, and the right part the set $\bigcap_{i=1}^5 F_i$. $\square$

Example 10. For any given integers c, K, there are n and sets $F_1, \ldots, F_n \subseteq \mathbb{R}^4$ such that the intersection of any at most c of them is a union of two convex sets, while $\bigcap_{i=1}^n F_i$ cannot be expressed as a union of fewer than K convex sets.

Proof. Let G be a K-chromatic graph such that any subgraph of G with at most c edges is 2-colorable (the existence of such a graph follows from [Er], say). Let C be a cyclic polytope in $\mathbb{R}^4$ with $|V(G)|$ vertices, and suppose that its vertices are identified with the vertex set V of G. Any two vertices of C are connected by an edge (one-dimensional face) of C. Let $e_1, \ldots, e_n$ be the edges of G. For $e_i = \{u, v\}$, we let F_i be C minus the

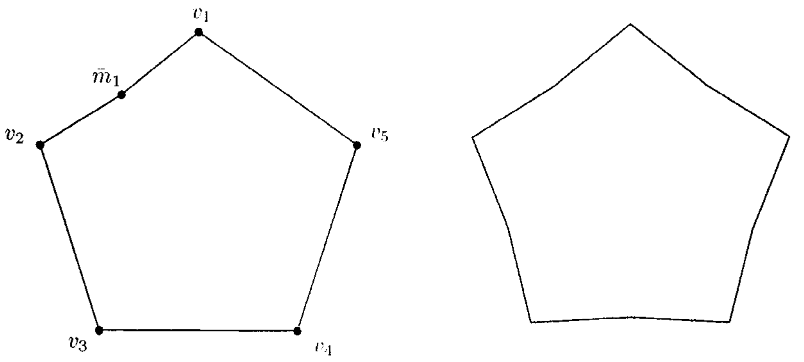

Fig. 2. Illustration to Example 9.

relative interior of the edge uv of the polytope C. The set $\bigcap_{i=1}^{n} F_i$ cannot be expressed as a union of fewer than K convex sets, since this would induce a proper coloring of G by less than K colors. On the other hand, if $X = \bigcap_{e_i \in E'} F_i$, where E' is a set of at most c edges of G, we let $\chi\colon V \to \{1, 2\}$ be a 2-coloring of the graph with edge set E'. For $j = 1, 2$ we set $A_j = X \backslash \{v \in V; \chi(v) \neq j\}$. As in the previous example, it may be checked that the A_j are convex and cover X. A modification with all F_i closed or all F_i open is again possible. $\square$

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