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Lattice Coverings and Gaussian Measures of n-Dimensional Convex Bodies*

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Abstract. Let $\|\cdot\|$ be the euclidean norm on $\mathbf{R}^n$ and let γ_n be the (standard) Gaussian measure on $\mathbf{R}^n$ with density $(2\pi)^{-n/2}e^{-\|x\|^2/2}$. Let ϑ ($\simeq$ 1.3489795) be defined by $\gamma_1([-\vartheta/2, \vartheta/2]) = \frac{1}{2}$ and let L be a lattice in $\mathbf{R}^n$ generated by vectors of norm $\leq \vartheta$. Then, for any closed convex set V in $\mathbf{R}^n$ with $\gamma_n(V) \geq \frac{1}{2}$, we have $L + V = \mathbf{R}^n$ (equivalently, for any $a \in \mathbf{R}^n$, $(a + L) \cap V \neq \emptyset$). The above statement can also be viewed as a "nonsymmetric" version of the Minkowski theorem.

Let U, V be a pair of convex sets in $\mathbf{R}^n$ containing the origin in the interior. We define $\beta(U, V)$ as the smallest $r > 0$ satisfying the following condition: to each sequence $u_1, \ldots, u_n \in U$ there correspond signs $\varepsilon_1, \ldots, \varepsilon_n = \pm 1$ such that $\varepsilon_1 u_1 + \cdots + \varepsilon_n u_n \in rV$. Upper and lower bounds for $\beta(U, V)$ for various sets U and V (usually centrally symmetric) were investigated by several authors. We mention some of their results once the appropriate notation is introduced, see also the references in [3].

Let L be a lattice in $\mathbf{R}^n$, i.e., an additive subgroup of $\mathbf{R}^n$ generated by n linearly independent vectors. The quantities (again, usually defined for centrally symmetric sets)

$$\lambda_n(L, U) = \min\{r > 0: \dim \operatorname{span}(L \cap rU) = n\},$$
$$\mu(L, V) = \min\{r > 0: L + rV = \mathbf{R}^n\}$$

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are called the nth minimum and the covering radius of L with respect to U and V, respectively; sometimes $\mu(L, V)$ is called "the nth covering minimum" and denoted $\mu_n(L, V)$. We define

$$\alpha(U, V) = \sup_L \frac{\mu(L, V)}{\lambda_n(L, U)},$$

where the supremum is taken over all lattices L in $\mathbf{R}^n$. A standard elementary argument shows that $\alpha(U, V) \leq \beta(U, V)$ (see, e.g., Lemma 4 in [3]).

By B_n we denote the closed euclidean unit ball in $\mathbf{R}^n$. Let E be an n-dimensional ellipsoid in $\mathbf{R}^n$ with center at zero and principal semiaxes $\alpha_1, \ldots, \alpha_n$. The result of [4], that closed connected additive subgroups of nuclear locally convex spaces are linear subspaces, was essentially based on the fact that

$$\alpha(B_n, E) = \tfrac{1}{2}(\alpha_1^2 + \cdots + \alpha_n^2)^{1/2}.$$

Then it was proved in [2] that

$$\beta(B_n, E) = (\alpha_1^2 + \cdots + \alpha_n^2)^{1/2}.$$

Let K_n be the unit cube in $\mathbf{R}^n$. Consider the rectangular parallelepiped

$$P = \{(x_1, \ldots, x_n) \in \mathbf{R}^n\colon |x_k| \leq \alpha_k \text{ for } k = 1, \ldots, n\},$$

where $\alpha_1, \ldots, \alpha_n > 0$. This paper was motivated by an attempt to give possibly best upper bounds for $\alpha(B_n, P)$ and $\beta(B_n, P)$ as functions of $\alpha_1, \ldots, \alpha_n$ (for $\beta(K_n, P)$, see [5] and [9] where it was, in particular, proved that $\beta(K_n, K_n) = O(\sqrt{n})$ as $n \to \infty$; see also [1]). In particular, we were interested in the so-called Komlós conjecture which asserts that $\beta(B_n, K_n)$ remains bounded as $n \to \infty$.

We denote by γ_n the (standard) Gaussian measure on $\mathbf{R}^n$ with density $(2\pi)^{-n/2}e^{-\|x\|^2/2}$, where $\|x\|$ is the euclidean norm of x. Let ϑ ($\simeq 1.3489795$) be the positive number given by $\gamma_1([-\vartheta/2, \vartheta/2]) = \frac{1}{2}$, i.e.,

$$\int_0^{\vartheta/2} e^{-t^2/2}\, dt = \frac{\sqrt{2\pi}}{4}.$$

By a ϑ-coset in $\mathbf{R}^n$ we mean a coset modulo a lattice L generated by vectors of Euclidean norm $\leq \vartheta$, i.e., satisfying $\lambda_n(L, B_n) \leq \vartheta$. The aim of this paper is to prove the following fact:

Theorem. *If V is a closed convex set in $\mathbf{R}^n$ with $\gamma_n(V) \geq \frac{1}{2}$, then V intersects every ϑ-coset.*

Corollary. *If V is as in the theorem, then $\alpha(B_n, V) \leq \vartheta^{-1}$. In particular $\alpha(B_n, K_n) = O(\sqrt{\log n})$ as $n \to \infty$.*

We point out that, in full generality, the theorem is sharp and that, similarly, the first part of the corollary cannot be significantly improved. However, it is conceivable that $\alpha(B_n, \cdot)$ may be replaced by $\beta(B_n, \cdot)$ in the corollary; see the conjecture at the end of this paper.

For the proof we need the following.

Lemma. *If V is a closed convex set in $\mathbf{R}^n$ with $\gamma_n(V) \geq \frac{1}{2}$ and M is a linear subspace of $\mathbf{R}^n$ of dimension m, then $\gamma_m(V \cap M) \geq \frac{1}{2}$.*

Remark 1. An analysis of the proof shows that unless V is a half-space, or an infinite cylinder orthogonal to M, the inequality in the assertion of the lemma is strict.

We need some preparation for the proofs of the lemma and the theorem. For a convex set V in $\mathbf{R}^n$ and $x \in \mathbf{R}$ denote

$$V_x = \{(x_1, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}: (x_1, \ldots, x_{n-1}, x) \in V\}. \tag{1}$$

Recall now an inequality of Ehrhard (see Theorem 3.2 of [6]). If A, B are nonempty convex Borel subsets of $\mathbf{R}^n$ and $0 \leq \lambda \leq 1$, then

$$\Phi^{-1}(\gamma_n(\lambda A + (1-\lambda)B)) \geq \lambda\Phi^{-1}(\gamma_n(A)) + (1-\lambda)\Phi^{-1}(\gamma_n(B)), \tag{2}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy, \qquad x \in \mathbf{R},$$

is the (standard) Gaussian cumulative distribution function. It follows in particular that $g(x) = \Phi^{-1}(\gamma_{n-1}(V_x))$ is a concave function of x on the interval $I = \{x: \gamma_{n-1}(V_x) > 0\}$. Consequently,

$$W = \{(x, y) \in \mathbf{R}^2: x \in I \text{ and } y \leq g(x)\} \tag{3}$$

is a closed convex subset of $\mathbf{R}^2$. Note that $\gamma_1(W_x) = \gamma_1((-\infty, g(x)]) = \gamma_{n-1}(V_x)$ for $x \in \mathbf{R}$, where W_x is defined analogously to V_x; in particular, $\gamma_n(V) = \gamma_2(W)$.

Proof of the Lemma. Clearly, it is enough to consider the case $m = n-1$ and (by the rotationary invariance of the Gaussian measure) $M = \{(x_1, \ldots, x_n): x_n = 0\}$. For V with $\gamma_n(V) \geq \frac{1}{2}$ we construct $W \subset \mathbf{R}^2$ as above, the assertion of the lemma is then equivalent to $\gamma_1(W_0) \geq \frac{1}{2}$ or $(0, 0) \in W$. To conclude the argument it remains to note that $(0, 0) \notin W$, together with W being closed and convex, would imply $\frac{1}{2} > \gamma_2(W) = \gamma_n(V)$, a contradiction. □

Remark 2. For the proof of the theorem we use the lemma with $n = 2$ and $m = 1$, a special case that can be proved without appealing to Ehrhard's inequality (2). However, the proof of the theorem itself does use Ehrhard's inequality. See also [10] for results related to the lemma.

Proof of the Theorem. We use induction on n. For $n = 1$, the theorem is rather trivial. So, suppose that for a certain $n \geq 2$ the theorem is true for all dimensions strictly less than n. Take an arbitrary ϑ-coset H in $\mathbf{R}^n$ and a convex set V in $\mathbf{R}^n$ disjoint with H. We are to prove that $\gamma_n(V) < \frac{1}{2}$.

Fix some $u \in H$ and consider the lattice $L = H - u$. By assumption, we have $\lambda_n(L, B_n) \leq \vartheta$. Choose $a_1, \ldots, a_n \in L \cap \vartheta B_n$ generating L and let M be the linear span of $a_1, \dots, a_{n-1}$. As before, we may assume that $M = \{(x_1, \dots, x_n): x_n = 0\}$. Let H' be the orthogonal projection of H onto the nth coordinate axis of $\mathbf{R}^n$ (i.e., onto the orthogonal complement of M). Clearly, H' is a ϑ-coset. Additionally, if $x \in H'$, then, by our inductive hypothesis, $\gamma_{n-1}(V_x) < \frac{1}{2}$ and so $(x, 0) \notin W$ (V_x and W have the same meaning here as in (1) and (3)). The case $n = 1$ of the theorem now yields that $\gamma_1(W \cap \{(x, 0): x \in \mathbf{R}\}) < \frac{1}{2}$ and the lemma then implies that $\frac{1}{2} > \gamma_2(W) = \gamma_n(V)$, as required. □

Conjecture. *There exists a function f on $(0, 1)$ such that for each symmetric convex set V in $\mathbf{R}^n$ one has $\beta(B_n, V) \le f(\gamma_n(V))$.*

Remark 3. Let T be a bounded linear operator from a Hilbert space H to a Banach space X. We say that T is *tight* if the image of every connected additive subgroup of H is dense in its linear span in X. If X is a Hilbert space, then T is tight if and only if it is a Hilbert–Schmidt operator; sufficiency was proved in [4], the proof of necessity can easily be obtained by standard methods. The argument of [4] together with the theorem proved above imply that ℓ-operators are tight (for the definition of ℓ-operators, see p. 38 of [8]). An interesting problem, closely connected with the Komlós conjecture, is to describe tight diagonal operators from l_2 to c_0.

Remark 4. In connection with Problem 1 of McMullen and Wills [7, p. 263] it is worth noting the following fact. Let S be an arbitrary n-dimensional simplex in $\mathbf{R}^n$ disjoint with the integer lattice $\mathbf{Z}^n$ and let $r(S)$ be the radius of the inscribed ball. Then it follows immediately from our theorem that $r(S) < c(1 + \log n)^{1/2}$ where $c > 0$ is a numerical constant. Hence, by the result of Steinhagen mentioned on p. 255 of [7], the minimal width of S is less than $c_1 n^{1/2}(1 + \log n)^{1/2}$.

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