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Aggregate and fractal tessellations

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Abstract. Consider a sequence of stationary tessellations $\{\Theta^n\},\ n = 0, 1, \dots$ of $\mathbb{R}^d$ consisting of cells $\{C^n(x_i^n)\}$ with the nuclei $\{x_i^n\}$. An aggregate cell of level one, $C_0^1(x_i^0)$, is the result of merging the cells of Θ^1 whose nuclei lie in $C^0(x_i^0)$. An aggregate tessellation Θ_0^n consists of the aggregate cells of level n, $C_0^n(x_i^0)$, defined recursively by merging those cells of Θ^n whose nuclei lie in $C^{n-1}(x_i^0)$.

We find an expression for the probability for a point to belong to a typical aggregate cell, and obtain bounds for the rate of its expansion. We give necessary conditions for the limit tessellation to exist as $n \to \infty$ and provide upper bounds for the Hausdorff dimension of its fractal boundary and for the spherical contact distribution function in the case of Poisson-Voronoi tessellations $\{\Theta^n\}$.

1. Motivation

A *tessellation* of $\mathbb{R}^d$ is a countable collection of closed bounded sets called *cells* such that

(a) union of all cells is the whole space;
(b) intersection of any two different cells has d-Lebesgue measure zero;
(c) each bounded set intersects a finite number of cells.

Tessellations are widely used to model different cellular systems (see, e. g., [7] and references therein).

We assume that each cell C_i is associated with a unique *nucleus* $x(C_i)$ according to a certain rule satisfying an obvious compatibility condition: $\theta x(C_i) = x(\theta C_i)$ for any shift transformation θ in $\mathbb{R}^d$. For example, the *Voronoi tessellation* has cells defined as

$$C(x_i) = \{x \in \mathbb{R}^d\ :\ \|x - x_i\| \le \|x - x_j\|,\ \forall j \ne i\},$$

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where $\|\cdot\|$ is the Euclidean norm. Thus, the cell with nucleus x_i consists of the points that are closer to x_i than to any other nucleus. A random tessellation with nuclei can be viewed as a marked point process $M = \{(x_i, C(x_i))\}$. In this paper we deal with *stationary* tessellations, i. e., M is stationary with respect to shifts θ.

Recently, random Voronoi tessellations were used as models of service zones of telecommunication stations. The main advantage of these models is that they dramatically reduce the number of structuring parameters of the model to just a few parameters of the underlying stochastic process and often allow for an analytical treatment of complex networks characteristics (see [1], [2]).

In many cases, however, models using Voronoi tessellation over-simplify the complex geometry of service zones. For instance, in the case of *wireless* communications the base station that will handle a call from a mobile terminal is determined by the signal strength rather than Euclidean distance to the stations. Affected by the wave attenuation phenomena, the zones' boundaries have extremely irregular, distorted shapes.

This calls for development of more complex tessellation models that are still described in terms of a small number of parameters and simple enough to be analytically treatable.

For this we introduce an operation of *aggregation* on independent stationary tessellations equipped with nuclei. Let $\Theta^0 = \{C^0(x_i^0)\}$ and $\Theta^1 = \{C^1(x_i^1)\}$ be two such tessellations. Define the aggregate cells of $\Theta_0^1 = \Theta^0 \circ \Theta^1$ as

$$C_0^1(x_i^0) = \bigcup_{j:\, x_j^1 \in C^0(x_i^0)} C^1(x_j^1)\,.$$

In words, $C_0^1(x_i^0)$ is the union of all the cells of Θ^1 whose nuclei lie in $C^0(x_i^0)$. Due to the independence and stationarity assumptions, with probability 1 every x_j^1 lies in unique cell of Θ^0. Hence, Θ_0^1 is again a tessellation, though some of its cells $C_0^1(x_i^0)$ may be empty. It is easy to verify that the operation of aggregation is associative and that the aggregate tessellation Θ_0^1 is itself stationary. Let $\{\Theta^n\}_{n\in\mathbb{N}}$ be a sequence of independent stationary tessellations with the nuclei sets $\Pi_n = \{x_i^n\},\ n \in \mathbb{N}$. The aggregation of the first n terms of the sequence yields the *aggregate tessellation of order* n: $\Theta_0^n = \Theta^0 \circ \Theta^1 \circ \ldots \circ \Theta^n$ with the nuclei set $\Pi_0 = \{x_i^0\}$. The cells of this tessellation will be called *aggregate* n*-cells* and denoted by $C_0^n(x_i^0)$.

Figure 1 shows simulated *Poisson-Voronoi aggregate tessellations*, quoted as PVAT in the sequel, for which the elements of the sequence $\{\Theta^n\}_{n\in\mathbb{N}}$ are all Voronoi tessellations generated, respectively, by mutually independent homogeneous Poisson point processes $\Pi_n,\ n \in \mathbb{N}$. The diagrams were produced by computer program `pvat`[1] written by one of the authors. It can be seen that the cells of PVAT are, in general, neither convex nor connected; nor do they need to contain the nucleus.

It is clear that the higher the growth rate of the successive intensities, the less the boundary of cell $C_0^{n+1}(x_i^0)$ deviates from the boundary of $C_0^n(x_i^0)$. On the other hand, for close intensity values the boundary becomes very irregular, cells are more likely to split, and quite often there is no point of Π_{n+1} in $C_0^n(x_i^0)$, so that

[1] Available from http://www.stams.strath.ac.uk/~sergei

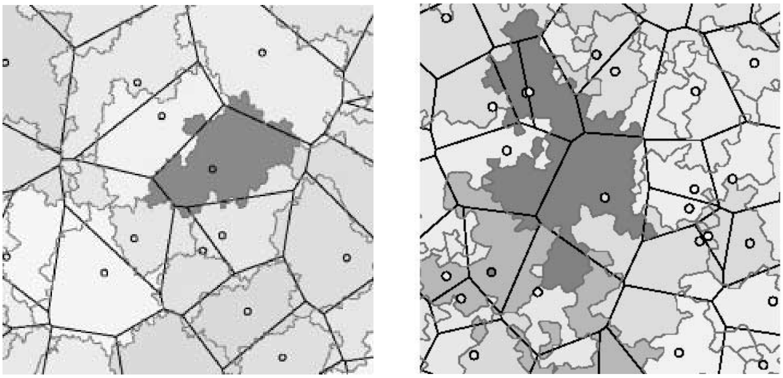

Fig. 1. Initial Voronoi tessellation and n-level aggregate cells in PVAT model with geometrically growing intensities $\lambda_n = \lambda^n$. Left image: $n = 3$, $\lambda = 10$. Right image: $n = 25$, $\lambda = 1.2$

$C_0^{n+1}(x_i^0)$ is empty. Using an analogy with branching processes, we may think of the nuclei of the Θ^n-cells that make up the aggregate cell $C_0^n(x_i^0)$ as of n-generation *offspring* of a 0-level parent nucleus x_i^0. If we connect by segments the nuclei Π_n of each level n with their offsprings at the next level Π_{n+1}, we will obtain a family of *spanning trees* studied for Poisson-Voronoi case in [3]. In the present paper we address properties of the aggregate *cells* rather than those of the spanning trees.

New phenomena appear in the limit, when n tends to infinity. As we have seen above, there are models in which with positive probability some of the aggregate cells are empty. *A priori* it is not clear if we do not end up with all the 0-nuclei Π_0 dying out with probability 1. Even if we do not, will Θ_0^n converge in some sense to a limit, say, Θ_0^∞ that *is* a tessellation? It is easy to imagine that Θ_0^∞ may have a fractal boundary (it also may not! – see Section 3) and thus it is unclear if the boundary has d-Lebesgue measure 0 and whether only a finite number of the limit aggregate cells hits a bounded set.

The structure of the paper is the following. Section 2 presents the main results. We find an expression for the *coverage probability* function for a typical aggregate n-cell (the one with nucleus at the origin under the Palm distribution of the process Π_0) via the corresponding characteristics of $\Theta^0, \ldots, \Theta^n$. This result is valid for *any* stationary aggregate tessellation although a closed form expression can be obtained only in a few cases. Further we deal mainly with PVAT. We find uniform upper bounds on the diameter of a typical cell and on the range of variation of its boundary. Next, we show that with positive probability there is a ball contained in *all* n-level aggregate cells $C_0^n(0)$. This property is sufficient in the self-similar case for the limit cells, defined as the set lower limit of $\{C_0^n(x_i^0)\}_{n\in\mathbb{N}}$, to form a tessellation. Defined by a simple recursive procedure, the boundary of the limit PVAT has an intricate self-similar structure at any scale allowing us to call it fractal. To characterize its degree of irregularity, we provide an upper bound for its Hausdorff dimension which is based on the analysis of the boundary contact distribution function. Note that the parts of the cell's fractal boundary are highly dependent making most of previously developed techniques for random fractals inapplicable to PVAT (a presentation of modern methods used in studying fractals can be found, e. g., in K. Falconer's book [5]). Section 3 contains examples of aggregate tessellations manifesting various properties. Finally, Section 4 contains the proofs of the main statements of Section 2. Other proofs and further details can be found in [9].

The following notation is used throughout the paper. By **P** we denote the *distribution* in a probability space carrying the sequence of *independent stationary* point processes $\Pi_0, \Pi_1, \dots$, and by $\mathbf{P}_n^0$, the *Palm distribution* with respect to the process of nuclei Π_n. Most frequently we consider the Palm distribution with respect to Π_0, for which we simply write $\mathbf{P}^0$ instead of $\mathbf{P}_0^0$. Similar notation is used for the corresponding expectations. $\Pi_n(B)$ stands for the number of points of Π_n in a Borel set $B \subset \mathbb{R}^d$. The *intensity* of Π_n is denoted by λ_n, so that $\mathbf{E}_n\, \Pi_n(B) = \lambda_n |B|$, and it is the only parameter characterizing a homogeneous Poisson process. We also assume that $\lambda_0 = 1$, which is just a matter of scale choice. Finally, $b(x, r)$ is the closed ball centered in x with radius r, and $b_d = \pi^{d/2}/\Gamma(d/2 + 1)$ is the volume of a *unit ball* in $\mathbb{R}^d$.

2. Main results

2.1. Coverage probability

Consider a tessellation Θ^n of fixed level n. Under Palm probability $\mathbf{P}_n^0$, there is almost surely a point of Π_n at the origin 0. Since the density of cells is λ_n, the volume of a typical cell is $\mathbf{E}_n^0\, |C^n(0)| = \lambda_n^{-1}$ (see, e. g., [6], Corollary 5.2, equation (5.6)). Therefore,

$$\lambda_n \int \mathbf{P}_n^0\big\{y \in C^n(0)\big\}\, dy = \lambda_n\, \mathbf{E}_n^0 \int \mathbb{I}(y \in C^n(0))\, dy = \lambda_n\, \mathbf{E}_n^0\, |C^n(0)|,$$

which is 1, so that the function $f_n(y) = \lambda_n \mathbf{P}_n^0\big\{y \in C^n(0)\big\}$ is a distribution density in $\mathbb{R}^d$.

The next statement provides a formula for calculating the probability that a point $y \in \mathbb{R}^d$ is covered by a typical aggregate cell of level n.

Theorem 1. *Let $f_{0,n}(y) = \mathbf{P}^0\big\{y \in C_0^n(0)\big\}$. Then for each natural n, it equals the convolution*

$$f_{0,n}(y) = f_0 * f_1 * \cdots * f_n(y).$$

Let us take a closer look at the coverage probability for Poisson-Voronoi aggregate tessellations. For PVAT we have:

$$f_n(y) = \lambda_n \exp\big\{-\lambda_n b_d \|y\|^d\big\}. \tag{1}$$

In $\mathbb{R}^2$ this corresponds to the normal distribution with zero mean and the covariance matrix

$$\begin{pmatrix} (2\pi\lambda_n)^{-1} & 0 \\ 0 & (2\pi\lambda_n)^{-1} \end{pmatrix}. \tag{2}$$

Therefore, the convolution also corresponds to a normal r.v. with zero mean and the covariance matrix being the sum of (2). The corresponding density is thus

$$f_{0,n}(y) = L_n \exp\left\{-L_n\pi \|y\|^2\right\}, \quad \text{where } L_n^{-1} = \sum_{i=0}^{n} \lambda_i^{-1},$$

which is the same as for a typical cell in the ordinary Voronoi tessellation with the nuclei intensity L_n. Such "mean field" approximation is valid only in this planar case, the reason being the stability of the distributions 1 in $d = 2$. The general formula for the characteristic function of $f_{0,n}$ in $\mathbb{R}^d$ and explicit expression for the density in $\mathbb{R}^1$ can be found in [9].

2.2. Evolution of PVAT cells

Here we investigate the behavior of the typical aggregate cell $C_0^n(0)$ as n tends to infinity on the Palm space of the process Π_0. The maximal and the minimal distance from a point z to the cell's boundary can be defined, respectively, as

$$R_n(z) = \begin{cases} \min\{r : b(z,r) \supset C_0^n(0)\} & \text{if } C_0^n(0) \neq \emptyset, \\ 0 & \text{otherwise;} \end{cases}$$
$$r_n(z) = \begin{cases} \max\{r : b(z,r) \subset C_0^n(0)\} & \text{if } z \in C_0^n(0), \\ 0 & \text{otherwise.} \end{cases}$$

The definition takes into account that the aggregate cell of order $n \geq 1$ might not contain z or might be empty. Our aim is to characterize the distribution of

$$R_\infty(z) = \sup_n R_n(z), \qquad r_\infty(z) = \inf_n r_n(z).$$

Theorem 2. *Let $\phi(y)$ be the inverse of the function $y(x) = xe^x$. Assume that*

$$c = \sum_{n=1}^{\infty} \left(\frac{\phi(\lambda_n)}{\lambda_n}\right)^{1/d} = \sum_{n=1}^{\infty} e^{-\phi(\lambda_n)/d} < \infty. \tag{3}$$

Then for PVAT the following inequalities hold for all $\rho > c\sqrt{d}$ and any $z \in \mathbb{R}^d$:

$$\mathbf{P}^0\left\{R_\infty(z) > \rho + \|z\|\right\} \leq a_1 \sum_{n=1}^{\infty} e^{-\phi(\lambda_n)A(\rho)}, \tag{4}$$

$$\mathbf{P}^0\left\{r_\infty(z) = 0 \,\middle|\, r_0(z) > \rho\right\} \leq a_1 \sum_{n=1}^{\infty} e^{-\phi(\lambda_n)A(\rho)}, \tag{5}$$

where $a_1 = 2\big((3/2)^d - 1\big)\, b_d d^{d/2} c^{d-1}$ and $A(\rho) = (\rho/c\sqrt{d})^d + 1/d - 1$.

Remark 1. By definition, $\log\phi(x)+\phi(x)=\log x$. Therefore, for $x\geq 1$ one has $\phi(x)\leq \log x$. Since $e^{-\phi(\lambda_n)}=\phi(\lambda_n)/\lambda_n$ and $\lambda_n>1$ for all sufficiently large n, the conditions

$$\sum_{n=1}^{\infty}\Big(\frac{\log\lambda_n}{\lambda_n}\Big)^{1/d}<\infty \quad \text{or} \quad \sum_{n=1}^{\infty}\lambda_n^{-1/d+\varepsilon}<\infty \tag{6}$$

for some $0<\varepsilon<1/d$ are sufficient for (3) to hold.

Corollary 1. *For PVAT with exponentially growing intensities $\lambda_n=\lambda^n$ for some $\lambda>1$, one has for all $\rho>c_*$ and any $z\in\mathbb{R}^d$,*

$$\mathbf{P}^0\big\{R_\infty(z)>\rho+\|z\|\big\}\leq c_1\exp\big\{-c_2\rho^d\big\}, \tag{7}$$

$$\mathbf{P}^0\big\{r_\infty(z)=0\,\big|\,r_0(z)>\rho\big\}\leq c_1\exp\big\{-c_2\rho^d\big\}\,. \tag{8}$$

One may take $c_*=((\log c_1)/c_2)^{1/d}$,

$$c_1=a_1\bigg(1+d\,\frac{1+\phi(\lambda)}{\phi(\lambda)\log\lambda}\bigg)e^{\phi(\lambda)}, \quad \textit{and} \quad c_2=\phi(\lambda)\,(c\sqrt{d})^{-d}\,.$$

Corollary 2. *Under conditions of Theorem 2, with probability one, each family of cells $\{C_0^n(x_i^0)\}_{n\in\mathbb{N}}$ is uniformly bounded in $\mathbb{R}^d$.*

Corollary 3. *Under conditions of Theorem 2, for any $y\in\mathbb{R}^d$ with positive probability, there exists $x_i^0\in\Pi_0$ such that $y\in\operatorname{int}\big(\cap_n C_0^n(x_i^0)\big)$. The lower bound for such probability is given in (21).*

Bounds for the probability of a cell's extinction can be found in [9].

The next theorem reinforces, in some sense, Theorem 2 showing that not only the radius but also the range of the cells' boundaries evolution is related to the sum in (3). Recall the definitions of Minkowski operations $\oplus$ and $\ominus$ for two sets A, B in a vector space:

$$A\oplus B=\{a+b\,:\,a\in A,\ b\in B\}, \qquad A\ominus B=(A^c\oplus B)^c\,.$$

For $0\leq m<n\leq\infty$ introduce the variable γ_m^n, which shows how far the boundary of $C_0^n(0)$ stretches from the boundary of its predecessor $C_0^m(0)$. Put

$$\gamma_m^n=\inf\{r\,:\,C_0^m(0)\ominus b(0,r)\subseteq C_0^n(0)\subseteq C_0^m(0)\oplus b(0,r)\}$$

if both $C_0^m(0)$ and $C_0^n(0)$ are non-empty, otherwise put $\gamma_m^n=0$.

Theorem 3. *Let $\phi(y)$ be the inverse of the function $y(x)=xe^x$ and let*

$$c(m,n)=\sum_{k=m+1}^{n}e^{-\phi(\lambda_k)/d}.$$

Then for $1 \le m < n \le \infty$, *one has for all* $\rho > c(m,n)\sqrt{d}$,

$$\mathbf{P}^0\big\{\gamma_m^n > \rho \,\big|\, \operatorname{diam} C_0^m(0) \le r_0\big\} \le b_d d^{d/2} c^d(m,n)\Big(\frac{3}{2} + \frac{r_0}{\rho}\Big)^d \sum_{k=m+1}^{n} e^{-\phi(\lambda_k)\,A(\rho,m,n)},$$

where $A(\rho,m,n) = \big(\rho/\sqrt{d}c(m,n)\big)^d - 1$.

The proof mimics the one of Theorem 2 and can be found in [9].

Corollary 4. *Assume that (3) is satisfied. Then* $\lim_{n\to\infty} \gamma_n^{n+k} = 0$ *in probability uniformly in* $k \ge 1$.

2.3. Limit tessellation

Heuristic arguments suggest that the difference between two successive aggregate cells becomes smaller and smaller if the intensities of the point processes grow sufficiently fast. In the case of PVAT, Corollary 4 shows that it is indeed the case. Moreover, by Corollary 3 with positive probability the family of cells $\{C_0^n(x_i^0)\}$ centered in the same nucleus x_i^0 possesses a non-empty "core" $\operatorname{int}\big(\cap_n C_0^n(x_i^0)\big)$. Therefore, it is important to know, whether there exists and in what sense a *limit* object for the process of aggregation, and if the answer is positive, whether this limit object is itself a tessellation.

It is possible to define the limit cells in various other ways. We show that under a suitable condition the most natural definitions are equivalent and yield the same limit tessellation.

Let $\{A_n\}$ be a sequence of closed subsets of $\mathbb{R}^d$. Recall the definitions of Painlevé-Kuratowski lower and upper set limits:

$$\liminf_n A_n = \{x \,:\, \exists\{x_n\} \text{ such that } x_n \in A_n \text{ and } x = \lim_n x_n\},$$
$$\limsup_n A_n = \{x \,:\, \exists\{x_{n_k}\} \text{ such that } \lim_k n_k = \infty,\ x_{n_k} \in A_{n_k},\ \text{and } x = \lim_k x_{n_k}\}.$$

In words, a point belongs to $\liminf_n A_n$ if and only if any of its neighborhoods intersects with all sets A_n starting from some m; a point belongs to $\limsup_n A_n$ if and only if any its neighborhoods intersects infinitely many sets A_n. Both limits are closed sets and their coincidence is equivalent to convergence in the so-called *Fell* topology on closed sets (see, e. g., [4]). Define the sets

$$C_0^\infty(x_i^0) = \operatorname{cl}\big(\cup_m \cap_{n\ge m} C_0^n(x_i^0)\big), \quad D_0^\infty(x_i^0) = \operatorname{cl}\big(\cap_m \cup_{n\ge m} C_0^n(x_i^0)\big), \tag{9}$$
$$E_0^\infty(x_i^0) = \liminf_n C_0^n(x_i^0), \quad F_0^\infty(x_i^0) = \limsup_n C_0^n(x_i^0). \tag{10}$$

Let $C_m^n(x_i^m)$ be the cells of the aggregate tessellation Θ_m^n defined as

$$\Theta_m^n = \Theta^m \circ \Theta^{m+1} \circ \ldots \circ \Theta^n.$$

Theorem 4. *Assume that condition*

$$\mathbf{P}\{\exists m, x_j^m \in \Pi_m \;:\; 0 \in \operatorname{int}\big(\cap_{n\geq m} C_m^n(x_j^m)\big)\} = 1\,. \tag{11}$$

is satisfied. Then the sets $C_0^\infty(x_i^0)$, $D_0^\infty(x_i^0)$, $E_0^\infty(x_i^0)$, and $F_0^\infty(x_i^0)$ defined by (9–10) have coinciding interiors and constitute the same tessellation Θ_0^∞.

Remark 2. This theorem enables us to define cells of the limit tessellation Θ_0^∞ as the sets $A_i \equiv \operatorname{cl}\big(\operatorname{int} C_0^\infty(x_i^0)\big)$. Since the interiors of the sets (9-10) and of A_i coincide, a natural question is whether the sets themselves coincide. Another related question is whether there is a convergence of the boundaries

$$\Gamma_0^n = \mathbb{R}^d \setminus \cup_i \operatorname{int}\left(C_0^n(x_i^0)\right) \quad \text{and} \quad \Gamma_0^\infty = \mathbb{R}^d \setminus \cup_i \operatorname{int} A_i\,. \tag{12}$$

The answers are, in general, negative: A_i may be strictly smaller than (9-10). The difference is that the latter sets are not necessarily *regular*, i. e., they do not necessarily coincide with the closures of their interiors. As to the second question, we show in the proof of Theorem 4 that there *is* a limit $\Gamma' \equiv \lim_n \Gamma_0^n$ in Fell topology, one has $\Gamma_0^\infty \subseteq \Gamma'$, but they need not coincide. An expression for Γ' is also given.

Corollary 5. *Under condition (11), for each $y \in \mathbb{R}^d$,*

$$\lim_n \mathbf{P}^0\left\{y \in C_0^n(0)\right\} = \mathbf{P}^0\left\{y \in C_0^\infty(0)\right\},$$
$$\lim_n |C_0^n(0)| = |C_0^\infty(0)| \quad a.s.$$

and thus $\mathbf{E}^0\,|C_0^\infty(0)| = \lambda_0$.

Call a sequence of tessellations Θ^n *self-similar* if there exists $\nu > 0$ such that for all $n \in \mathbb{N}$ the scaled tessellation $\nu\Theta^n$, defined through its boundary $\nu\Gamma(\Theta^n) = \{\nu y \;:\; y \in \Gamma(\Theta^n)\}$, has the same distribution as Θ^{n-1}.

Theorem 5. *For a sequence of self-similar tessellations*

$$\mathbf{P}\left\{\exists x_i^0 \in \Pi_0 \;:\; 0 \in \operatorname{int}(\cap_{n=0}^\infty C_0^n(x_i^0))\right\} > 0\,, \tag{13}$$

implies (11); in particular, (13) implies the statements of Theorem 4 and its corollaries.

Corollary 6. *Theorem 5 together with Example 2 imply that $\{C_0^\infty(x_i^0)\}$ in self-similar case constitute a tessellation if and only if (13) holds.*

Remark 3. Since the sequence of Poisson-Voronoi tessellations with exponentially growing intensities $\lambda_n = \lambda^n$ for $\lambda > 1$ are self-similar (with $\nu = \lambda^{1/d}$), then in view of Corollary 3, the condition (13) is satisfied, so that the family $\{C_0^\infty(x_i^0)\}$ constitutes a tessellation of $\mathbb{R}^d$. Closely examining the proof of Theorem 5 it is seen that the key step is to find a vanishing upper bound for the LHS of (22). For instance, for PVAT with super-exponential growth of intensities: $\liminf_n \lambda_{n+1}/\lambda_n > 1$, it will again be bounded by p^n, since $\mathbf{P}\{T_n < \infty \,\big|\, T_{n-1} = k\}$ is not increasing in k for all sufficiently large k, so that (11) also holds for that case and thus a limit tessellation Θ_0^∞ exists.

2.4. Fractal boundary of the limit cells

From now on we confine ourselves to PVAT with exponentially growing intensities. As we have shown in Remark above, the limit Poisson-Voronoi tessellations exist and the boundary of the limit cells Γ_0^∞ is a random closed set defined by (12). As the distributions of Γ_0^∞ and Γ_n^∞ scaled by $\lambda^{n/d}$ coincide, the boundary of the limit tessellation is statistically self-similar. This property is rather different from geometrical self-similarity in the sense of Γ_0^∞ being a union of scaled copies of self. However, by construction, Γ_{n-1}^∞ consists of parts of Γ_n^∞, and therefore, Γ_0^∞ has a similar structure at any scale of observation, which allows us to call it a fractal.

One of its important characteristics of the boundary is the *spherical contact distribution function* $H(r)$, defined as

$$H(r) = \mathbf{P}\{\Gamma_0^\infty \cap b(0,r) \neq \emptyset \,\big|\, 0 \notin \Gamma_0^\infty\}, \qquad r \geq 0.$$

Here, as Theorem 5 shows, the probability of the condition is one, thus $H(0) = 0$. Some information on the degree of variability of the cell boundary can be derived from the rate at which $H(r)$ decreases as r tends to zero.

Theorem 6. *For PVAT with exponentially growing intensities:* $\lambda_n = \lambda^n$ *for some* $\lambda > 1$*, there exist constants* $K > 0$ *and* $q \in (0,1)$ *such that for all* $r \geq 0$*,*

$$H(r) = \mathbf{P}\{b(0,r) \cap \Gamma_0^\infty \neq \emptyset\} \leq K r^{dq}.$$

The values of q *and* K *are given in* (30).

The primary characteristic of a fractal is its dimension, which can be defined in several ways. We will be interested in the *Hausdorff dimension* of Γ_0^∞ (see, e. g., [5, p. 20–23] for definitions of different dimensions that we use here).

Theorem 7. *Let* q *be the constant defined in (30). Then for PVAT with exponentially growing intensities:* $\lambda_n = \lambda^n$ *for some* $\lambda > 1$*, one has*

$$\dim_H \Gamma_0^\infty = \mathbf{E} \dim_H \Gamma \qquad a.s.,$$
$$\dim_H \Gamma_0^\infty < d(1-q) \qquad a.s.$$

3. Examples and counterexamples

The examples in this section are all in $\mathbb{R}^1$, but d-dimensional analogues can be constructed by taking Cartesian product of independent realizations of the described tessellations.

Example 1.
(i) aggregate cells may never die;
(ii) asymptotic behavior of aggregate cells depends on the nuclei choice;

(iii) limit tessellation may not be fractal;

Consider a stationary tessellation Θ^n of $\mathbb{R}^1$ obtained by shifting the regular mesh of intervals of length λ_n^{-1} with a vertex at the origin by a random vector uniformly distributed in $[0, \lambda_n^{-1}]$.

Assume that $\{\lambda_n\}$ is a non-decreasing sequence and $\lambda_0 = 1$. Choose $\alpha \in [0, 1]$ and define the nucleus of each cell (segment) to be the point dividing it in proportion $\alpha : (1-\alpha)$ from left to right. It is easy to see by induction that the size of any Θ_0^n-cell along each coordinate axis is at least λ_n^{-1}, so there is always at least one nucleus of Π_{n+1} in each cell. As a result, all aggregate cells are all segments and the cells never die. Moreover, for each n, the sizes of the aggregate cells along the line do not change if λ_{n+1}/λ_n is a natural number and change periodically if λ_{n+1}/λ_n is a rational one. By construction, the boundaries of the Θ^n-cells have coordinates $\lambda_n^{-1}(k + u_n),\ k \in \mathbb{Z}$, where u_n are independent uniformly distributed in $(0, 1)$ random variables describing the shifts.

It is straightforward to verify that the evolution of the boundaries of Θ_0^n-cells are given by the following recursion

$$a_k^n = a_k^{n-1} + d_k^n\,, \quad \text{where}$$
$$d_k^n = \lambda_n^{-1}\big(1 - \alpha - \langle \lambda_n a_k^{n-1} - \alpha - u_n \rangle\big)$$

with $\langle z \rangle = z - \max\{k \in \mathbb{Z} :\ k \le z\}$ being the fractional part of a real number z. Note that for any z and for any u uniformly distributed in $(0, 1)$, the r.v. $\langle z + u \rangle$ is again uniform in $(0, 1)$. Therefore,

$$\mathbf{E}\, d_k^n = \lambda_n^{-1}(1/2 - \alpha)\,, \qquad \mathbf{var}\, d_k^n = 1/(12\lambda_n^2),$$

so that there is a systematic drift to the right or to the left if $\alpha < 1/2$ or $\alpha > 1/2$, respectively. We have $|d_k^n| \le 1$. By the well known result on random series convergence, the boundaries of the aggregate cells almost surely stabilize as $n \to \infty$ if and only if both series $(1/2 - \alpha)\sum_n \lambda_n^{-1}$ and $\sum_n \lambda_n^{-2}$ converge. We see a noticeable dependence on the nuclei choice when, say, $\lambda_n = n$. In this case the cells stabilize only if $\alpha = 1/2$ and float to plus or minus infinity depending on whether α is smaller or greater than $1/2$.

Example 2.

(i) it is possible that $E_0^\infty(x_i^0) = F_0^\infty(x_i^0)$ for all x_i^0, but these cells do not constitute a tessellation;

(ii) existence of limit of coverage probabilities does not imply existence of a limit tessellation;

Let $\hat{\Theta}$ be a tessellation of $\mathbb{R}^1$ with non-connected cells $i + [-5/6, -3/6] \cup [-1/6, 1/6] \cup [3/6, 5/6]$ and nuclei $i \in \mathbb{Z}$. Put $\Theta^n = 3^{-n}(\hat{\Theta} + u_n)$, where u_n are independent uniformly distributed in $[0, 1]$ r.v.'s. It is easy to see that $\Theta_0^n = \hat{\Theta}_0^n + \sum_{i=0}^n 3^{-i} u_i$, where $\hat{\Theta}_0^n = \hat{\Theta} \circ 3^{-1}\hat{\Theta} \circ \cdots \circ 3^{-n}\hat{\Theta}$. The cells of $\hat{\Theta}_0^n$, consist of 3^{n+1} segments:

$$i - 1 + \bigcup_{k=0}^{3^{n+1}-1} \Big[\frac{4k+1}{2\cdot 3^{n+1}}, \frac{4k+3}{2\cdot 3^{n+1}}\Big] \qquad i \in \mathbb{Z}\,.$$

Geometrically, next generation aggregate cells are obtained by swapping neighboring thirds between the segments at their boundary points. Therefore, $D_0^\infty(i) = E_0^\infty(i) = F_0^\infty(i) = i + [-1, 1] + 2v$, where v is uniform in $[-1, 1]$, so that almost each point y is covered by 2 limit cells: $E_0^\infty(i)$ with $|y - i| \le 1$ and thus it is not a tessellation. In contrast, $C_0^\infty(i) = \emptyset$ for all i. Note also that $\mathbf{P}\{y \in C_0^n(0)\}$ converges to a non-degenerate distribution.

4. Proofs

4.1. Theorem 1

Proof. By definition, for $n = 0$, obviously, $f_0(y) = \mathbf{P}^0\big\{y \in C^0(0)\big\}$. Suppose, the statement of theorem holds for $n-1$. By the Campbell theorem (see, e. g., [8, p. 119]),

$$\begin{aligned}\mathbf{P}^0\big\{y \in C_0^n(0)\big\} &= \mathbf{E}^0 \sum_{x_i^n \in \Pi_n} \mathbb{1}\big(x_i^n \in C_0^{n-1}(0)\big)\, \mathbb{1}\big(y \in C^n(x_i^n)\big) \\ &= \lambda_n \int_{\mathbb{R}^d} \mathbf{P}^0\big\{z \in C_0^{n-1}(0)\big\}\, \mathbf{P}_n^0\big\{y - z \in C^n(0)\big\}\, dz.\end{aligned}$$

It is easy to see that this expression corresponds to what was stated:

$$f_{0,n}(y) = \int_{\mathbb{R}^d} f_{0,n-1}(z)\, f_n(y-z)\, dz.$$

4.2. Theorem 2

Proof. We begin with inequality (4). Since

$$\mathbf{P}^0\big\{R_\infty(z) > \rho + \|z\|\big\} \le \mathbf{P}^0\big\{R_\infty(0) > \rho\big\},$$

it is sufficient to prove (4) for $z = 0$. Let $\{\rho_n\}$ be a monotonously increasing sequence of positive numbers converging to ρ. Then we have

$$\{R_\infty(0) > \rho\} \subset \cup_{n=0}^\infty \{R_n(0) > \rho_n\}.$$

Next, we use the following inequality: if $B \subset \cup_{n=0}^\infty B_n$, then

$$\mathbf{P}(B) \le \mathbf{P}(B_0) + \sum_{n=1}^\infty \mathbf{P}(B_n \cap \bar{B}_{n-1}) \le \mathbf{P}(B_0) + \sum_{n=1}^\infty \mathbf{P}(B_n \,\big|\, \bar{B}_{n-1}) \,.$$

Hence,

$$\mathbf{P}^0\big\{R_\infty(0) > \rho\big\} \le \mathbf{P}^0\big\{R_0 > \rho_0\big\} + \sum_{n=1}^\infty \mathbf{P}^0\big\{R_n(0) > \rho_n \,\big|\, R_{n-1}(0) \le \rho_{n-1}\big\}. \tag{14}$$

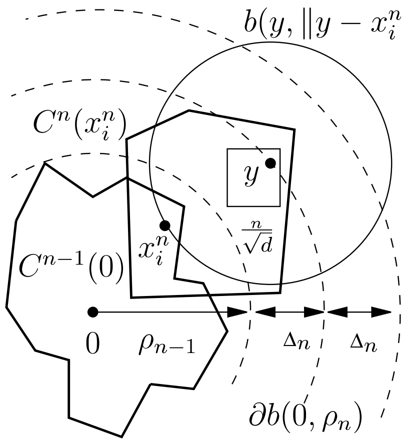

Fig. 2. Large increase in $R_n(0)$ implies existence of a large empty ball

Given that $R_{n-1}(0) \le \rho_{n-1}$, the event $\{R_n(0) > \rho_n\}$, implies the existence of a Voronoi cell $C^n(x_i^n)$ with the nucleus inside of the ball $b(0, \rho_{n-1})$ containing some point y on the sphere $\partial b(0, \rho_n)$. Therefore, the interior of the ball $b(y, \|y - x_i^n\|)$ contains no points of Π_n (see Figure 2).

Denote $\Delta_n = \rho_n - \rho_{n-1}$. Take $u \in [-1/2, 1/2]^d$ and consider the collection of mesh cubes of side $\Delta_n/\sqrt{d}$ whose centers z_m are those grid points $(\mathbb{Z}^d + u)\Delta_k/\sqrt{d}$ that lie inside the annulus $b(0, \rho_n + \Delta_n/2) \setminus b(0, \rho_n - \Delta_n/2)$. Denote by $N = N(u)$ their number. If $z(y)$ is the center of the cube containing y, then for any point $x \in K(z(y), u)$ one has

$$\|x - y\| \le \|x - z(y)\| + \|z(y) - y\| \le \Delta_k/2 + \Delta_k/2 = \Delta_k \le \|y - x_i^k\|.$$

Thus the cube $K(z(y), u)$ lies in the closure of $b(y, \|y - x_i^k\|)$ and contains no point of Π_n inside. This yields for the summands in (14) the following bound:

$$1 - \Big(1 - \exp\big\{-\lambda_n(\Delta_n/\sqrt{d})^d\big\}\Big)^N, \tag{15}$$

which is the probability that at least one of the mesh cubes contains no points of Π_n.

To estimate N, let u be uniformly distributed in $[0, 1]^d$. Then $(\mathbb{Z}^d + u)\Delta_k/\sqrt{d}$ is a stationary point process with intensity $(\sqrt{d}/\Delta_k)^d$. Therefore, the mean number of its points inside the annulus equals

$$\mathbf{E}_u N = (\sqrt{d}/\Delta_k)^d b_d\big[(\rho_n + \Delta_n/2)^d - (\rho_n - \Delta_n/2)^d\big].$$

Hence there exists u_0 such that

$$\begin{aligned} N = N(u_0) &\le (\sqrt{d}/\Delta_k)^d b_d\big[(\rho_n + \Delta_n/2)^d - (\rho_n - \Delta_n/2)^d\big] \\ &= b_d \Delta_n (\sqrt{d}/\Delta_k)^d \sum_{k=0}^{d-1} (\rho_n + \Delta_n/2)^k (\rho_n - \Delta_n/2)^{d-k-1} \end{aligned}$$

$$
\begin{aligned}
&< b_d \Delta_n (\sqrt{d}/\Delta_k)^d \sum_{k=0}^{d-1} (3/2\rho)^k \rho^{d-k-1} \\
&= 2\big((3/2)^d - 1\big)\, b_d d^{d/2} \, (\rho/\Delta_n)^{d-1} \,.
\end{aligned}
$$

Since $1-(1-a)^N < aN$ for any $0 < a < 1$ and $N \geq 1$, the value of (15) does not exceed

$$
2\big((3/2)^d - 1\big)\, b_d d^{d/2} (\rho/\Delta_n)^{d-1} \exp\big\{ -\lambda_n (\Delta_n/\sqrt{d})^d \big\} \,. \tag{16}
$$

Next, choose a specific sequence $\{\rho_n\}$ with the increments $\Delta_n = \rho c^{-1} e^{-\phi(\lambda_n)/d}$ with c defined in (3). It is easy to see that ρ_n monotonously converges to ρ. For such $\{\rho_n\}$, from (16) it follows that the right-hand side of (14) is bounded by

$$
a_1 c^{1-d} \sum_{n=0}^{\infty} \exp\big\{ -\big(\rho/c\sqrt{d}\big)^d \phi(\lambda_n) + (d-1)\big(\log c + \phi(\lambda_n)/d\big)\big\}, \tag{17}
$$

which is equivalent to (4). We have used here the definition of ϕ, due to which $e^{-\phi(\lambda_n)} = \phi(\lambda_n)/\lambda_n$. The function $A(\rho) = O(\rho^d)$ increases to infinity and is greater than $1/d$ for all $\rho > c\sqrt{d}$. Therefore, for such ρ, the series in (4) converges and the whole bound tends to 0 as $\rho \to \infty$ providing the almost sure finiteness of $R_\infty(z)$.

Inequality (5) is proved much in the same manner. Fix a small $0 < \varepsilon < 1$ and consider a sequence $\{\rho'_n\}$ with $\rho'_0 = \rho$ that monotonously decreases to $\varepsilon\rho$. First, from

$$
\{r_\infty(z) = 0,\ r_0(z) > \rho\} \subset \cup_{n=1}^{\infty} \{r_n(z) < \rho'_n,\ r_0(z) > \rho\}
$$

it follows that

$$
\begin{aligned}
&\mathbf{P}^0 \big\{ r_\infty(z) = 0 \,\big|\, r_0(z) > \rho \big\} \\
&\quad \leq \sum_{n=1}^{\infty} \mathbf{P}^0 \big\{ r_n(z) < \rho'_n,\ r_{n-1}(z) \geq \rho'_{n-1} \,\big|\, r_0(z) > \rho \big\}.
\end{aligned} \tag{18}
$$

The event $\{r_n(z) < \rho'_n,\ r_{n-1}(z) \geq \rho'_{n-1}\}$ implies the existence of a Voronoi cell $C^n(x_i^n)$ with the nucleus outside of the ball $b(z, \rho'_{n-1})$ having some point $y \in C^n(x_i^n)$ inside $b(z, \rho'_n)$. Hence, there exists a ball of radius at least $\Delta'_n = \rho'_{n-1} - \rho'_n$ centered on the sphere $\partial b(z, \rho'_{n-1})$. Note that this event is independent of the event $\{r_0(z) > \rho\}$. Thus the summands in (18) can be bounded as in (15) with $\Delta'_n = (1-\varepsilon)\Delta_n$. With that choice of Δ'_n the right hand side of (18) is bounded by an expression similar to (17) with c replaced by $(1-\varepsilon)c$. Due to the arbitrariness of ε, expression (17) also provides an upper bound. The rest of the proof remains unchanged.

4.3. Corollary 1

Proof. Recall the following integral estimate: $\sum_{n=1}^{\infty} h(n) \le h(1) + \int_1^{\infty} h(x)dx$ for any non-increasing positive function $h(x)$. We have

$$\sum_{n=1}^{\infty} e^{-\phi(\lambda^n)A(\rho)} \le e^{-\phi(\lambda)A(\rho)} + \int_1^{\infty} e^{-\phi(\lambda^x)A(\rho)}dx$$
$$= e^{-\phi(\lambda)A(\rho)} + \frac{1}{\log\lambda}\int_{\phi(\lambda)}^{\infty}(1+y^{-1})e^{-yA(\rho)}dy$$

after the variable change $y = \phi(\lambda^x)$. Next, since $1 + y^{-1} \le 1 + \phi(\lambda)^{-1}$ on the integration domain, the whole expression can be bounded by

$$e^{-\phi(\lambda)A(\rho)} + \frac{1+\phi(\lambda)^{-1}}{A(\rho)\log\lambda}e^{-\phi(\lambda)A(\rho)} < \Big(1 + \frac{1+\phi(\lambda)^{-1}}{d^{-1}\log\lambda}\Big)e^{-\phi(\lambda)A(\rho)},$$

so that (4-5) become (7-8), respectively. It can be immediately verified that these estimates become nontrivial if $\rho > c_*$ and that $c_* > c\sqrt{d}$.

4.4. Corollary 2

Proof. Let τ_x be the stationary shift defined on the probability space Ω such that $\Pi_n(\tau_x\omega)(B) = \Pi_n(\omega)(B - x)$ for any Borel set $B \subset \mathbb{R}^d$ and any n. In this notation, $\tau_{x_i^0}R_n(0)$ for $x_i^0 \in \Pi_0$ is the maximal distance from x_i^0 to the boundary of $C_0^n(x_i^0)$ that corresponds to the above definition of $R_n(0)$ with $C_0^n(0)$ replaced with $C_0^n(x_i^0) = \tau_{x_i^0}C_0^n(0)$. The probability that there exists an unbounded family of cells with the nucleus in a ball $b(0, N)$ equals

$$\mathbf{P}\bigcup_{x_i\in\Pi_0\cap b(0,N)}\big\{\tau_{x_i^0}R_\infty(0) = \infty\big\} \le \mathbf{E}\sum_{x_i\in\Pi_0\cap b(0,N)}\mathbb{1}\big\{\tau_{x_i^0}R_\infty(0) = \infty\big\}$$
$$= b_d N^d\,\mathbf{E}^0\,\mathbb{1}\{R_\infty(0) = \infty\} = 0,$$

where we have used the Campbell theorem and (4). Letting N grow to infinity proves the assertion.

4.5. Corollary 3

Proof. It is sufficient that the distance from y to the boundary of the Π_0-cell containing y is sufficiently large so that the boundaries of the progressing n-cells never reach y. The probability of the latter event is positive by (5).

Due to stationarity, we may put $y = 0$. Consider the following representation:

$$\mathbf{P}\big\{\exists x_i^0\ :\ 0 \in \mathrm{int}\,\big(\cap_n C_0^n(x_i^0)\big)\big\} = \mathbf{E}\sum_{x_i^0\in\Pi_0}\mathbb{1}\Big(0 \in \mathrm{int}\,\big(\cap_n C_0^n(x_i^0)\big)\Big).$$

By the Campbell theorem and by the isotropy, the right-hand side equals

$$\int_{\mathbb{R}^d} \mathbf{P}^0\big\{ -z \in \operatorname{int}\big(\cap_n C_0^n(0)\big)\big\}\, dz$$
$$\geq \int_{\mathbb{R}^d} \mathbf{P}^0\big\{z \in \operatorname{int}\big(\cap_n C_0^n(0)\big)\,\big|\, b(z,\rho) \subseteq C^0(0)\big\}\mathbf{P}^0\big\{b(z,\rho) \subseteq C^0(0)\big\}\, dz \quad (19)$$

for arbitrary $\rho > 0$. By Theorem 2, the first factor under the integral in (19) is greater than $1 - a_1 \sum_{n=1}^{\infty} \big(\phi(\lambda_n)/\lambda_n\big)^{A(\rho)}$ provided that $\rho \geq c\sqrt{d}$. The second factor is the probability that no points of Π_0 lie in the figure $\cup_{\|z-z'\|=\rho} b(z', \|z'\|)$. This figure is obtained by rotation of a cardioid around its symmetry axis; by construction, it is contained in the ball $b(z, \|z\| + 2\rho)$. Therefore,

$$\mathbf{P}^0\big\{b(z,\rho) \subseteq C^0(0)\big\} > \exp\big\{-b_d(\|z\| + 2\rho)^d\big\}\,.$$

Using this estimate we get

$$\begin{aligned}\int_{\mathbb{R}^d} \mathbf{P}^0\big\{b(z,\rho) \subseteq C^0(0)\big\}\, dz &> \int_{\mathbb{R}^d} \exp\big\{-b_d(\|z\| + 2\rho)^d\big\}\, dz \\ &> \int_0^{\rho} r^{d-1} d b_d \exp\big\{-b_d(3\rho)^d\big\}\, dr \\ &= b_d \rho^d \exp\big\{-b_d(3\rho)^d\big\}. \end{aligned} \quad (20)$$

Hence

$$\mathbf{P}\big\{\exists x_i^0 \,:\, y \in \operatorname{int}\big(\cap_n C_0^n(x_i^0)\big)\big\}$$
$$> \sup_{\rho \geq c\sqrt{d}} \left[b_d \rho^d \exp\big\{-b_d(3\rho)^d\big\} \left(1 - a_1 \sum_{n=1}^{\infty} \left(\frac{\phi(\lambda_n)}{\lambda_n}\right)^{A(\rho)}\right)\right] > 0\,. \quad (21)$$

4.6. Theorem 4

Proof. The proof consists of two parts: in the first one we prove that the sets $C_0^{\infty}(x_i^0)$ constitute a tessellation, in the second we show that the interiors of the alternatively defined limit cells coincide.

Part I. Recall the definition of a tessellation from Section 1. We start by verifying condition *(b)*. Consider disjoint open sets $O_i = \cup_m \operatorname{int}\big(\cap_{n \geq m} C_0^n(x_i^0)\big)$. Every $x_j^m \in \Pi_m$ is contained in an almost surely unique aggregate cell $C_0^{m-1}(x_i^0)$. Thus if (11) holds, the set $\Gamma' \equiv \mathbb{R}^d \setminus \cup_i O_i$ a.s. misses the origin, which implies that $|\Gamma'| = 0$. Since $O_i \subset C_0^{\infty}(x_i^0)$, we have $C_0^{\infty}(x_i^0) \cap C_0^{\infty}(x_j^0) \subset \Gamma_0^{\infty} \subseteq \Gamma'$ for $i \neq j$.

Let us verify *(c)*. For a bounded Borel set $B \subset \mathbb{R}^d$ introduce a random variable

$$N(B) = \limsup_{n\to\infty} \sum_{x_i^n \in \Pi_n} \mathbb{1}\{C_n^{\infty}(x_i^n) \cap B \neq \emptyset\}.$$

Obviously, if $N(B) < \infty$, only a finite number of cells $C_0^{\infty}(x_i^0)$ intersects B. Denote by σ_n the σ-algebra generated by the sequence of processes $\{\Pi_k,\, k \geq n\}$, and note that the event $\{N(B) = \infty\}$ belongs to the tail sigma-algebra $\sigma_\infty = \cap_n \sigma_n$. Since all the processes Π_k are independent, the zero-one law applies so that $\mathbf{P}\{N(B) = \infty\} = 0$ or 1. From (11) it follows that for some $\varepsilon > 0$,

$$\mathbf{P}\big\{\exists x_j^m \in \Pi_m \;:\; b(0,\varepsilon) \in \operatorname{int}(\cap_{n=m}^{\infty} C_m^n(x_j^m))\big\} > 0\,,$$

and hence, $\mathbf{P}\{N(b(0,\varepsilon)) = \infty\} = 0$. Every bounded set $B \subset \mathbb{R}^d$ can be covered by a finite family $\{b(t_k,\varepsilon)\}_{k\le K}$ of copies of $b(0,\varepsilon)$ shifted by t_k. Because of stationarity, $N(b(0,\varepsilon))$ and $N(b(t_k,\varepsilon))$ have the same distribution for each $t_k \in \mathbb{R}^d$, therefore

$$\mathbf{P}\{N(B) = \infty\} \le \sum_{k=1}^{K} \mathbf{P}\{N\big(b(t_k,\varepsilon)\big) = \infty\} = 0.$$

In order to prove *(a)*, we need to show that the set $\Phi = \mathbb{R}^d \setminus \cup_i C_0^\infty(x_i^0)$ is a.s. empty. Observe that $\Phi \subseteq \Gamma_0^\infty$, and therefore Φ contains only boundary points if non-empty. For such a point $y \in \Phi$, there exists a sequence $\{y_k\} \subset \cup_i C_0^\infty(x_i^0)$ converging to y. Being itself a bounded set, this sequence visits only a finite number of limit cells $C_0^\infty(x_i^0)$. At least one of these cells contains an infinite subsequence $\{y_{k_n}\}$, it contains also y because cells are closed sets. We come to a contradiction with the non-emptiness of Φ.

Part II. We will show that the interiors of the cells defined in (9-10) coincide with $\operatorname{int}(\operatorname{cl}(O_i))$ a.s. Because $\operatorname{cl}(O_i) \subseteq C_0^\infty(x_i^0) \subseteq D_0^\infty(x_i^0) \subseteq F_0^\infty(x_i^0)$ and $\operatorname{cl}(O_i) \subseteq E_0^\infty(x_i^0) \subseteq F_0^\infty(x_i^0)$, it is sufficient to show that $\operatorname{int}(\operatorname{cl}(O_i)) = \operatorname{int}\big(F_0^\infty(x_i^0)\big)$.

Suppose $x \notin \operatorname{int}(\operatorname{cl}(O_i))$. Every open neighborhood $v(x)$ of x contains a point $y \notin \operatorname{cl}(O_i)$. Therefore, $v(x)$ contains an open subset $v(y)$ disjoint of $\operatorname{cl}(O_i)$. Since $|\Gamma'| = 0$, there exists a point $z \in v(y) \cap O_j$ for some $j \ne i$. But $O_j \cap F_0^\infty(x_i^0) = \emptyset$ and thus $x \notin \operatorname{int}\big(F_0^\infty(x_i^0)\big)$.

Hence, the boundary Γ_0^∞ is common for the tessellations defined by (9–10).

We now prove that Γ' is the limit of the boundaries Γ_0^n as it was mentioned in Remark 2. Let us verify that $\Gamma' \subseteq \liminf_n \Gamma_0^n$. If $x \in \Gamma'$, then any neighborhood $v(x)$ of x hits at least two disjoint sets O_i and O_j. Therefore, $v(x)$ hits $\cap_{n\ge m} C_0^n(x_i^0)$ and $\cap_{n\ge m} C_0^n(x_j^0)$ starting from some m. Then it must also hit Γ_0^n for all $n \ge m$.

Now we prove that $\limsup_n \Gamma_0^n \subseteq \Gamma'$. Let n_k be a sequence of natural numbers such that $\lim_k n_k = \infty$. Suppose $x_{n_k} \in \Gamma_0^{n_k}$ and $x = \lim_k x_{n_k}$. If $x \notin \Gamma'$, then $x \in O_i$ for some i. Then $x \in \operatorname{int} \cap_{n\ge m} C_0^n(x_i^0)$ for some m, and the sequence of $x_{n_k} \in \Gamma_0^{n_k}$ cannot converge to x. From this contradiction it follows that $x \in \Gamma'$.

4.7. Corollary 5

Proof. By Theorem 4, $\mathbf{P}^0\big\{y \in C_0^\infty(0)\big\} = \mathbf{P}^0\big\{y \in D_0^\infty(0)\big\}$. Using the continuity property of the probability measures, we obtain

$$\mathbf{P}^0\big\{y \in C_0^\infty(0)\big\} = \lim_m \mathbf{P}^0\big\{y \in \cup_{n\ge m} C_0^n(0)\big\} \ge \lim_n \mathbf{P}^0\big\{y \in C_0^n(0)\big\},$$

$$\mathbf{P}^0\big\{y \in C_0^\infty(0)\big\} = \lim_m \mathbf{P}^0\big\{y \in \cap_{n\ge m} C_0^n(0)\big\} \le \lim_n \mathbf{P}^0\big\{y \in C_0^n(0)\big\}.$$

The second statement is obtained by simply replacing $\mathbf{P}^0\big\{y \in \cdot\big\}$ by $|\cdot|$ above.

4.8. Theorem 5

Proof. Suppose, the tessellations Θ^n are self-similar with the coefficient $\nu > 0$. Let $v(0)$ be an open neighborhood of the origin, and let $x^n(0)$ be the point of Π_n such that $0 \in C^n(x^n(0))$. Consider the following r.v.'s:

$$
\begin{aligned}
T_1 &= \min\big\{k > 0 \,:\, v(0) \not\subset C_0^k\big(x^0(0)\big)\big\},\\
T_2 &= \min\big\{k > T_1 \,:\, v(0) \not\subset \nu^{T_1+1} C^k_{T_1+1}\big(x^{T_1+1}(0)\big)\big\},\\
&\dots\\
T_{n+1} &= \min\big\{k > T_n \,:\, v(0) \not\subset \nu^{T_n+1} C^k_{T_n+1}\big(x^{T_n+1}(0)\big)\big\}, \text{ etc.}
\end{aligned}
$$

Thus defined r.v.'s are stopping times and the distribution of T_{n+1} given $\{T_n = k\}$ depends only on $\{\Pi_n\}_{n>k}$. By self-similarity we have $\mathbf{P}\{T_{n+1} < \infty \,\big|\, T_n < \infty\} = \mathbf{P}\{T_1 < \infty\} = p$, which is strictly smaller than 1 by assumption (13). Therefore,

$$
\mathbf{P}\{T_1 < \infty, \dots, T_n < \infty\} = p^n . \tag{22}
$$

Thus $\mathbf{P}\{\forall k,\ T_k < \infty\} = 0$ and a.s. there exists n such that $T_n < \infty$, $T_{n+1} = \infty$, meaning that condition (11) is satisfied with $m = T_n + 1$.

4.9. Theorem 6

Proof. Consider the cells of the limit tessellation $\{C_n^\infty(x_i^n)\}$ defined in the same way as in (9). Let $X(n, r)$ be the nuclei of those cells whose boundary crosses the ball $b(0, r)$, i. e.,

$$
X(n, r) = \{x_i^n \in \Pi_n \,:\, \partial C_n^\infty(x_i^n) \cap b(0, r) \neq \emptyset\}.
$$

We will first prove the estimate: for each $n \geq 1$ and for each $s > 0$,

$$
\mathbf{P}\big\{b(0, r) \cap \Gamma_0^\infty \neq \emptyset\big\} < f(r, n, s)^n , \tag{23}
$$

where $f(r, n, s) = 1 - \mathbf{P}\big\{X(n, r) \subseteq b(0, s)\big\}\mathbf{P}\big\{b(0, s) \subseteq C^{n-1}(x^{n-1}(0))\big\}$, and $x^{n-1}(0)$ denotes the closest point of Π_{n-1} to 0.

Consider the events $E(m, r) = \{b(0, r) \cap \Gamma_m^\infty \neq \emptyset\}$, where Γ_m^∞ is the boundary of the tessellation $\{C_m^\infty(x_i^m)\}$. Note that $E(0, r)$ is the event in the left-hand side of (23). Since $\Gamma_m^\infty \subseteq \Gamma_{m+1}^\infty$, we have $E(m, r) \subseteq E(m + 1, r)$ and therefore,

$$
\begin{aligned}
\mathbf{P}\big\{E(0, r)\big\} &= \mathbf{P}\big\{E(1, r)\big\}\mathbf{P}\big\{E(0, r) \,\big|\, E(1, r)\big\}\\
&= \mathbf{P}\big\{E(n, r)\big\} \prod_{m=1}^{n} \mathbf{P}\big\{E(m-1, r) \,\big|\, E(m, r)\big\}\\
&< \prod_{m=1}^{n} \mathbf{P}\big\{E(m-1, r) \,\big|\, E(m, r)\big\}.
\end{aligned} \tag{24}
$$

We assert that for every m and $s_m > 0$,

$$
\mathbf{P}\big\{E(m-1, r) \,\big|\, E(m, r)\big\} \leq f(r, m, s_m). \tag{25}
$$

Indeed, if $b(0,r) \cap \Gamma_m^\infty \neq \emptyset$ and

$$X(m,r) \subseteq b(0,s_m) \subseteq C^{m-1}(x^{m-1}(0)),$$

then the cells $C_m^\infty(x_i^m)$ for which $x_i^m \in X(m,r)$ join in $C_{m-1}^\infty(x^{m-1}(0))$ so that $b(0,r) \cap \Gamma_{m-1}^\infty = \emptyset$.

Since the intensity of each Π_n equals λ^n, the distributions of

$$(\Pi_n, \Pi_{n+1}, \dots) \quad \text{and} \quad (\lambda^{1/d}\Pi_{n+1}, \lambda^{1/d}\Pi_{n+2}, \dots)$$

coincide. Consequently, the sets $X(m+1,r)$ and $\lambda^{-1/d}X(m,\lambda^{1/d}r)$ have the same distribution, and

$$\begin{aligned}\mathbf{P}\big\{X(m+1,r) \subseteq b(0,s_m)\big\} &= \mathbf{P}\big\{\lambda^{-1/d}X(m,\lambda^{1/d}r) \subseteq b(0,s_m)\big\} \\ &= \mathbf{P}\big\{X(m,\lambda^{1/d}r) \subseteq b(0,\lambda^{1/d}s_m)\big\} \\ &< \mathbf{P}\big\{X(m,r) \subseteq b(0,\lambda^{1/d}s_m)\big\},\end{aligned}$$

as $X(m,r) \subseteq X(m,\lambda^{1/d}r)$. Also

$$\mathbf{P}\big\{b(0,s_m) \subseteq C^m(x^m(0))\big\} = \mathbf{P}\big\{b(0,\lambda^{1/d}s_m) \subseteq C^{m-1}(x^{m-1}(0))\big\}$$

so that $f(r,m-1,\lambda^{1/d}s_m) < f(r,m,s_m)$. Alternatively, $f(r,m,s_m) < f(r,m+1,\lambda^{-1/d}s_m) < f(r,n,\lambda^{-(n-m)/d}s_m)$ by induction. Thus, taking $s_m = \lambda^{(n-m)/d}s$ in (25), by (24) we obtain (23).

Next, we find a bound for $f(r,n,s)$. We have

$$\begin{aligned}&1 - \mathbf{P}\big\{X(n,r) \subseteq b(0,s)\big\} \\ &\quad = 1 - \mathbf{P}\big\{X(0,\lambda^{n/d}r) \subseteq b(0,\lambda^{n/d}s)\big\} \\ &\quad < \mathbf{P}\big\{\exists x_i^0 \,:\, x_i^0 \notin b(0,\lambda^{n/d}s) \text{ and } C_0^\infty(x_i^0) \cap b(0,\lambda^{n/d}r) \neq \emptyset\big\} \\ &\quad < \mathbf{E}\sum_{x_i^0 \in \Pi_0} \mathbb{1}\big(\|x_i^0\| > \lambda^{n/d}s \text{ and } C_0^\infty(x_i^0) \cap b(0,\lambda^{n/d}r) \neq \emptyset\big).\end{aligned} \tag{26}$$

Recall the definition of $R_\infty(z)$ from Section 2.2. By the Campbell theorem, the last expectation in (26) equals

$$\begin{aligned}&\int_{\|z\|>\lambda^{n/d}s} \mathbf{P}^0\big\{C_0^\infty(0) \cap b(-z,\lambda^{n/d}r) \neq \emptyset\big\}\,dz \\ &\quad = \int_{\|z\|>\lambda^{n/d}s} \mathbf{P}^0\big\{R_\infty(0) > \|z\| - \lambda^{n/d}r\big\}\,dz \\ &\quad = \lambda^n \int_{\|z\|>s} \mathbf{P}^0\big\{R_\infty(0) > \lambda^{n/d}(\|z\| - r)\big\}\,dz\,.\end{aligned} \tag{27}$$

Choose $s = 2r$. Then by (7), the right-hand side of (27) does not exceed

$$\begin{aligned}c_1\lambda^n \int_{\|z\|>2r} e^{-c_2\lambda^n(\|z\|-r)^d}\,dz &= c_1\lambda^n \int_{2r}^\infty db_d\rho^{d-1}e^{-c_2\lambda^n(\rho-r)^d}\,d\rho \\ &< c_1\lambda^n \int_{2r}^\infty db_d(2\rho-2r)^{d-1}e^{-c_2\lambda^n(\rho-r)^d}\,d\rho \\ &= (2^{d-1}c_1/c_2)e^{-c_2\lambda^n r^d}.\end{aligned}$$

Denote $a = (1/c_2)\log(2^d c_1/c_2)$ and choose $n = \lfloor d \log_\lambda(a/r)\rfloor + 1$, where $\lfloor\cdot\rfloor$ is the integer part of a number. With such n, inequality $a \le \lambda^n r^d \le \lambda a$ holds and $(2^{d-1}c_1/c_2)e^{-c_2\lambda^n r^d} \le 1/2$. Therefore,

$$\mathbf{P}\big\{X(n,r) \subseteq b(0,s)\big\} \ge 1/2. \tag{28}$$

Since there is only one Voronoi cell that may contain a ball, we may write

$$\begin{aligned}\mathbf{P}\big\{b(0,s) \subseteq C^{n-1}(x^{n-1}(0))\big\} &= \mathbf{P}\big\{b(0,\lambda^{(n-1)/d}s) \subseteq C^0(x^0(0))\big\} \\ &= \mathbf{E}\sum_{x_i^0\in\Pi_0} \mathbb{I}\big\{b(0,\lambda^{(n-1)/d}s) \subseteq C^0(x_i^0)\big\} \\ &= \int \mathbf{P}^0\big\{b(z,\lambda^{(n-1)/d}s) \subseteq C^0(0)\big\}\,dz\,.\end{aligned}$$

Using the lower bound from (20), for s and n chosen as above we obtain

$$\begin{aligned}\mathbf{P}\big\{b(0,s) \subseteq C^{n-1}(x^{n-1}(0))\big\} &> b_d\lambda^{n-1}s^d \exp\big\{-b_d\lambda^{n-1}(3s)^d\big\} \\ &\ge (2^d a b_d/\lambda)\exp\big\{-6^d a b_d\big\}.\end{aligned} \tag{29}$$

Putting together (28) and (29) we get from (23)

$$\begin{aligned}\mathbf{P}\big\{b(0,r)\cap\Gamma_0^\infty \ne \emptyset\big\} &\le \Big[1-(2^{d-1}ab_d/\lambda)\exp\big\{-6^d a b_d\big\}\Big]^{\lfloor d\log_\lambda(a/r)\rfloor+1} \\ &\le K r^{dq},\end{aligned}$$

where

$$q = -\log_\lambda\Big[1-(2^{d-1}ab_d/\lambda)\exp\big\{-6^d a b_d\big\}\Big] \quad \text{and} \quad K = a^{-dq}. \tag{30}$$

Since $\lambda > 1$, we have $0 < q < 1$. The theorem is proved.

4.10. Theorem 7

Proof. Consider the collection of mesh cubes of size M in $\mathbb{R}^d$ and let $\{\theta_\alpha\}$ be the family of shifts translating the cube at the origin $[0,M)^d$ by the vector $M\alpha$, where $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{Z}^d$. Introduce also

$$\Gamma(M) = \Gamma_0^\infty \cap [0,M)^d\,, \qquad \theta_\alpha\Gamma(M) = \Gamma_0^\infty \cap \theta_\alpha[0,M)^d.$$

Since $\theta_\alpha\Gamma(M) \subseteq \Gamma_0^\infty$, with probability 1 for all $\alpha\in\mathbb{Z}^d$, we have

$$\dim_H \Gamma_0^\infty \ge \dim_H \theta_\alpha\Gamma(M).$$

Denote $\Lambda_N = \{\alpha : |\alpha_i| \le N,\ i = 1,\dots,d\}$. Obviously,

$$\dim_H \Gamma_0^\infty \ge \sup_{\alpha\in\Lambda_N} \dim_H \theta_\alpha\Gamma(M).$$

Now by the ergodic theorem,

$$\begin{aligned}\dim_H \Gamma_0^\infty &\geq \lim_{N\to\infty} \sup_{\alpha\in\Lambda_N} \dim_H \theta_\alpha \Gamma(M) \\ &\geq \lim_{N\to\infty} \frac{1}{(2N)^d} \sum_{\alpha\in\Lambda_N} \dim_H \theta_\alpha \Gamma(M) = \mathbf{E} \dim_H \Gamma(M).\end{aligned}$$

Letting $M \to \infty$ in this inequality and using the property of monotonicity of the Hausdorff dimension, we get $\dim_H \Gamma_0^\infty \geq \mathbf{E} \dim_H \Gamma$, which implies the first equality of the theorem.

To prove the second inequality, we make use of the estimate of the Hausdorff dimension of a set by its upper box dimension (see, e. g.,[5, p. 24]). Let $N_\varepsilon(B)$ be the smallest number of closed balls of radius ε that cover B. Then

$$\dim_H \Gamma_0^\infty(M) \leq \limsup_{\varepsilon\to 0} \frac{\log N_\varepsilon(\Gamma_0^\infty(M))}{-\log\varepsilon}\,.$$

Take expectations at both sides of this inequality. It can be easily verified that $[0, M)^d$, and hence $\Gamma_0^\infty(M)$, can be covered by a family $\{b_i\}$ of less than $\left(M\sqrt{d}/2\varepsilon\right)^d$ balls of radius ε. Thus the function in the right-hand side under the limit is bounded by a constant not depending on ε, and therefore, we can exchange the limit and the expectation. Moreover, the function $\log(\cdot)$ is concave, hence $\mathbf{E}\log(\cdot) \leq \log \mathbf{E}(\cdot)$. Therefore,

$$\mathbf{E} \dim_H \Gamma_0^\infty(M) \leq \limsup_{\varepsilon\to 0} \frac{\log \mathbf{E}\, N_\varepsilon(\Gamma(M))}{-\log\varepsilon}.$$

Recalling the definition of the contact distribution $H(r)$ from the previous section, we get

$$\mathbf{E}\, N_\varepsilon(\Gamma_0^\infty(M)) \leq \mathbf{E} \sum_i \mathbb{I}(b_i \cap \Gamma_0^\infty(M)) \neq \emptyset) \leq \left(M\sqrt{d}/2\varepsilon\right)^d H(\varepsilon).$$

From Theorem 6 it follows that

$$\begin{aligned}&\mathbf{E} \dim_H \Gamma_0^\infty(M) \\ &\quad\leq \limsup_{\varepsilon\to 0} \frac{d\log(M\sqrt{d}/2) - d\log\varepsilon + \log K + dq\log\varepsilon}{-\log\varepsilon} = d(1-q)\,,\end{aligned}$$

and it remains to let $M \to \infty$ to obtain the second statement of the theorem.

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